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Direct and inverse spectral problems for discrete Sturm-Liouville problem with generalized function potential

Bayram Bala[1], Abdullah Kablan[1] and Manaf Dzh Manafov[2*]

In memory of GSh Guseinov (1951-2015)

*Correspondence:
mmanafov@adiyaman.edu.tr
[2]Faculty of Arts and Sciences, Department of Mathematics, Adıyaman University, Adıyaman, 02040, Turkey
Full list of author information is available at the end of the article

Abstract

In this work, we study the inverse problem for difference equations which are constructed by the Sturm-Liouville equations with generalized function potential from the generalized spectral function (GSF). Some formulas are given in order to obtain the matrix J, which need not be symmetric, by using the GSF and the structure of the GSF is studied.

MSC: Primary 39A12; 34A55; 34L15

Keywords: difference equation; inverse problems; generalized spectral function

1 Introduction

In this paper we deal with the $N \times N$ tridiagonal matrix

$$J = \begin{bmatrix} b_0 & a_0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & a_1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_M & a_M & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & c_M & d_{M+1} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & d_{N-3} & c_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & c_{N-3} & d_{N-2} & c_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & c_{N-2} & d_{N-1} \end{bmatrix}, \tag{1.1}$$

where $a_n, b_n \in \mathbb{C}$, $a_n \neq 0$ and

$$c_n = a_n/\alpha, \quad n \in \{M, M+1, \dots, N-2\},$$
$$d_n = b_n/\alpha, \quad n \in \{M+1, M+2, \dots, N-1\},$$

and $\alpha \neq 1$ is a positive real number.

The definitions and some properties of GSF are given in [1–6]. The inverse problem for the infinite Jacobi matrices from the GSF was investigated in [3–6], see also [7]. The inverse spectral problem for $N \times N$ tridiagonal symmetric matrix has been studied in [8] and the inverse spectral problem with spectral parameter in the initial conditions has been studied in [9]. The goal of this paper is to study the almost symmetric matrix J of the form (1.1). Almost symmetric here means that the entries above and below the main diagonal are the same except the entries a_M and c_M.

The eigenvalue problem we consider in this paper is $Jy = \lambda y$, where $y = \{y_n\}_{n=0}^{N-1}$ is a column vector. There exists a relation between this matrix eigenvalue problem and the second order linear difference equation

$$\begin{aligned} &a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda \rho_n y_n, \quad n \in \{0,1,\dots,M,\dots,N-1\}, \\ &a_{-1} = c_{N-1} = 1, \end{aligned} \tag{1.2}$$

for $\{y_n\}_{n=-1}^{N}$, with the boundary conditions

$$y_{-1} = y_N = 0, \tag{1.3}$$

where ρ_n is a constant defined by

$$\rho_n = \begin{cases} 1, & 0 \le n \le M, \\ \alpha, & M < n \le N-1, \end{cases} \quad 1 \ne \alpha > 0. \tag{1.4}$$

These expressions are equivalent. The problem (1.2), (1.3) is a discrete form of the Sturm-Liouville operator with discontinuous coefficients

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}y(x)\right] + q(x)y(x) = \lambda \rho(x) y(x), \quad x \in [a,b], \tag{1.5}$$

$$y(a) = y(b) = 0, \tag{1.6}$$

where $\rho(x)$ is a piecewise function defined by

$$\rho(x) = \begin{cases} 1, & a \le x \le c, \\ \alpha^2, & c < x \le b, \end{cases} \quad \alpha^2 \ne 1,$$

$[a,b]$ is a finite interval, α is a real number, and c is a discontinuity point in $[a,b]$. On eigenvalues and eigenfunctions of such an equation, see [10], and the inverse problem for this kind equation has been investigated in [11].

2 Generalized spectral function

In this section, we find the characteristic polynomial for the matrix J and then give the existence of linear functional which is defined from the ring of all polynomials in λ of degree $\le 2N$ with the complex coefficients to $\mathbb{C}$. Let us denote by $\{P_n(\lambda)\}_{n=-1}^{N}$, the solution of equation (1.2) together with the initial data

$$y_{-1} = 0, \qquad y_0 = 1. \tag{2.1}$$

By starting with (2.1), we can derive from equation (1.2) iteratively the polynomials $P_n(\lambda)$ of order n, for $n = \overline{1,N}$. In this way we obtain the unique solution $\{P_n(\lambda)\}_{n=0}^{N}$ of the following recurrence relations:

$$\begin{aligned}
&b_0P_0(\lambda) + a_0P_1(\lambda) = \lambda P_0(\lambda), \quad c_{N-1} = 1,\\
&a_{n-1}P_{n-1}(\lambda) + b_nP_n(\lambda) + a_nP_{n+1}(\lambda) = \lambda P_n(\lambda), \quad n \in \{1,2,\ldots,M\},\\
&c_{n-1}P_{n-1}(\lambda) + d_nP_n(\lambda) + c_nP_{n+1}(\lambda) = \lambda P_n(\lambda), \quad n \in \{M+1,\ldots,N-1\},
\end{aligned} \tag{2.2}$$

subject to the initial condition

$$P_0(\lambda) = 1. \tag{2.3}$$

Lemma 1 *The following equality holds*:

$$\det(J - \lambda I) = (-1)^N a_0a_1\cdots a_M c_{M+1}\cdots c_{N-1}P_N(\lambda). \tag{2.4}$$

Therefore, the roots of the polynomial $P_N(\lambda)$ and the eigenvalues of the matrix J are coincident.

