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Generalized q-Taylor formulas

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Abstract

In this paper, new generalized q-Taylor formulas involving both Riemann-Liouville and Caputo q-difference operators are constructed. Some applications with solutions of fractional q-difference equations are also given.

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1 Introduction

A q-analogue of Taylor series was introduced by Jackson [1]:

$$f(x)=\sum_{n=0}^{\infty}\frac{(1-q)^n}{(q;q)_n}D_q^n f(a)[x-a]_n, \tag{1.1}$$

where $0<q<1$, D_q is the q-derivative, and

$$[x-a]_n:=(x-a)(x-qa)\cdots\left(x-q^{n-1}a\right),\quad n\ge 1,[x-a]_0:=1.$$

Al-Salam and Verma [2] introduced the following q-interpolation series:

$$f(x)=\sum_{n=0}^{\infty}(-1)^n q^{-n(n-1)/2}\frac{(1-q)^n}{(q;q)_n}D_q^n f\left(aq^{-n}\right)[x-a]_n. \tag{1.2}$$

Al-Salam and Verma gave only formal proofs for (1.2); see [1, 2]. Analytic proofs of (1.1) and (1.2) were given in [3].

Results of generalized Taylor formulas involving the classical fractional derivative may be found in [4, 5]. In [5], a generalized Taylor formula involving the classical Riemann-Liouville fractional derivative of order α is deduced, whereas the generalized Taylor formula in [4] contains Caputo fractional derivative of order α, where $0<\alpha\le 1$.

In [6], a q-Taylor formula in terms of Riemann-Liouville fractional q-derivative $D_{q,a}^{\alpha}$ of order α is obtained. This result can be stated as follows.

Theorem A ([6]) *Let f be a function defined on $(0,b)$ and $\alpha \in (0,1)$. Then f can be expanded in the form*

$$\begin{aligned} f(x) &= \sum_{k=0}^{n-1} \frac{(D_{q,a}^{\alpha+k} f)(c)}{\Gamma_q(\alpha+k+1)} (x-c)^{(\alpha+k)} \\ &\quad + \frac{1}{\Gamma_q(\alpha)} \int_a^c (x-c)^{(\alpha-1)} \bigl(D_{q,a}^{\alpha} f\bigr)(t)\, d_q t \\ &\quad - K(a)(x-c)^{(\alpha-1)} + \bigl(I_{q,c}^{\alpha+n} D_{q,a}^{\alpha+n} f\bigr)(x), \end{aligned} \tag{1.3}$$

where $0 < a < c < x < b$, and $K(a)$ does not depend on x.

Also, in [7], a generalized q-Taylor formula in fractional q-calculus is established and used in deriving certain q-generating functions for the basic hyper-geometric functions.

In this paper, we give generalized Taylor formulas involving Riemann-Liouville fractional q-derivatives of order α and Caputo fractional q-derivatives of order α; see (4.3) and (4.4). We also give sufficient conditions that guarantee that the remainders of these formulas vanish to get infinite expansions.

In the following section, we give a brief account of the q-notations and notions that will be used throughout this paper. In Section 3, we give q-analogues of mean value theorems on $[0,a]$. In Section 4, we give generalized q-Taylor formulas involving both Riemann-Liouville fractional q-derivative and Caputo fractional q-derivative. Then conditions for infinite expansion for some functions are given. In the last section, we apply the obtained results in solving certain q-difference equations.

2 Notation and preliminaries

In the following, q is a positive number, $q < 1$. We follow [8] for the definition of the q-shifted factorial, Jackson q-integral, q-derivative, q-gamma function $\Gamma_q(z)$, and q-beta function $B_q(\alpha,\beta)$. Also, we follow [9] for the definition of the q-derivative at zero and the q-regular at zero functions.

The following q-integral is useful and will be used in the sequel:

$$\int_0^x (qt/x;q)_{\beta-1} t^{\alpha-1}\, d_q t = x^{\alpha} B_q(\alpha,\beta), \quad \alpha,\beta,x > 0; \tag{2.1}$$

it can be proved by setting $\xi = t/x$.

By $L_q^1(0,a)$, $a > 0$, we mean the Banach space of all functions defined on $(0,a]$ such that

$$\|f\| := \int_0^a \bigl|f(t)\bigr|\, d_q t < \infty, \tag{2.2}$$

where two functions in $L_q^1(0,a)$ are considered to be the same function if they have the same values at the sequence $\{aq^n\}_{n=0}^{\infty}$.

