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Stability analysis of discrete-time systems with variable delays via some new summation inequalities

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Abstract

This paper proposes an improved stability condition of discrete-time systems with variable delays. Based on some mathematical techniques, a series of new summation inequalities are obtained. These new inequalities are less conservative than the Jensen inequality. Based on these new summation inequalities and the reciprocally convex combination inequality, a novel sufficient criterion on asymptotical stability of discrete-time systems with variable delays is obtained by constructing a new Lyapunov-Krasovskii functional. The advantage of the proposed inequality in this paper is demonstrated by a classical example from the literature.

Keywords: Jensen's inequality; summation inequality; stability; discrete-time system

1 Introduction

Time delay is usually encountered in many practical situations such as signal processing, image processing etc. There has been an increasing research activity on time-delay systems during the past years [1–16]. The problem of the delay-dependent stability analysis of time-delay systems has become a hot research topic in the control community [17, 18] due to the fact that stability criteria can provide a maximum admissible upper bound of time delay. The maximum admissible upper bound can be regarded as an important index for the conservatism of stability criteria [19–23]. To our knowledge, Jensen's inequality has been mostly used as a powerful mathematical tool in the stability analysis of time-delay systems. However, Jensen's inequality neglected some terms, which unavoidably introduced some conservatism. In order to investigate the stability of a linear discrete systems with constant delay, Zhang and Han [24] established the following Abel lemma-based finite-sum inequality, which improved the Jensen inequality to some extent.

Theorem A [24] *For a constant matrix $R \in R^{n\times n}$ with $R = R^T > 0$, and two integers r_1 and r_2 with $r_2 - r_1 > 1$, the following inequality holds*:

$$\sum_{j=r_1}^{r_2-1} \eta^T(j)R\eta(j) \geq \frac{1}{\rho_1}\nu_1^T R\nu_1 + \frac{3\rho_2}{\rho_1\rho_3}\nu_2^T R\nu_2, \tag{1}$$

where $\eta(j) = x(j+1) - x(j)$, $\nu_1 = x(r_2) - x(r_1)$, $\nu_2 = x(r_2) + x(r_1) - \frac{2}{r_2-r_1-1}\sum_{j=r_1+1}^{r_2-1} x(j)$, $\rho_1 = r_2 - r_1$, $\rho_2 = r_2 - r_1 - 1$, $\rho_3 = r_2 - r_1 + 1$.

Seuret *et al.* [25] also obtained a new stability criterion for the discrete-time systems with time-varying delay via the following novel summation inequality.

Theorem B [25] *For a given symmetric positive definite matrix $R \in R^{n\times n}$ and any sequence of discrete-time variables z in $[-h,0] \cap Z \to R^n$, where $h \geq 1$, the following inequality holds*:

$$\sum_{i=-h+1}^{0} y^T(i)Ry(i) \geq \frac{1}{h}\begin{pmatrix}\Theta_0\\ \Theta_1\end{pmatrix}^T \begin{pmatrix}R & 0\\ 0 & 3(\frac{h+1}{h-1})R\end{pmatrix}\begin{pmatrix}\Theta_0\\ \Theta_1\end{pmatrix}, \tag{2}$$

where $y(i) = z(i) - z(i-1)$, $\Theta_0 = z(0) - z(-h)$, $\Theta_1 = z(0) + z(-h) - \frac{2}{h+1}\sum_{i=-h}^{0} z(i)$.

In fact, Theorem A is equivalent to Theorem B. These two summation inequalities encompass the Jensen inequality. It is worth mentioning that Theorem A and Theorem B can be regarded as a discrete time version of the Wirtinger-based integral inequality, which was proved in [26].

Recently, Park *et al.* [27] developed a novel class of integral inequalities for quadratic functions via some intermediate terms called auxiliary functions which improved the Wirtinger-based integral inequality. Based on the novel inequalities, some new stability criteria are presented for systems with time-varying delays by constructing some appropriate Lyapunov-Krasovskii functionals in [27].