Proof We will consider the proof in three cases. For each $n = \overline{1,M}$, let us define the determinant $\triangle_n(\lambda)$ as follows:

$$\triangle_n(\lambda) = \begin{vmatrix}
b_0-\lambda & a_0 & 0 & \cdots & 0 & 0 & 0\\
a_0 & b_1-\lambda & a_1 & \cdots & 0 & 0 & 0\\
0 & a_1 & b_2-\lambda & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & b_{n-3}-\lambda & a_{n-3} & 0\\
0 & 0 & 0 & \cdots & a_{n-3} & b_{n-2}-\lambda & a_{n-2}\\
0 & 0 & 0 & \cdots & 0 & a_{n-2} & b_{n-1}-\lambda
\end{vmatrix}.$$

Then expanding $\triangle_n(\lambda)$ by adding a row and column and finding the determinant of $\triangle_{n+1}(\lambda)$ by the elements of the last row, we obtain

$$\triangle_{n+1}(\lambda) = (b_n - \lambda)\triangle_n(\lambda) - a_{n-1}^2\triangle_{n-1}(\lambda), \quad n = \overline{1,M}, \triangle_0(\lambda) = 1. \tag{2.5}$$

Now for $n = \overline{M+2,N}$, let us define $\triangle_n(\lambda)$ as follows:

$$\begin{vmatrix}
b_0-\lambda & a_0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0\\
a_0 & b_1-\lambda & a_1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0\\
0 & a_1 & b_2-\lambda & \cdots & 0 & 0 & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & b_M-\lambda & a_M & \cdots & 0 & 0 & 0\\
0 & 0 & 0 & \cdots & c_M & d_{M+1}-\lambda & \cdots & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & d_{n-3}-\lambda & c_{n-3} & 0\\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & c_{n-3} & d_{n-2}-\lambda & c_{n-2}\\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & c_{n-2} & d_{n-1}-\lambda
\end{vmatrix}.$$

By using the same method, we get

$$\triangle_{n+1}(\lambda) = (d_n - \lambda)\triangle_n(\lambda) - c_{n-1}^2 \triangle_{n-1}(\lambda), \tag{2.6}$$

and finally, for $n = M + 1$, we find

$$\triangle_{M+2}(\lambda) = (d_{M+1} - \lambda)\triangle_{M+1}(\lambda) - a_M c_M \triangle_M(\lambda). \tag{2.7}$$

Dividing (2.5) and (2.7) by the product $a_0 \cdots a_{n-1}$, (2.6) by the product $a_0 \cdots a_{n-2} c_{n-1}$, we can easily show that the sequence

$$\begin{aligned} &h_{-1} = 0, \qquad h_0 = 1, \qquad h_n = (-1)^n (a_0 \cdots a_{n-1})^{-1} \triangle_n(\lambda), \quad n = \overline{1, M+1}, \\ &h_n = (-1)^n (a_0 \cdots c_{M+1} \cdots c_{n-1})^{-1} \triangle_n(\lambda), \quad n = \overline{M+2, N}, \end{aligned}$$

satisfies (1.2), (2.1). Then h_n is solution of (1.2), (2.1). We can show it by $P_n(\lambda)$ for $n = \overline{0, N}$. Since $\triangle_N(\lambda)$ is also equal to $\det(J - \lambda I)$ if we combine (2.5), (2.6) and (2.7), we obtain (2.4), for all $n \in \{0, 1, \ldots, M, M+1, \ldots, N\}$. □

Theorem 1 *There exists a unique linear functional* $\Omega : \mathbb{C}_{2N}[\lambda] \to \mathbb{C}$ *such that the following relations hold*:

$$\Omega\bigl(P_m(\lambda) P_n(\lambda)\bigr) = \frac{\delta_{mn}}{\eta}, \quad m, n \in \{0, 1, \ldots, M, \ldots, N-1\}, \tag{2.8}$$

$$\Omega\bigl(P_m(\lambda) P_N(\lambda)\bigr) = 0, \quad m \in \{0, 1, \ldots, M, \ldots, N\}, \tag{2.9}$$

where δ_{mn} *is the Kronecker delta*, η *is defined by*

$$\eta = \begin{cases} 1, & m, n \le M, \\ \alpha, & m, n > M, \end{cases} \tag{2.10}$$

and $\Omega(P(\lambda))$ *shows the value of* Ω *on any polynomial* $P(\lambda)$.