Let $\mathcal{L}_q^1(0,a)$ denote the space of all functions f defined on $(0,a]$ such that $f \in L_q^1(0,x)$ for all $x \in (qa,a]$. The space $\mathcal{AC}_q[0,a]$ is the space of all functions f defined on $[0,a]$ such that

f is q-regular at zero and

$$\sum_{j=0}^{\infty}\left|f\left(tq^{j}\right)-f\left(tq^{j+1}\right)\right|<\infty, \quad t\in(qa,a]. \tag{2.3}$$

A characterization of the space $\mathcal{AC}_q[0,a]$ is given as follows (see [9]).

Theorem B *Let f be a function defined on $[0,a]$. Then $f\in\mathcal{AC}_q[0,a]$ if and only if there exist a constant c and a function ϕ in $\mathcal{L}_q^1[0,a]$ such that*

$$f\in\mathcal{AC}_q[0,a] \quad\Longleftrightarrow\quad f(x)=c+\int_0^x \phi(u)\,d_qu, \quad x\in[0,a]. \tag{2.4}$$

Moreover, c and ϕ are uniquely determined by $c=f(0)$ and $\phi(x)=D_qf(x)$ for all $x\in(0,a]$.

The Riemann-Liouville fractional q-integral operator is introduced in [10] by Al-Salam through

$$I_q^{\alpha}f(x):=\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}f(t)\,d_qt, \quad \alpha\notin\{-1,-2,\ldots\}. \tag{2.5}$$

In [6], the generalized Riemann-Liouville fractional q-integral operator for $\alpha\in\mathbb{R}^+$ is given as

$$I_{q,a}^{\alpha}f(x):=\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_a^x (qt/x;q)_{\alpha-1}f(t)\,d_qt. \tag{2.6}$$

Using the definition of the q-integral, (2.5) reduces to

$$I_q^{\alpha}f(x)=x^{\alpha}(1-q)^{\alpha}\sum_{n=0}^{\infty}q^n\frac{(q^{\alpha};q)_n}{(q;q)_n}f\left(xq^n\right), \tag{2.7}$$

which is valid for all α. For example,

$$I_q^{\alpha}x^{\beta-1}=\frac{\Gamma_q(\beta)}{\Gamma_q(\beta+\alpha)}x^{\alpha+\beta-1}. \tag{2.8}$$

This basic Riemann-Liouville fractional q-integral was also given later by Agarwal [11]. In the same paper, he introduced the following semigroup property:

$$I_q^{\alpha}I_q^{\beta}f(x)=I_q^{\beta}I_q^{\alpha}f(x)=I_q^{\alpha+\beta}f(x), \quad \alpha,\beta\geq 0. \tag{2.9}$$

The generalized Riemann-Liouville fractional q-derivative is given in [6] by

$$D_{q,a}^{\alpha}f(x)=D_qI_{q,a}^{1-\alpha}f(x), \quad a\geq 0, \tag{2.10}$$

and $D_{q,0}^{\alpha}f(x)=D_q^{\alpha}f(x)$. The Caputo fractional q-derivative of order α, $0<\alpha\leq 1$, is (see[12])

$$^{c}D_q^{\alpha}f(x):=I_q^{1-\alpha}D_qf(x). \tag{2.11}$$

Let $\mathcal{AC}_q^{(k)}[0,a]$, $k \in N$, be the space of all functions f defined on $[0,a]$ such that $f, D_q f, \dots, D_q^{k-1} f$ are q-regular at zero and $D_q^{k-1} f \in \mathcal{AC}_q[0,a]$.

For $\alpha > 0$, let $k = \lceil \alpha \rceil$, where $\lceil \cdot \rceil$ is the ceiling function. Then the Riemann-Liouville fractional derivative $D_q^\alpha f(x)$ exists if (see [9])

$$f \in \mathcal{L}_q^1[0;a], \quad I_q^{k-\alpha} D_q^k f \in \mathcal{AC}_q^{(k)}[0,a],$$

and ${}^c D_q^\alpha f(x)$ exists if $f \in \mathcal{AC}_q^{(k)}[0,a]$.

The following results are proved in [12] for any $\alpha > 0$; the result for the case $0 < \alpha < 1$ is introduced in the following theorems without proof.

Theorem C *Assume that* $f \in \mathcal{L}_q^1[0;a]$ *and* $I_q^{1-\alpha} f \in \mathcal{AC}_q[0,a]$, *where* $0 < \alpha < 1$. *Then the Riemann-Liouville fractional derivative of order* α, $0 < \alpha < 1$, *exists, and*

$$I_q^\alpha D_q^\alpha f(x) = f(x) - \frac{I_q^{1-\alpha} f(0)}{\Gamma_q(\alpha)} x^{\alpha-1.} \tag{2.12}$$

Theorem D *If* $f \in \mathcal{AC}_q[0,a]$, *then*

$$I_q^{\alpha\, c} D_q^\alpha f(x) = I_q D_q f(x) = f(x) - f(0) \tag{2.13}$$

for $0 < \alpha < 1$.