The Lyapunov-Krasovskii functional method is the most commonly used method in the investigation of the stability of delayed systems. The conservativeness of this approach is mainly from the construction of the Lyapunov-Krasovskii functional and the estimation of its time derivative. In order to get less conservative results, Jensen's integral inequality, Wirtinger's integral inequality, and a free-matrix-based integral inequality are proposed to obtain a tighter upper bound of the integrals occurring in the time derivative of the Lyapunov-Krasovskii functional. Many papers have focused on integral inequalities and their applications in stability analysis of continuous-time-delayed systems. However, only a few papers have studied the summation inequalities and their application in stability analysis of discrete-time systems with variable delays. The summation inequalities in Theorem A and Theorem B are used to obtain a bound for $\sum_{j=r_1}^{r_2-1}\eta^T(j)R\eta(j)$ or $\sum_{i=-h+1}^{0} y^T(i)Ry(i)$.

Motivated by the above works, in order to provide a tighter bound for $\sum_{j=r_1}^{r_2-1}\eta^T(j)R\eta(j)$ or $\sum_{i=-h+1}^{0} y^T(i)Ry(i)$, this paper is aimed at establishing some novel summation inequalities as the discrete-time versions of the integral inequalities obtained in [27]. In this paper, we will extend the two summation inequalities given in [24, 25]. Some new summation inequalities are proposed to provide a sharper bound than the summation inequalities in [24, 25]. The inequalities in Theorem A and Theorem B are a special case of Corollary 6 in our paper. Moreover, a novel estimation to the double summation as $\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\Delta x(k)^T R\Delta x(k)$ is also given in this paper. Based on these new summation inequalities, the reciprocally convex combination inequality, and a new Lyapunov-Krasovskii functional, a less conservative sufficient criterion on asymptotical stability of discrete-time systems with variable delays is obtained.

Notations Throughout this paper, R^n and $R^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all $n \times m$ real matrices. For real symmetric matrices X and

Y, the notation $X \geq Y$ (or $X > Y$) means that the matrix $X - Y$ is a positive semi-definite (or positive definite). The symbol $*$ within a matrix represents the symmetric term of the matrix.

2 Novel summation inequalities

Theorem 1 *For a positive definite matrix $R > 0$, any sequence of discrete-time variables $y : [-h, 0] \cap Z \to R^n$, and any sequence of discrete-time variables $p : [-h, 0] \cap Z \to R$ satisfying $\sum_{k=-h+1}^{0} p(k) = 0$, the following inequality holds*:

$$\begin{aligned} &\sum_{k=-h+1}^{0} p^2(k) \sum_{i=-h+1}^{0} y(i)^T R y(i) \\ &\quad \geq \frac{1}{h} \Theta_0^T R \Theta_0 \sum_{k=-h+1}^{0} p^2(k) + \left[\sum_{i=-h+1}^{0} y(i)p(i)\right]^T R \left[\sum_{i=-h+1}^{0} y(i)p(i)\right], \end{aligned} \tag{3}$$

where $\Theta_0 = \sum_{k=-h+1}^{0} y(k)$.

Proof Let $z(i) = y(i) - \frac{1}{h}\Theta_0 - p(i)v$, where $v \in R^n$ is to be defined later. Then find a vector $\hat{v}$ to minimize the following energy function $J(v)$:

$$J(v) = \sum_{i=-h+1}^{0} z(i)^T R z(i). \tag{4}$$

Obviously,

$$\begin{aligned} J'(v) &= -2 \sum_{i=-h+1}^{0} \left[y(i)p(i) - \frac{1}{h}\Theta_0 p(i) - p^2(i)v\right]^T R \\ &= -2 \sum_{i=-h+1}^{0} \left[y(i)p(i) - p^2(i)v\right]^T R + \frac{2}{h}\Theta_0^T \sum_{i=-h+1}^{0} p(i)R \\ &= -2 \sum_{i=-h+1}^{0} y(i)^T p(i)R + 2v^T R \sum_{i=-h+1}^{0} p^2(i). \end{aligned} \tag{5}$$

If $\sum_{k=-h+1}^{0} p^2(k) > 0$, solving the equation $J'(v) = 0$ gives

$$\hat{v} = \sum_{i=-h+1}^{0} y(i)p(i) \left[\sum_{i=-h+1}^{0} p^2(i)\right]^{-1}. \tag{6}$$