Proof In order to show the uniqueness of Ω we assume that there exists such a linear functional Ω, satisfying (2.8) and (2.9). Let us define the $2N + 1$ polynomials as follows:

$$P_n(\lambda) \quad (n = \overline{0, N-1}), \qquad P_m(\lambda) P_N(\lambda) \quad (m = \overline{0, N}). \tag{2.11}$$

It is clear that this polynomial set is a basis for the linear space $\mathbb{C}_{2N}[\lambda]$. Indeed the polynomials defined by (2.11) are linearly independent and their number is equal to dimension of $\mathbb{C}_{2N}[\lambda]$. On the other hand, by using (2.8) and (2.9), the quantities of the polynomials given in (2.11) under the functional Ω can be found as finite values:

$$\Omega\bigl(P_n(\lambda)\bigr) = \frac{\delta_{0n}}{\eta}, \quad n \in \{0, 1, \ldots, M, \ldots, N-1\}, \tag{2.12}$$

$$\Omega\bigl(P_m(\lambda) P_N(\lambda)\bigr) = 0, \quad m \in \{0, 1, \ldots, N\}. \tag{2.13}$$

Therefore, by linearity, the functional Ω defined on $\mathbb{C}_{2N}[\lambda]$ is unique.

To show the existence of Ω, let us define it on the polynomials (2.11) by (2.12), (2.13) and then we expand Ω to over the whole space $\mathbb{C}_{2N}[\lambda]$ by using the linearity of Ω. It can be shown that the functional Ω satisfies (2.8), (2.9). Denote

$$\Omega\bigl(P_m(\lambda)P_n(\lambda)\bigr) = B_{mn}, \quad m,n \in \{0,1,\ldots,M,\ldots,N\}. \tag{2.14}$$

It is clear that $B_{mn} = B_{nm}$, for $m,n \in \{0,1,\ldots,N\}$. From (2.12) and (2.13), we get

$$B_{m0} = B_{0m} = \delta_{m0}, \quad m \in \{0,1,\ldots,M\}, \tag{2.15}$$

$$B_{m0} = B_{0m} = \frac{\delta_{m0}}{\alpha}, \quad m \in \{M+1,\ldots,N\}, \tag{2.16}$$

$$B_{mN} = B_{Nm} = 0, \quad m \in \{0,1,\ldots,N\}. \tag{2.17}$$

Since $\{P_n(\lambda)\}_0^N$ is the solution of (2.2), we derive from the first equation of (2.2), using (2.3),

$$\lambda = b_0 + a_0 P_1(\lambda).$$

Inserting this into the remaining equations in (2.2), we get

$$a_{n-1}P_{n-1}(\lambda) + b_n P_n(\lambda) + a_n P_{n+1}(\lambda) = b_0 P_n(\lambda) + a_0 P_1(\lambda)P_n(\lambda), \quad n \in \{1,2,\ldots,M\},$$
$$c_{n-1}P_{n-1}(\lambda) + d_n P_n(\lambda) + c_n P_{n+1}(\lambda) = b_0 P_n(\lambda) + a_0 P_1(\lambda)P_n(\lambda), \quad n \in \{M+1,\ldots,N-1\}.$$

If we apply the linear functional Ω to both sides of the last two equations, by taking into account (2.15), (2.16), and (2.17), we get

$$B_{n1} = B_{1n} = \delta_{n1}, \quad n \in \{0,1,\ldots,M\}, \tag{2.18}$$

$$B_{n1} = B_{1n} = \frac{\delta_{n1}}{\alpha}, \quad n \in \{M+1,\ldots,N\}. \tag{2.19}$$

Further, recalling the definition of ρ_n in (1.4), we write

$$a_{m-1}P_{m-1}(\lambda) + b_m P_m(\lambda) + a_m P_{m+1}(\lambda) = \lambda\rho_m P_m(\lambda), \quad m \in \{1,2,\ldots,M,\ldots,N-1\},$$
$$a_{n-1}P_{n-1}(\lambda) + b_n P_n(\lambda) + a_n P_{n+1}(\lambda) = \lambda\rho_n P_n(\lambda), \quad n \in \{1,2,\ldots,M,\ldots,N-1\}.$$

If the first equality is multiplied by $P_n(\lambda)$ and the second equality is multiplied by $P_m(\lambda)$, then the second result is subtracted from the first, we obtain:

for $m,n \in \{1,2,\ldots,M\}$,

$$\begin{aligned} &a_{m-1}P_{m-1}(\lambda)P_n(\lambda) + b_m P_m(\lambda)P_n(\lambda) + a_m P_{m+1}(\lambda)P_n(\lambda) \\ &\quad = a_{n-1}P_{n-1}(\lambda)P_m(\lambda) + b_n P_n(\lambda)P_m(\lambda) + a_n P_{n+1}(\lambda)P_m(\lambda), \end{aligned}$$

for $m \in \{1,2,\ldots,M\}$, $n \in \{M+1,\ldots,N-1\}$,

$$\begin{aligned} &a_{m-1}P_{m-1}(\lambda)P_n(\lambda) + b_m P_m(\lambda)P_n(\lambda) + a_m P_{m+1}(\lambda)P_n(\lambda) \\ &\quad = c_{n-1}P_{n-1}(\lambda)P_m(\lambda) + d_n P_n(\lambda)P_m(\lambda) + c_n P_{n+1}(\lambda)P_m(\lambda), \end{aligned}$$