It is worth mentioning that the key point in the proofs of Theorems C and D is the q-integration by parts formula:

$$\int_0^b f(t) D_q g(t)\, d_q t = (fg)(b) - \lim_{n\to\infty} (fg)\bigl(bq^n\bigr) - \int_0^b D_q f(t) g(qt)\, d_q t.$$

Hence, if fg is q-regular at zero, then the limit on the right-hand side is nothing but $(fg)(0)$.

3 Generalized q-mean value theorems on $[0,a]$

In this section, we introduce two q-analogues of the mean value theorems. The first one is for q-integrals on an interval of the form $[0,a]$, and the second is a mean value theorem with both of Riemann-Liouville fractional q-derivative and Caputo fractional q-derivative on $[0,a]$. The first one can be stated as follows.

Theorem 3.1 (Mean value theorem for q-integrals) *Let g be a continuous function defined on* $[0,a]$, *and h be a nonnegative function defined on* $[0,a]$ *and q-regular at zero. Then*

$$\int_0^a g(t)h(t)\, d_q t = g(\xi) \int_0^a h(t)\, d_q t \tag{3.1}$$

for some $\xi \in [0,a]$.

Proof The proof is similar to the classical case (see [13], p.139) and is omitted. □

The derivations of the main results of this paper mainly depend on Theorem 3.1.

Remark 3.2

(1) We cannot replace the lower end point of the q-integrals in (3.1) by arbitrary nonzero number because the inequality

$$\left|\int_c^a f(t)\,d_qt\right| \leq \int_c^a \left|f(t)\right|\,d_qt,$$

holds only for $c \in \{0, aq^n, n \in \mathbb{N}_0\}$. In this case, (3.1) is also true.

(2) There are q-analogues of mean value theorems on $[a,b]$ in [14], but all these analogues are valid only for certain values of q. For example, one of the mean value theorems for q-integrals in [14] is the following:

Let f, g be continuous functions on $[a,b]$. Then there exists $\widehat{q} \in (0,1)$ such that

$$\big(\forall q \in (\widehat{q}, 1)\big)\ \big(\exists \xi \in [a,b]\big): \quad \int_a^b g(t)f(t)\,d_qt = g(\xi)\int_a^b f(t)\,d_qt.$$

The second theorem is a q-analogue of the mean value theorem for derivative on $[0,a]$. Throughout the rest of this article, we assume that $0 < \alpha < 1$.

Theorem 3.3

(1) *If $f \in \mathcal{L}_q^1[0;a]$, $I_q^{1-\alpha}f \in \mathcal{AC}_q[0,a]$, and $x^{1-\alpha}D_q^\alpha f \in C[0,a]$, then*

$$f(x) = \frac{I_q^{1-\alpha}f(0)}{\Gamma_q(\alpha)}x^{\alpha-1} + \frac{\Gamma_q(\alpha)\xi^{1-\alpha}D_q^\alpha f(\xi)}{\Gamma_q(2\alpha)}x^{2\alpha-1}. \tag{3.2}$$

(2) *If $f \in \mathcal{AC}_q[0,a]$ and ${}^cD_q^\alpha f \in C[0,a]$, then*

$$f(x) = f(0) + \frac{{}^cD_q^\alpha f(\xi)}{\Gamma_q(\alpha)}x^\alpha \tag{3.3}$$

for some ξ lying in the interval $[0,x]$ and all $x \in (0,a]$.

Proof We first prove (3.2). Since (see [15], p.494)

$$B_q(\alpha,\beta) = \frac{\Gamma_q(\alpha)\Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)},$$

from (2.5), Theorem 3.1, and (2.1) we get

$$\begin{aligned} I_q^\alpha D_q^\alpha f(x) &= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}t^{\alpha-1}t^{1-\alpha}D_q^\alpha f(t)\,d_qt \\ &= \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\xi^{1-\alpha}D_q^\alpha(\xi)\int_0^x (qt/x;q)_{\alpha-1}t^{\alpha-1}\,d_qt \\ &= \frac{\Gamma_q(\alpha)\xi^{1-\alpha}D_q^\alpha f(\xi)}{\Gamma_q(2\alpha)}x^{2\alpha-1} \end{aligned}$$

for $0 \leq \xi \leq x$. Hence, (3.2) follows from (2.12). Similarly, using (2.13), we can prove (3.3). □