Substituting $\hat{v}$ for v in $J(v)$, we get

$$\begin{aligned} J(\hat{v}) &= \sum_{i=-h+1}^{0} \left[y(i) - \frac{1}{h}\Theta_0 - p(i)\hat{v}\right]^T R \left[y(i) - \frac{1}{h}\Theta_0 - p(i)\hat{v}\right] \\ &= \sum_{i=-h+1}^{0} \left[y(i) - \frac{1}{h}\Theta_0\right]^T R \left[y(i) - \frac{1}{h}\Theta_0\right] \end{aligned}$$

$$
\begin{aligned}
&-2\sum_{i=-h+1}^{0}\left[y(i)-\frac{1}{h}\Theta_0\right]^T Rp(i)\hat{v}+\sum_{i=-h+1}^{0}p^2(i)\hat{v}^T R\hat{v} \\
&=\sum_{i=-h+1}^{0}\left[y(i)-\frac{1}{h}\Theta_0\right]^T R\left[y(i)-\frac{1}{h}\Theta_0\right]-2\sum_{i=-h+1}^{0}y(i)^T Rp(i)\hat{v} \\
&\quad+\sum_{i=-h+1}^{0}p^2(i)\hat{v}^T R\hat{v} \\
&=\sum_{i=-h+1}^{0}\left[y(i)-\frac{1}{h}\Theta_0\right]^T R\left[y(i)-\frac{1}{h}\Theta_0\right]-2\sum_{i=-h+1}^{0}p(i)y(i)^T R\hat{v} \\
&\quad+\sum_{i=-h+1}^{0}p^2(i)\hat{v}^T R\hat{v} \\
&=\sum_{i=-h+1}^{0}\left[y(i)-\frac{1}{h}\Theta_0\right]^T R\left[y(i)-\frac{1}{h}\Theta_0\right]-2\sum_{i=-h+1}^{0}p^2(i)\hat{v}^T R\hat{v} \\
&\quad+\sum_{i=-h+1}^{0}p^2(i)\hat{v}^T R\hat{v} \\
&=\sum_{i=-h+1}^{0}\left[y(i)-\frac{1}{h}\Theta_0\right]^T R\left[y(i)-\frac{1}{h}\Theta_0\right]-\sum_{i=-h+1}^{0}p^2(i)\hat{v}^T R\hat{v} \\
&=\sum_{i=-h+1}^{0}y(i)^T Ry(i)-2\sum_{i=-h+1}^{0}y(i)^T R\frac{1}{h}\Theta_0+\frac{1}{h^2}\sum_{i=-h+1}^{0}\Theta_0^T R\Theta_0 \\
&\quad-\sum_{i=-h+1}^{0}p(i)^2\hat{v}^T R\hat{v} \\
&=\sum_{i=-h+1}^{0}y(i)^T Ry(i)-\frac{1}{h}\Theta_0^T R\Theta_0-\sum_{i=-h+1}^{0}p^2(i)\hat{v}^T R\hat{v}.
\end{aligned}
\tag{7}
$$

By the non-negative characteristic of the energy function $J(v)$, we have

$$
\sum_{i=-h+1}^{0}y(i)^T Ry(i)\geq\frac{1}{h}\Theta_0^T R\Theta_0+\frac{1}{\sum_{i=-h+1}^{0}p^2(i)}\left[\sum_{i=-h+1}^{0}y(i)p(i)\right]^T R\left[\sum_{i=-h+1}^{0}y(i)p(i)\right]. \tag{8}
$$

If $\sum_{k=-h+1}^{0}p^2(k)=0$, obviously, inequality (3) holds.

This completes the proof of Theorem 1. □

By choosing an appropriate sequence $p(k)$, we get the following corollaries.

Corollary 1 *For a positive definite matrix $R>0$ and any sequence of discrete-time variables $y:[-h,0]\cap Z\to R^n$, the following inequality holds*:

$$
\sum_{i=-h+1}^{0}y(i)^T Ry(i)\geq\frac{1}{h}\Theta_0^T R\Theta_0+\frac{3}{h}\frac{h+1}{h-1}\Omega_1^T R\Omega_1, \tag{9}
$$

where $\Theta_0=\sum_{k=-h+1}^{0}y(k)$, $\Omega_1=\sum_{s=-h+1}^{0}y(s)-\frac{2}{h+1}\sum_{k=-h+1}^{0}\sum_{s=-h+1}^{k}y(s)$.