for $m \in \{M+1,\dots,N-1\}$, $n \in \{1,2,\dots,M\}$,

$$\begin{aligned} &c_{m-1}P_{m-1}(\lambda)P_n(\lambda) + d_mP_m(\lambda)P_n(\lambda) + c_mP_{m+1}(\lambda)P_n(\lambda) \\ &\quad = a_{n-1}P_{n-1}(\lambda)P_m(\lambda) + b_nP_n(\lambda)P_m(\lambda) + a_nP_{n+1}(\lambda)P_m(\lambda), \end{aligned}$$

for $m, n \in \{M+1,\dots,N-1\}$,

$$\begin{aligned} &c_{m-1}P_{m-1}(\lambda)P_n(\lambda) + d_mP_m(\lambda)P_n(\lambda) + c_mP_{m+1}(\lambda)P_n(\lambda) \\ &\quad = c_{n-1}P_{n-1}(\lambda)P_m(\lambda) + d_nP_n(\lambda)P_m(\lambda) + c_nP_{n+1}(\lambda)P_m(\lambda). \end{aligned}$$

If the functional Ω is applied to both sides of these equations, and using (2.15)-(2.19), we obtain for B_{mn} the following boundary value problems:

for $m, n \in \{1,2,\dots,M\}$,

$$a_{m-1}B_{m-1,n} + b_mB_{mn} + a_mB_{m+1,n} = a_{n-1}B_{n-1,m} + b_nB_{nm} + a_nB_{n+1,m}, \tag{2.20}$$

for $m \in \{1,2,\dots,M\}$, $n \in \{M+1,\dots,N-1\}$,

$$a_{m-1}B_{m-1,n} + b_mB_{mn} + a_mB_{m+1,n} = c_{n-1}B_{n-1,m} + d_nB_{nm} + c_nB_{n+1,m}, \tag{2.21}$$

for $m \in \{M+1,\dots,N-1\}$, $n \in \{1,2,\dots,M\}$,

$$c_{m-1}B_{m-1,n} + d_mB_{mn} + c_mB_{m+1,n} = a_{n-1}B_{n-1,m} + b_nB_{nm} + a_nB_{n+1,m}, \tag{2.22}$$

for $m, n \in \{M+1,\dots,N-1\}$,

$$c_{m-1}B_{m-1,n} + d_mB_{mn} + c_mB_{m+1,n} = c_{n-1}B_{n-1,m} + d_nB_{nm} + c_nB_{n+1,m}, \tag{2.23}$$

for $n \in \{0,1,\dots,M\}$,

$$B_{n0} = B_{0n} = \delta_{n0}, \qquad B_{n1} = B_{1n} = \delta_{n1}, \qquad B_{Nn} = B_{nN} = 0, \tag{2.24}$$

for $n \in \{M+1,\dots,N\}$,

$$B_{n0} = B_{0n} = \frac{\delta_{n0}}{\alpha}, \qquad B_{n1} = B_{1n} = \frac{\delta_{n1}}{\alpha}, \qquad B_{Nn} = B_{nN} = 0. \tag{2.25}$$

After starting with boundary values (2.24), (2.25) and using equations (2.20)-(2.23), we can find all B_{mn} uniquely as follows:

$$\begin{aligned} &B_{mn} = \delta_{mn}, \quad m, n \in \{0,1,\dots,M\}, \\ &B_{mn} = \frac{\delta_{mn}}{\alpha}, \quad m, n \in \{M+1,\dots,N-1\}, \\ &B_{mN} = 0, \quad m \in \{0,1,\dots,M,M+1,\dots,N\}. \end{aligned}$$

□

Definition 1 The linear functional Ω defined by Theorem 1 is called the GSF of the matrix J given in (1.1).

3 Inverse problem from the generalized spectral function

In this section, we solve the inverse spectral problem of reconstructing the matrix J by its GSF and we give the structure of GSF. The inverse spectral problem may be stated as follows: determine the reconstruction procedure to construct the matrix J from a given GSF and find the necessary and sufficient conditions for a linear functional Ω on $\mathbb{C}_{2N}[\lambda]$, to be the GSF for some matrix J of the form (1.1). For the investigation of necessary and sufficient conditions for a given linear functional to be the GSF, we will refer to Theorems 2 and 3 in [8]. In this paper, we only find the formulas to construct the matrix J.