4 Generalized q-Taylor formula

In this section, we introduce generalized q-Taylor formulas for functions in terms of the sequential Riemann-Liouville q-derivative and the sequential Caputo fractional q-derivatives, where the sequential Riemann-Liouville q-derivative $\mathcal{D}_q^{n\alpha}$ and Caputo fractional q-derivative ${}^c\mathcal{D}_q^{n\alpha}$, $n \in \mathbb{N}$, are

$$\mathcal{D}_q^{n\alpha} = D_q^{\alpha} \cdots D_q^{\alpha} \quad \text{and} \quad {}^c\mathcal{D}_q^{n\alpha} = {}^cD_q^{\alpha} \cdots {}^cD_q^{\alpha} \quad (n \text{ times}),$$

respectively. The following lemma is important to get these formulas.

Lemma 4.1

(1) *If* $\mathcal{D}_q^{k\alpha} f \in \mathcal{L}_q^1[0,a]$ *and* $I_q^{1-\alpha}\mathcal{D}_q^{k\alpha} f \in \mathcal{AC}_q[0,a]$, $k = 0,1,\ldots,n$, *then*

$$I_q^{n\alpha}\mathcal{D}_q^{n\alpha} f(x) - I_q^{(n+1)\alpha}\mathcal{D}_q^{(n+1)\alpha} f(x) = \frac{I_q^{1-\alpha}\mathcal{D}_q^{n\alpha} f(0)}{\Gamma_q((n+1)\alpha)} x^{(n+1)\alpha-1}. \tag{4.1}$$

(2) *If* ${}^c\mathcal{D}_q^{k\alpha} f \in \mathcal{AC}_q[0,a]$, $k = 0,1,\ldots,n$, *then*

$$I_q^{n\alpha}\,{}^c\mathcal{D}_q^{n\alpha} f(x) - I_q^{(n+1)\alpha}\,{}^c\mathcal{D}_q^{(n+1)\alpha} f(x) = \frac{{}^c\mathcal{D}_q^{n\alpha} f(0)}{\Gamma_q(n\alpha+1)} x^{n\alpha}. \tag{4.2}$$

Proof We give a proof of (4.1), and the proof of (4.2) can be obtained similarly. Applying (2.12) and (2.8), we obtain

$$\begin{aligned} I_q^{n\alpha}\mathcal{D}_q^{n\alpha} f(x) - I_q^{(n+1)\alpha}\mathcal{D}_q^{(n+1)\alpha} f(x) &= I_q^{n\alpha}\left(\mathcal{D}_q^{n\alpha} f(x) - I_q^{\alpha} D_q^{\alpha}\left(\mathcal{D}_q^{n\alpha} f(x)\right)\right) \\ &= I_q^{n\alpha}\left(\frac{I_q^{1-\alpha}\mathcal{D}_q^{n\alpha} f(0)}{\Gamma_q(\alpha)} x^{\alpha-1}\right) \\ &= \frac{I_q^{1-\alpha}\mathcal{D}_q^{n\alpha} f(0)}{\Gamma_q(\alpha)} I_q^{n\alpha}\left(x^{\alpha-1}\right) \\ &= \frac{I_q^{1-\alpha}\mathcal{D}_q^{n\alpha} f(0)}{\Gamma_q((n+1)\alpha)} x^{(n+1)\alpha-1}, \end{aligned}$$

and the lemma follows. □

Theorem 4.2 (Generalized q-Taylor formulas)

(1) *Suppose that* $\mathcal{D}_q^{k\alpha} f \in \mathcal{L}_q^1[0,a]$, $I_q^{1-\alpha}\mathcal{D}_q^{k\alpha} f \in \mathcal{AC}_q[0,a]$, $k = 0,1,\ldots,n-1$, *and* $x^{1-\alpha}\mathcal{D}_q^{n\alpha} f \in C[0,a]$. *Then*

$$f(x) = \sum_{k=1}^{n-1} \frac{I_q^{1-\alpha}\mathcal{D}_q^{k\alpha} f(0)}{\Gamma_q((k+1)\alpha)} x^{(k+1)\alpha-1} + \frac{\Gamma_q(\alpha)\xi^{1-\alpha}\mathcal{D}_q^{n\alpha} f(\xi)}{\Gamma_q((n+1)\alpha)} x^{(n+1)\alpha-1}. \tag{4.3}$$

(2) *Suppose that* ${}^c\mathcal{D}_q^{k\alpha} f \in \mathcal{AC}_q[0,a]$, $k = 0,1,\ldots,n-1$, *and* ${}^cD_q^{n\alpha} f \in C[0,a]$. *Thus,*

$$f(x) = \sum_{k=0}^{n-1} \frac{{}^c\mathcal{D}_q^{k\alpha} f(0)}{\Gamma_q(k\alpha+1)} x^{k\alpha} + \frac{{}^c\mathcal{D}_q^{n\alpha} f(\xi)}{\Gamma_q(n\alpha+1)} x^{n\alpha}, \tag{4.4}$$

where $0 \le \xi \le x$.