Proof Let $p(k) = h - 1 + 2k$, then $\sum_{k=-h+1}^{0} p(k) = 0$ and $\sum_{i=-h+1}^{0} p^2(i) = \frac{(h-1)h(h+1)}{3}$,

$$\begin{aligned}\sum_{i=-h+1}^{0} y(i)p(i) &= \sum_{i=-h+1}^{0} (h-1+2i)y(i)\\ &= -2\sum_{k=-h+1}^{-1}\sum_{s=-h+1}^{k} y(s) + (h-1)\sum_{s=-h+1}^{0} y(s)\\ &= -2\sum_{k=-h+1}^{0}\sum_{s=-h+1}^{k} y(s) + (h+1)\sum_{s=-h+1}^{0} y(s)\\ &= (h+1)\left[\sum_{s=-h+1}^{0} y(s) - \frac{2}{h+1}\sum_{k=-h+1}^{0}\sum_{s=-h+1}^{k} y(s)\right]\\ &= (h+1)\Omega_1.\end{aligned} \tag{10}$$

By using Theorem 1, inequality (9) holds. □

Let $\Omega_1^* = \sum_{s=-h+1}^{0} y(s) - \frac{2}{h+1}\sum_{k=-h+1}^{0}\sum_{s=k}^{0} y(s)$. Due to

$$\sum_{k=-h+1}^{0}\sum_{s=-h+1}^{k} y(s) + \sum_{k=-h+1}^{0}\sum_{s=k}^{0} y(s) = (h+1)\sum_{s=-h+1}^{0} y(s), \tag{11}$$

we have

$$\begin{aligned}\Omega_1 &= \sum_{s=-h+1}^{0} y(s) - \frac{2}{h+1}\sum_{k=-h+1}^{0}\sum_{s=-h+1}^{k} y(s)\\ &= -\sum_{s=-h+1}^{0} y(s) + \frac{2}{h+1}\sum_{k=-h+1}^{0}\sum_{s=k}^{0} y(s)\\ &= -\Omega_1^*.\end{aligned} \tag{12}$$

Hence, Corollary 1 is equivalent to Corollary 2.

Corollary 2 *For a positive definite matrix $R > 0$ and any sequence of discrete-time variables $y : [-h, 0] \cap Z \to R^n$, the following inequality holds*:

$$\sum_{i=-h+1}^{0} y(i)^T R y(i) \ge \frac{1}{h}\Theta_0^T R \Theta_0 + \frac{3}{h}\frac{h+1}{h-1}\Omega_1^{*T} R \Omega_1^*, \tag{13}$$

where $\Theta_0 = \sum_{k=-h+1}^{0} y(k)$, $\Omega_1^* = \sum_{s=-h+1}^{0} y(s) - \frac{2}{h+1}\sum_{k=-h+1}^{0}\sum_{s=k}^{0} y(s)$.

Corollary 3 *For a positive definite matrix $R > 0$ and any sequence of discrete-time variables $x : [-h, 0] \cap Z \to R^n$, the following inequality holds*:

$$\sum_{i=-h+1}^{0} \Delta x(i)^T R \Delta x(i) \ge \frac{1}{h}\Theta_0^T R \Theta_0 + \frac{1}{h}\Theta_1^T \frac{3(h+1)}{h-1} R \Theta_1, \tag{14}$$

where $\Delta x(i) = x(i) - x(i-1)$, $\Theta_0 = x(0) - x(-h)$, $\Theta_1 = x(0) + x(-h) - \frac{2}{h+1}\sum_{k=-h}^{0} x(k)$.