Recall that $P_n(\lambda)$ is a polynomial of degree n, so it can be expressed as

$$P_n(\lambda) = \gamma_n\left(\lambda^n + \sum_{k=0}^{n-1}\chi_{nk}\lambda^k\right), \quad n \in \{0,1,\dots,M,\dots,N\}. \tag{3.1}$$

where γ_n and χ_{nk} are constants. Inserting (3.1) in (2.2) and using the equality of the polynomials, we can find the following equalities between the coefficients a_n, b_n, c_n, d_n and the quantities γ_n, χ_{nk}:

$$\begin{aligned} a_n &= \frac{\gamma_n}{\gamma_{n+1}} \quad (0 \le n \le M), \gamma_0 = 1, \\ c_n &= \frac{\gamma_n}{\gamma_{n+1}} \quad (M < n \le N-2), \qquad c_M = \frac{\gamma_M}{\alpha\gamma_{M+1}}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} b_n &= \chi_{n,n-1} - \chi_{n+1,n} \quad (0 \le n \le M), \chi_{0,-1} = 0, \\ d_n &= \chi_{n,n-1} - \chi_{n+1,n} \quad (M < n \le N-1). \end{aligned} \tag{3.3}$$

It is easily shown that there exists an equivalence between (2.8), (2.9), and

$$\Omega\left(\lambda^m P_n(\lambda)\right) = \frac{\delta_{mn}}{\eta\gamma_n}, \quad m = \overline{0,n}, n \in \{0,1,\dots,M,\dots,N-1\}, \tag{3.4}$$

$$\Omega\left(\lambda^m P_N(\lambda)\right) = 0, \quad m = \overline{0,N}, \tag{3.5}$$

respectively. Indeed, from (3.1), we can write

$$\Omega\left(P_m(\lambda)P_n(\lambda)\right) = \gamma_m\Omega\left(\lambda^m P_n(\lambda)\right) + \gamma_m\sum_{j=0}^{m-1}\chi_{mj}\Omega\left(\lambda^j P_n(\lambda)\right). \tag{3.6}$$

Then, since

$$\lambda^j = \sum_{i=0}^{j} c_i^{(j)} P_i(\lambda), \quad j \in \{0,1,\dots,N\},$$

it follows from (3.6) that (3.4), (3.5) hold if we have (2.8), (2.9) and conversely if (3.4), (3.5) hold, then (2.8), (2.9) can be obtained from (3.6) and (3.1).

Now, let us introduce

$$t_l = \Omega\left(\lambda^l\right), \quad l \in \{0,1,\dots,2N\}, \tag{3.7}$$

which are called 'power moments' of the functional Ω.

Writing the expansion (3.1) in (3.4) and (3.5) instead of $P_n(\lambda)$ and $P_N(\lambda)$, respectively, and using the notation in (3.7), we get

$$t_{n+m}+\sum_{k=0}^{n-1}\chi_{nk}t_{k+m}=0,\quad m=\overline{0,n-1},n\in\{1,2,\ldots,N\},\tag{3.8}$$

$$t_{2N}+\sum_{k=0}^{N-1}\chi_{Nk}t_{k+N}=0,\tag{3.9}$$

$$t_{2n}+\sum_{k=0}^{n-1}\chi_{nk}t_{k+n}=\frac{1}{\eta\gamma_n^2},\quad n\in\{0,1,\ldots,N-1\},\tag{3.10}$$

where η is defined in (2.10).

As a result of all discussions above, we write the procedure to construct the matrix in (1.1). In turn, in order to find the entries a_n, b_n, c_n, d_n of the required matrix J, it suffices to know only the quantities γ_n, χ_{nk}. Given the linear functional Ω which satisfies the conditions of Theorem 2 in [8] on $\mathbb{C}_{2N}[\lambda]$, we can use (3.7) to find the quantities t_l and write down the inhomogeneous system of linear algebraic equations (3.8) with the unknowns $\chi_{n0},\chi_{n1},\ldots,\chi_{n,n-1}$, for every fixed $n\in\{1,2,\ldots,N\}$. After solving this system uniquely and using (3.10), we find the quantities γ_n and so we obtain a_n, b_n, c_n, d_n, recalling (3.2), (3.3). Therefore, we can construct the matrix J.

Using the numbers t_l defined in (3.7), let us present the determinants

$$D_n=\begin{vmatrix} t_0 & t_1 & \cdots & t_n\\ t_1 & t_2 & \cdots & t_{n+1}\\ \vdots & \vdots & \ddots & \vdots\\ t_n & t_{n+1} & \cdots & t_{2n}\end{vmatrix},\quad n=\overline{0,N}.\tag{3.11}$$

From the definition of determinants in (3.11), it can be shown that the determinant of system (3.8) is D_{n-1}. Then, solving system (3.8) by using Cramer's rule, we obtain

$$\chi_{nk}=-\frac{D_{n-1}^{(k)}}{D_{n-1}},\quad k=\overline{0,n-1},\tag{3.12}$$

where $D_m^{(k)}$ ($k=\overline{0,m}$) is the determinant formed by exchanging in D_m the $(k+1)$th column by the vector $(t_{m+1},t_{m+2},\ldots,t_{2m+1})^T$. Next, substituting equation (3.12) of χ_{nk} into the left-hand side of (3.10), we find

$$\gamma_n^{-2}=\frac{\eta D_n}{D_{n-1}},\tag{3.13}$$

where η is defined in (2.10). Now if we set $D_m^{(m)}=\triangle_m$, then we obtain from (3.2), (3.3), by using (3.12), (3.13),

$$a_n=\pm\frac{\sqrt{D_{n-1}D_{n+1}}}{D_n}\quad(0\le n\le M-1),D_{-1}=1,\tag{3.14}$$