Proof For (4.3), applying (4.1), we obtain

$$\begin{aligned}&\sum_{k=0}^{n-1}\bigl[I_q^{k\alpha}\mathcal{D}_q^{k\alpha}f(x)-I_q^{(k+1)\alpha}\mathcal{D}_q^{(k+1)\alpha}f(x)\bigr]\\&\quad=\sum_{k=0}^{n-1}\frac{I_q^{1-\alpha}\mathcal{D}_q^{k\alpha}f(0)}{\Gamma_q((k+1)\alpha)}x^{(k+1)\alpha-1},\end{aligned}\tag{4.5}$$

that is,

$$f(x)=\sum_{k=0}^{n-1}\frac{I_q^{1-\alpha}\mathcal{D}_q^{k\alpha}f(0)}{\Gamma_q((k+1)\alpha)}x^{(k+1)\alpha-1}+I_q^{n\alpha}\mathcal{D}_q^{n\alpha}f(x).\tag{4.6}$$

Applying the q-integral mean value theorem and (2.8) yield

$$\begin{aligned}I_q^{n\alpha}\mathcal{D}_q^{n\alpha}f(x)&=\frac{x^{n\alpha-1}}{\Gamma_q(n\alpha)}\int_0^x(qt/x;q)_{n\alpha-1}t^{\alpha-1}t^{1-\alpha}\mathcal{D}_q^{n\alpha}f(t)\,d_qt\\&=\frac{x^{n\alpha-1}}{\Gamma_q(n\alpha)}\xi^{1-\alpha}\mathcal{D}_q^{n\alpha}f(\xi)\int_0^x(qt/x;q)_{n\alpha-1}t^{\alpha-1}\,d_qt\\&=\frac{\Gamma_q(\alpha)\xi^{1-\alpha}\mathcal{D}_q^{n\alpha}f(\xi)}{\Gamma_q((n+1)\alpha)}x^{(n+1)\alpha-1}\end{aligned}\tag{4.7}$$

for some $\xi\in[0,x]$. Combining (4.6) and (4.7) yields (4.3).

By using (4.2), (4.4) can be treated similarly. □

A natural question arises: can we expand a function f in terms of q-fractional derivatives? That is,

$$f(x)=x^{\alpha-1}\sum_{k=0}^{\infty}c_kx^{k\alpha}\quad\text{or}\quad f(x)=\sum_{k=0}^{\infty}c_kx^{k\alpha}?$$

The following theorem gives the answer for such expansions with sufficient conditions for the uniform convergence.

Theorem 4.3 *Assume that* $f\in\mathcal{L}_q^1[0,a]$ *and* $x^{1-\alpha}\mathcal{D}_q^{n\alpha}f\in C[0,a]$ *for all* $n\in\mathbb{N}$. *If*

$$\bigl|x^{1-\alpha}\mathcal{D}_q^{n\alpha}f(x)\bigr|\le cA^{n\alpha},\quad\forall x\in[0,a],n\in\mathbb{N},$$

where c *is a positive constant, and* A *is a positive number satisfying* $A<\frac{1}{a(1-q)}$, *then* f *has the expansion*

$$f(x)=\sum_{k=0}^{\infty}\frac{I_q^{1-\alpha}\mathcal{D}_q^{k\alpha}f(0)}{\Gamma_q((k+1)\alpha)}x^{(k+1)\alpha-1}.\tag{4.8}$$

Moreover, the series $\sum_{k=0}^{\infty}\frac{I_q^{1-\alpha}\mathcal{D}_q^{k\alpha}f(0)}{\Gamma_q((k+1)\alpha)}x^{k\alpha}$ *converges uniformly to* $x^{1-\alpha}f(x)$ *on* $[0,a]$.

Proof Using (4.3), we obtain

$$\begin{aligned}
&\left| x^{1-\alpha} f(x) - \sum_{k=0}^{n-1} \frac{I_q^{1-\alpha} \mathcal{D}_q^{k\alpha} f(0)}{\Gamma_q((k+1)\alpha)} x^{k\alpha} \right| \\
&\quad \le c\Gamma_q(\alpha) \frac{(aA)^{n\alpha}}{\Gamma_q((n+1)\alpha)} \\
&\quad = \frac{c\Gamma_q(\alpha)(q^{(n+1)\alpha};q)_\infty}{(q;q)_\infty} \frac{(aA)^{n\alpha}}{(1-q)^{1-(n+1)\alpha}} \\
&\quad = \frac{c\Gamma_q(\alpha)(q^{(n+1)\alpha};q)_\infty}{(q;q)_\infty (1-q)^{1-\alpha}} \big(aA(1-q)\big)^{n\alpha} \longrightarrow 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, the result follows. □

Theorem 4.4 *Assume that* ${}^c\mathcal{D}_q^{n\alpha} f \in C[0,a]$ *for* $n \in \mathbb{N}$. *If*

$$\left| {}^c\mathcal{D}_q^{n\alpha} f(x) \right| \le cA^{n\alpha}, \quad \forall x \in [0,a], n \in \mathbb{N},$$

where c *is a positive constant, and* A *is a positive number satisfying* $A < \frac{1}{a(1-q)}$, *then* f *has the expansion*

$$f(x) = \sum_{k=0}^{\infty} \frac{{}^c\mathcal{D}_q^{k\alpha} f(0)}{\Gamma_q(k\alpha + 1)} x^{k\alpha}, \tag{4.9}$$

and the series on the right-hand side of (4.9) *converges uniformly to* $f(x)$ *on* $[0,a]$.

Proof The proof is similar to the proof of Theorem 4.3 and is omitted. □

Remark 4.5

(1) If a function f has the expansion

$$f(x) = \sum_{k=0}^{\infty} a_k x^{(k+1)\alpha-1},$$

then we can deduce that

$$a_k = \frac{I_q^{1-\alpha} \mathcal{D}_q^{k\alpha} f(0)}{\Gamma_q((k+1)\alpha)}.$$

Also, if a function f has the expansion

$$f(x) = \sum_{k=0}^{\infty} b_k x^{k\alpha},$$

then we can deduce that

$$b_k = \frac{{}^c\mathcal{D}_q^{k\alpha} f(0)}{\Gamma_q(k\alpha + 1)}.$$

(2) The results of this paper are valid if f is a function defined on intervals of the form $[-a,a]$ or $[-a,0]$, where $a>0$. In these two cases, $\mathcal{L}_q^1[-a,b]$, $b=0$ or a, is the space of all functions defined on $[-a,b]$ such that

$$\sum_{k=0}^{\infty} q^k(1-q)\bigl|f(xq^k)\bigr| < \infty \quad \text{for all } x\in[-a,b].$$

The space $\mathcal{AC}_q[-a,b]$ is the space of all q-regular at zero functions that satisfy condition (2.3) for all $t\in[-a,b]$.

5 Examples

In this section, we apply the generalized q-Taylor formula to solve fractional q-difference equations with constant coefficients. A solution to this type of equations is introduced in [12] by using q-Laplace transforms. In the following examples, λ is a real number. We assume that the conditions of Theorems 4.3 and 4.4 are satisfied.

Example 5.1 Consider the q-initial value problem

$$^{c}D_q^{\alpha} y(x) = \lambda y(x), \quad y(0)=y_0, x>0. \tag{5.1}$$

We assume that $y\in C[0,a]$ for some $a>0$ to be determined later. By (5.1), $^{c}\mathcal{D}_q^{n\alpha}y(x) = \lambda^n y(x)$. Consequently,

$$\bigl|{}^{c}\mathcal{D}_q^{n\alpha}y(x)\bigr| \le c|\lambda|^n, \quad c:=\max_{x\in[0,a]}\bigl|y(x)\bigr|.$$

Hence, if we assume that $|\lambda a^{\alpha}(1-q)^{\alpha}|<1$, then $y(x)$ can be written as

$$y(x)=\sum_{n=0}^{\infty} {}^{c}\mathcal{D}_q^{n\alpha}y(0)\frac{x^{n\alpha}}{\Gamma_q(n\alpha+1)} = y_0 e_{\alpha,1}\bigl(\lambda x^{\alpha};q\bigr), \quad x\in[0,a], \tag{5.2}$$

where $e_{\nu,\mu}(z;q)$ is one of the q-Mittag-Leffler function defined by

$$e_{\nu,\mu}(z;q)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma_q(\nu k+\mu)}, \quad |z|<(1-q)^{\nu}.$$