$$a_M=\pm\frac{\sqrt{\alpha D_{M-1}D_{M+1}}}{D_M},\qquad c_M=\pm\frac{\sqrt{D_{M-1}D_{M+1}}}{\sqrt{\alpha}D_M},\tag{3.15}$$

$$c_n = \pm\frac{\sqrt{D_{n-1}D_{n+1}}}{D_n} \quad (M < n \le N-2), \tag{3.16}$$

$$b_n = \frac{\triangle_n}{D_n} - \frac{\triangle_{n-1}}{D_{n-1}} \quad (0 \le n \le M), \triangle_{-1} = 0, \tag{3.17}$$

$$d_n = \frac{\triangle_n}{D_n} - \frac{\triangle_{n-1}}{D_{n-1}} \quad (M < n \le N-1), \triangle_0 = t_1. \tag{3.18}$$

Hence , if Ω which satisfies the conditions of Theorem 3 in [8] is given, then the values a_n, b_n, c_n, d_n of the matrix J are obtained by equations (3.14)-(3.18), where D_n is defined by (3.11) and (3.7).

In the following theorem, we will show that the GSF of Jhas a special form and we will give a structure of the GSF. Let J be a matrix which has the form (1.1) and Ω be the GSF of J. Here we characterize the structure of Ω.

Theorem 2 *Let $\lambda_1, \dots, \lambda_p$ be all the eigenvalues with the multiplicities $m_1, \dots, m_p$, respectively, of the matrix J. These are also the roots of the polynomial* (2.4). *Then there exist numbers β_{kj} ($j = \overline{1, m_k}$, $k = \overline{1, p}$) uniquely determined by the matrix J such that for any polynomial $P(\lambda) \in \mathbb{C}_{2N}[\lambda]$ the following formula holds*:

$$\Omega\big(P(\lambda)\big) = \sum_{k=1}^{p}\sum_{j=1}^{m_k} \frac{\beta_{kj}}{(j-1)!} P^{(j-1)}(\lambda_k), \tag{3.19}$$

where $P^{(j-1)}(\lambda)$ denotes the $(j-1)$th derivative of $P(\lambda)$ with respect to λ.

Proof Let J be a matrix which has the form (1.1). Take into consideration the difference equation (1.2)

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda\rho_n y_n, \quad n \in \{0, 1, \dots, N-1\}, a_{-1} = c_{N-1} = 1, \tag{3.20}$$

where $\{y_n\}_{n=-1}^{N}$ is desired solution and

$$\rho_n = \begin{cases} 1, & 0 \le n \le M, \\ \alpha, & M < n \le N-1. \end{cases}$$

Denote by $\{P_n(\lambda)\}_{n=-1}^{N}$ and $\{Q_n(\lambda)\}_{n=-1}^{N}$ the solutions of (3.20) satisfying the initial conditions

$$P_{-1}(\lambda) = 0, \qquad P_0(\lambda) = 1, \tag{3.21}$$

$$Q_{-1}(\lambda) = -1, \qquad Q_0(\lambda) = 0. \tag{3.22}$$

For each $n \ge 0$, the degree of polynomial $P_n(\lambda)$ is n and the degree of polynomial $Q_n(\lambda)$ is $n-1$. It is clear that the entries $R_{nm}(\lambda)$ of the resolvent matrix $R(\lambda) = (J - \lambda I)^{-1}$ are of the form

$$R_{nm}(\lambda) = \begin{cases} \rho_n P_n(\lambda)[Q_m(\lambda) + M(\lambda)P_m(\lambda)], & 0 \le n \le m \le N-1, \\ \rho_n P_m(\lambda)[Q_n(\lambda) + M(\lambda)P_n(\lambda)], & 0 \le m \le n \le N-1, \end{cases} \tag{3.23}$$

where

$$M(\lambda) = -\frac{Q_N(\lambda)}{P_N(\lambda)}, \tag{3.24}$$

and ρ_n is defined in (1.4). Let $f = (f_1, f_2, \ldots, f_{N-1})^T \in \mathbb{C}^N$ be an arbitrary vector. Since

$$R(\lambda)f = -\frac{f}{\lambda} + O\left(\frac{1}{\lambda^2}\right),$$

as $|\lambda| \to \infty$, we get for each $n \in \{0, 1, \ldots, N-1\}$ and for a sufficiently large positive number r

$$f_n = -\frac{1}{2\pi i}\int_{\Gamma_r}\left\{\sum_{m=0}^{N-1} R_{nm}(\lambda)f_m\right\}d\lambda + \int_{\Gamma_r} O\left(\frac{1}{\lambda^2}\right)d\lambda, \tag{3.25}$$

where Γ_r is the circle in the λ-plane of radius r centered at the origin.

Let all the distinct zeros of $P_N(\lambda)$ in (2.4) be $\lambda_1, \ldots, \lambda_p$ with multiplicities $m_1, \ldots, m_p$, respectively. Then

$$P_N(\lambda) = c(\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_N)^{m_p}, \tag{3.26}$$

where c is a constant. We have $1 \le p \le N$ and $m_1 + \cdots + m_p = N$. By (3.26), we can write $\frac{Q_N(\lambda)}{P_N(\lambda)}$ as the sum of partial fractions:

$$\frac{Q_N(\lambda)}{P_N(\lambda)} = \sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{\beta_{kj}}{(\lambda - \lambda_k)^j}, \tag{3.27}$$

where β_{kj} are some uniquely determined complex numbers which depend on the matrix J. Inserting (3.23) in (3.25) and using (3.24), (3.27) we get, by the residue theorem and passing then to the limit $r \to \infty$,

$$f_n = \sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{\beta_{kj}}{(j-1)!}\left\{\frac{d^{j-1}}{d\lambda^{j-1}}\left[\rho_n F(\lambda)P_n(\lambda)\right]\right\}_{\lambda=\lambda_k}, \quad n \in \{0, 1, \ldots, N-1\}, \tag{3.28}$$

where

$$F(\lambda) = \sum_{m=0}^{N-1} f_m P_m(\lambda). \tag{3.29}$$

Now define the functional Ω on $\mathbb{C}_{2N}[\lambda]$ by the formula

$$\Omega\big(P(\lambda)\big) = \sum_{k=1}^{p}\sum_{j=1}^{m_k}\frac{\beta_{kj}}{(j-1)!}P^{(j-1)}(\lambda_k), \quad P(\lambda) \in \mathbb{C}_{2N}[\lambda]. \tag{3.30}$$

Thus, (3.28) can be written as follows:

$$\frac{f_n}{\rho_n} = \Omega\big(F(\lambda)P_n(\lambda)\big), \quad n \in \{0, 1, \ldots, N-1\}. \tag{3.31}$$

Now by using (3.29) in (3.31) and the arbitrariness of $\{f_m\}_{m=0}^{N-1}$, we see that the first relation in Theorem 1,

$$\Omega\bigl(P_m(\lambda)P_n(\lambda)\bigr) = \delta_{mn}, \quad m,n \in \{0,1,\ldots,M\}, \tag{3.32}$$

$$\Omega\bigl(P_m(\lambda)P_n(\lambda)\bigr) = \frac{\delta_{mn}}{\alpha}, \quad m,n \in \{M+1,\ldots,N-1\}, \tag{3.33}$$

holds. Moreover, from (3.26) and (3.30), we have also the second relation in Theorem 1,

$$\Omega\bigl(P_m(\lambda)P_N(\lambda)\bigr) = 0, \quad m \in \{0,1,\ldots,N\}. \tag{3.34}$$

These mean that the GSF of the matrix J has the form (3.19). □

Now, we shall work out two examples to illustrate our formulas. In the first example, in order to determine $\chi_{n,k}$ and γ_n, we will use (3.8)-(3.10).

Example 1 Take into consideration the case $N = 3$, $M = 1$, and the functional Ω described by the formula

$$\Omega\bigl(P(\lambda)\bigr) = \frac{1}{3}\bigl(P(0) + P(1) + P(2)\bigr).$$

It is clear that the functional defined above has the structure given in Theorem 2 and satisfies the conditions of Theorem 2 in [8]. So it can be chosen as a GSF. From (3.7) we calculate all t_l as follows:

$$\begin{aligned} &t_0 = 1, \quad t_1 = 1, \quad t_2 = \frac{5}{3}, \\ &t_3 = 3, \quad t_4 = \frac{17}{3}, \quad t_5 = 11, \quad t_6 = \frac{65}{3}. \end{aligned} \tag{3.35}$$

Then solving the system of equation (3.8) by using the values in (3.35), we get

$$\begin{aligned} &\chi_{1,0} = -1, \quad \chi_{2,0} = \frac{1}{3}, \quad \chi_{2,1} = -2, \\ &\chi_{3,0} = 0, \quad \chi_{3,1} = 2, \quad \chi_{3,2} = -3. \end{aligned} \tag{3.36}$$

Now inserting the quantities in (3.35) and (3.36) into equation (3.10), we obtain

$$\gamma_0 = 1, \quad \gamma_1 = \pm\sqrt{\frac{3}{2}}, \quad \gamma_2 = \pm\frac{3}{\sqrt{2\alpha}}. \tag{3.37}$$

Now it follows from (3.2) and (3.3) that

$$\begin{aligned} &a_0 = \pm\sqrt{\frac{2}{3}}, \quad a_1 = \pm\sqrt{\frac{\alpha}{3}}, \quad c_1 = \pm\sqrt{\frac{1}{3\alpha}}, \\ &b_0 = 1, \quad b_1 = 1, \quad d_2 = 1, \end{aligned}$$

where (3.36) and (3.37) are used. Consequently, we find the four matrices $J_\pm$ for Ω as follows:

$$J_\pm = \begin{bmatrix} b_0 & a_0 & 0 \\ a_0 & b_1 & a_1 \\ 0 & c_1 & d_2 \end{bmatrix} = \begin{bmatrix} 1 & \pm\sqrt{\frac{2}{3}} & 0 \\ \pm\sqrt{\frac{2}{3}} & 1 & \pm\sqrt{\frac{\alpha}{3}} \\ 0 & \pm\sqrt{\frac{1}{3\alpha}} & 1 \end{bmatrix}.$$

The characteristic polynomials which are determined by the matrices $J_\pm$ are obtained:

$$\det(J_\pm - \lambda I) = \lambda(\lambda - 1)(\lambda - 2).$$

In the following example, by using Theorem 3 in [8], it can be shown that the necessary and sufficient conditions for a given linear functional Ω to be the GSF hold and the matrix J can be constructed from (3.14)-(3.18).