Example 5.2 Consider the q-initial value problem

$$^{c}\mathcal{D}_q^{2\alpha}y(x)=-y(x), \quad y(0)=0, \qquad {}^{c}D_q^{\alpha}y(0)=1. \tag{5.3}$$

We assume that $y, {}^{c}D_q^{\alpha}y\in C[0,a]$ for some $a>0$ to be determined later. From (5.3), we conclude that

$$^{c}\mathcal{D}_q^{(2n+1)\alpha}y(x)=(-1)^{n}\,{}^{c}D_q^{\alpha}y(x), \qquad {}^{c}\mathcal{D}_q^{(2n)\alpha}y(x)=(-1)^n y(x), \quad n\in\mathbb{N}.$$

Hence, if $c=\max\{\max_{x\in[0,a]}|y(x)|, \max_{x\in[0,a]}|{}^{c}D_q^{\alpha}y(x)|\}$, then

$$\bigl|{}^{c}\mathcal{D}_q^{n\alpha}y(x)\bigr|\le c, \quad \forall n\in\mathbb{N}.$$

Therefore, by Theorem 4.3, if a is chosen such that $a < \frac{1}{(1-q)}$, then

$$\begin{aligned} y(x) &= \sum_{n=0}^{\infty} {}^{c}\mathcal{D}_q^{n\alpha} y(0) \frac{x^{n\alpha}}{\Gamma_q(n\alpha+1)} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{(2n+1)\alpha}}{\Gamma_q((2n+1)\alpha+1)} = x^{\alpha} e_{2\alpha,\alpha+1}\bigl(-x^{2\alpha};q\bigr). \end{aligned} \tag{5.4}$$

It is worth mentioning that if we set $\alpha = 1$ in (5.4), then we get the Jackson q-sine function introduced in [16]. Thus, we may consider the function in (5.4) as a fractional analogue of the Jackson q-sine function.

Example 5.3 Consider the q-initial value problem

$$D_q^{\alpha} y(x) = \lambda y(x), \qquad \bigl[x^{1-\alpha} y\bigr]\bigl(0^+\bigr) = \frac{y_0}{\Gamma_q(\alpha)}. \tag{5.5}$$

Hence, $\mathcal{D}_q^{n\alpha} y(x) = \lambda^n y(x)$. We seek a solution y such that $x^{1-\alpha} y(x) \in C[0,a]$ for some a. Then

$$\bigl|x^{1-\alpha}\mathcal{D}_q^{n\alpha} y(x)\bigr| \le c|\lambda|^n, \quad c := \max_{x\in[0,a]} \bigl|x^{1-\alpha} y(x)\bigr|.$$

We can show that

$$I_q^{1-\alpha} D_q^{\alpha} y(0) = \Gamma_q(\alpha)\bigl[x^{1-\alpha} y(x)\bigr]\bigl(0^+\bigr). \tag{5.6}$$

Consequently, $I_q^{1-\alpha}\mathcal{D}_q^{k\alpha} y(0) = \lambda^n y_0$. Therefore,

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty} \frac{I_q^{1-\alpha}\mathcal{D}_q^{k\alpha} y(0)}{\Gamma_q((k+1)\alpha)} x^{(k+1)\alpha-1} \\ &= y_0 x^{\alpha-1} \sum_{k=0}^{\infty} \frac{(\lambda x^{\alpha})^k}{\Gamma_q((k+1)\alpha)} = y_0 x^{\alpha-1} e_{\alpha,\alpha}\bigl(\lambda x^{\alpha};q\bigr), \end{aligned}$$

where $|\lambda a^{\alpha}(1-q)^{\alpha}| < 1$.

Example 5.4 Consider the q-initial value problem

$$\mathcal{D}_q^{2\alpha} y(x) = -\lambda y(x), \qquad \bigl[x^{1-\alpha} y\bigr]\bigl(0^+\bigr) = \frac{y_1}{\Gamma_q(\alpha)}, \qquad \bigl[x^{1-\alpha} D_q^{\alpha} y\bigr]\bigl(0^+\bigr) = \frac{y_2}{\Gamma_q(\alpha)}. \tag{5.7}$$

Thus,

$$\mathcal{D}_q^{2n\alpha} y(x) = (-\lambda)^n y(x), \qquad \mathcal{D}_q^{(2n+1)\alpha} y(x) = (-\lambda)^n D_q^{\alpha} y(x).$$

For a solution y such that $x^{1-\alpha} y(x), x^{1-\alpha} D_q^{\alpha} y(x) \in C[0,a]$ for some a, we have

$$\bigl|x^{1-\alpha}\mathcal{D}_q^{n\alpha} y(x)\bigr| \le c|\lambda|^n, \quad c := \max\Bigl\{\max_{x\in[0,a]} \bigl|x^{1-\alpha} y(x)\bigr|, \max_{x\in[0,a]} \bigl|x^{1-\alpha} D_q^{\alpha} y(x)\bigr|\Bigr\}.$$

Also,

$$I_q^{1-\alpha}\mathcal{D}_q^{2n\alpha}y(0) = (-\lambda)^n y_1, \qquad I_q^{1-\alpha}\mathcal{D}_q^{(2n+1)\alpha}y(0) = (-\lambda)^n y_2.$$

Consequently,

$$\begin{aligned} y(x) &= x^{\alpha-1}\sum_{k=0}^{\infty}\frac{I_q^{1-\alpha}\mathcal{D}_q^{2k\alpha}y(0)}{\Gamma_q((2k+)\alpha)}x^{2k\alpha} + x^{\alpha-1}\sum_{k=0}^{\infty}\frac{I_q^{1-\alpha}\mathcal{D}_q^{(2k+1)\alpha}y(0)}{\Gamma_q((2k+2)\alpha)}x^{(2k+1)\alpha} \\ &= y_1x^{\alpha-1}\sum_{k=0}^{\infty}\frac{(-\lambda)^k x^{2k\alpha}}{\Gamma_q((2k+1)\alpha)} + y_2x^{\alpha-1}\sum_{k=0}^{\infty}\frac{(-\lambda)^k x^{(2k+1)\alpha}}{\Gamma_q((2k+2)\alpha)} \\ &= y_1x^{\alpha-1}e_{2\alpha,\alpha}\left(-\lambda x^{2\alpha};q\right) + y_2x^{2\alpha-1}e_{2\alpha,2\alpha}\left(-\lambda x^{2\alpha};q\right), \end{aligned}$$

where $|\lambda a^\alpha(1-q)^\alpha| < 1$.

Example 5.5 Consider the initial value problem

$$D_q^\alpha y(x) = \lambda q^{\alpha(1-\alpha)}y\left(q^\alpha x\right), \qquad \left[x^{1-\alpha}y\right]\left(0^+\right) = \frac{1}{\Gamma_q(\alpha)}. \tag{5.8}$$

Applying

$$D_{q,x}^\alpha y(x\beta) = \beta\left(D_q^\alpha y\right)(x\beta), \tag{5.9}$$

on (5.8) $n-1$ times, we obtain

$$\mathcal{D}_q^{n\alpha}y(x) = \left(\lambda q^{\alpha(1-\alpha)}\right)^n q^{\frac{n(n-1)\alpha}{2}}y\left(xq^{n\alpha}\right). \tag{5.10}$$

For a solution y such that $x^{1-\alpha}y(x) \in C[0,a]$, we have

$$\left|x^{1-\alpha}\mathcal{D}_q^{n\alpha}y(x)\right| \le c|\lambda|^n q^{\frac{n(n-1)\alpha}{2}}, \quad c := \max_{x\in[0,a]}\left|x^{1-\alpha}y(x)\right|,$$

and

$$I_q^{1-\alpha}\mathcal{D}_q^{n\alpha}y(0) = \lambda^n q^{\frac{n(n-1)\alpha}{2}}.$$

Therefore,

$$\begin{aligned} y(x) &= \sum_{k=0}^{\infty}\frac{I_q^{1-\alpha}\mathcal{D}_q^{k\alpha}y(0)}{\Gamma_q((k+1)\alpha)}x^{(k+1)\alpha-1} \\ &= x^{\alpha-1}\sum_{k=0}^{\infty}q^{\frac{n(n-1)\alpha}{2}}\frac{(\lambda x^\alpha)^k}{\Gamma_q((k+1)\alpha)} \\ &= x^{\alpha-1}E_{\alpha,\alpha}\left(\lambda x^\alpha;q\right), \end{aligned}$$

where, in general, $E_{\alpha,\beta}(z;q)$ is a second q-analogue of Mittag-Leffler function defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}q^{\frac{n(n-1)\alpha}{2}}\frac{z^n}{\Gamma_q(n\alpha+\beta)}, \quad z \in \mathbb{R}.$$

Hence, a can be taken to be any positive value in this example. For some derived properties for these q-analogues of Mittag-Leffler functions, see [9] and the references therein.

Competing interests
The author declares that he has no competing interests.

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