Example 2 Let us consider the functional Ω defined by the formula for $N = 3$ and $M = 1$

$$\Omega\big(P(\mu)\big) = P(\mu) + 3P'(\mu) + 4P''(\mu),$$

where μ is any number. From (3.7), we obtain

$$t_0 = 1, \qquad t_l = \Omega\big(\mu^l\big) = \mu^l + 3l\mu^{l-1} + 4l(l-1)\mu^{l-2}, \tag{3.38}$$

and from (3.38), we get numbers t_l for $l = \overline{1,6}$ as follows:

$$\begin{aligned} &t_1 = \mu + 3, \qquad t_2 = \mu^2 + 6\mu + 8, \\ &t_3 = \mu^3 + 9\mu^2 + 24\mu, \qquad t_4 = \mu^4 + 12\mu^3 + 48\mu^2, \\ &t_5 = \mu^5 + 15\mu^4 + 80\mu^3, \qquad t_6 = \mu^6 + 18\mu^5 + 120\mu^4. \end{aligned} \tag{3.39}$$

By using (3.39) and recalling (3.11), we find

$$D_{-1} = 1, \qquad D_0 = t_0 = 1, \tag{3.40}$$

$$D_1 = \begin{vmatrix} t_0 & t_1 \\ t_1 & t_2 \end{vmatrix} = \begin{vmatrix} 1 & \mu + 3 \\ \mu + 3 & \mu^2 + 6\mu + 8 \end{vmatrix} = -1, \tag{3.41}$$

$$D_2 = \begin{vmatrix} 1 & \mu + 3 & \mu^2 + 6\mu + 8 \\ \mu + 3 & \mu^2 + 6\mu + 8 & \mu^3 + 9\mu^2 + 24\mu \\ \mu^2 + 6\mu + 8 & \mu^3 + 9\mu^2 + 24\mu & \mu^4 + 12\mu^3 + 48\mu^2 \end{vmatrix} = -512, \tag{3.42}$$

and similarly, after some basic operations, we get $D_3 = 0$. From the equality $D_m^{(m)} = \triangle_m$, we determine

$$\triangle_{-1} = 0, \qquad \triangle_0 = t_1 = \mu + 3, \tag{3.43}$$

$$\triangle_1 = \begin{vmatrix} 1 & \mu^2 + 6\mu + 8 \\ \mu + 3 & \mu^3 + 9\mu^2 + 24\mu \end{vmatrix} = -2\mu - 24, \tag{3.44}$$

$$\triangle_2 = \begin{vmatrix} 1 & \mu + 3 & \mu^3 + 9\mu^2 + 24\mu \\ \mu + 3 & \mu^2 + 6\mu + 8 & \mu^4 + 12\mu^3 + 48\mu^2 \\ \mu^2 + 6\mu + 8 & \mu^3 + 9\mu^2 + 24\mu & \mu^5 + 15\mu^4 + 80\mu^3 \end{vmatrix} = -1536\mu. \tag{3.45}$$

Now, it follows from (3.14), (3.15), and (3.16) that

$$a_0 = \pm i, \qquad a_1 = \pm 16i\sqrt{2\alpha}, \qquad c_1 = \pm\frac{16i\sqrt{2}}{\sqrt{\alpha}},$$

and from (3.17), (3.18) that

$$b_0 = \mu + 3, \qquad b_1 = \mu + 21, \qquad d_2 = \mu - 24,$$

where (3.40)-(3.45) are used. Consequently, we find the four matrices $J_\pm$ for Ω as follows:

$$J_\pm = \begin{bmatrix} b_0 & a_0 & 0 \\ a_0 & b_1 & a_1 \\ 0 & c_1 & d_2 \end{bmatrix} = \begin{bmatrix} \mu + 3 & \pm i & 0 \\ \pm i & \mu + 21 & \pm 16i\sqrt{2\alpha} \\ 0 & \pm\frac{16i\sqrt{2}}{\sqrt{\alpha}} & \mu - 24 \end{bmatrix}.$$

The characteristic polynomials which are determined by the matrices $J_\pm$ are obtained:

$$\det(J_\pm - \lambda I) = (\mu - \lambda)^3.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Author details
[1]Faculty of Arts and Sciences, Department of Mathematics, Gaziantep University, Gaziantep, 27310, Turkey. [2]Faculty of Arts and Sciences, Department of Mathematics, Adıyaman University, Adıyaman, 02040, Turkey.

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