Proof Let $y(i) = \Delta x(i) = x(i) - x(i-1)$ in Corollary 1. Then we have

$$\Theta_0 = \sum_{k=-h+1}^{0} y(k) = x(0) - x(-h) \tag{15}$$

and

$$\begin{aligned}
\Omega_1 &= \sum_{s=-h+1}^{0} y(s) - \frac{2}{h+1} \sum_{k=-h+1}^{0} \sum_{s=-h+1}^{k} y(s) \\
&= x(0) - x(-h) - \frac{2}{h+1} \sum_{k=-h+1}^{0} \bigl[x(k) - x(-h)\bigr] \\
&= x(0) - x(-h) - \frac{2}{h+1} \sum_{k=-h+1}^{0} x(k) + \frac{2h}{h+1} x(-h) \\
&= x(0) + x(-h) - \frac{2}{h+1} \sum_{k=-h}^{0} x(k) \\
&= \Theta_1.
\end{aligned} \tag{16}$$

Using Corollary 1, we have completed the proof of Corollary 3. □

Remark 1 Corollary 1 or Corollary 2 in this paper can be regard as a discrete version of Wirtinger-based integral inequality proved in [26]. Corollary 3 is a special case of Corollary 1 or Corollary 2. In fact, Corollary 3 is equivalent to Theorem A and Theorem B. So Corollary 1 or Corollary 2 in this paper implies Theorem A and Theorem B.

Generally, we have the following result which includes Corollary 3 as a special case.

Corollary 4 *For a positive definite matrix $R > 0$, any sequence of discrete-time variables $x : [-h, 0] \cap Z \to R^n$, and any sequence of discrete-time variables $p : [-h, 0] \cap Z \to R$ satisfying $\sum_{k=-h+1}^{0} p(k) = 0$, the following inequality holds*:

$$\begin{aligned}
&\sum_{i=-h+1}^{0} \Delta x(i)^T R \Delta x(i) \\
&\quad \geq \frac{1}{h} \Theta_0^T R \Theta_0 + \frac{1}{\sum_{i=-h+1}^{0} p^2(i)} \left[\sum_{i=-h+1}^{0} p(i) \Delta x(i)\right]^T R \left[\sum_{i=-h+1}^{0} p(i) \Delta x(i)\right],
\end{aligned} \tag{17}$$

where $\Theta_0 = x(0) - x(-h)$, $\Delta x(i) = x(i) - x(i-1)$.

To go a step further, suppose that $p_1(i) = h - 1 + 2i$, $p_2(i) = i^2 + (h-1)i + \frac{(h-1)(h-2)}{6}$, then
(1) $\sum_{i=-h+1}^{0} p_m(i) = 0$, $m = 1, 2$,
(2) $\sum_{i=-h+1}^{0} p_1(i) p_2(i) = 0$,
(3) $\sum_{i=-h+1}^{0} p_2^2(i) = \frac{(h-2)(h-1)h(h+1)(h+2)}{180}$.
Noting that

$$\sum_{i=-h+1}^{0} \sum_{k=i}^{0} y(k) = \sum_{i=-h+1}^{0} (h+i) y(i) \tag{18}$$

and

$$\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\sum_{m=k}^{0}y(m)=\sum_{i=-h+1}^{0}\frac{(h+i)(h+i+1)}{2}y(i). \tag{19}$$

Then we get

$$\begin{aligned}
&\sum_{i=-h+1}^{0}p_2(i)y(i)\\
&\quad=2\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\sum_{m=k}^{0}y(m)+\frac{(h+1)(h+2)}{6}\sum_{i=-h+1}^{0}y(i)-(h+2)\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)\\
&\quad=\frac{(h+1)(h+2)}{6}\Bigg[\sum_{i=-h+1}^{0}y(i)-\frac{6}{h+1}\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)\\
&\qquad+\frac{12}{(h+1)(h+2)}\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\sum_{m=k}^{0}y(m)\Bigg].
\end{aligned} \tag{20}$$

Let $p(k)=p_2(k)$ in Theorem 1, we have the following theorem.

Theorem 2 *For a positive definite matrix $R>0$ and any sequence of discrete-time variables $y:[-h,0]\cap Z\to R^n$ the following inequality holds*:

$$\sum_{i=-h+1}^{0}y(i)^TRy(i)\geq\frac{1}{h}\Theta_0^TR\Theta_0+\frac{5(h+1)(h+2)}{(h-2)(h-1)h}\Omega_2^TR\Omega_2, \tag{21}$$

where $\Omega_2=\sum_{i=-h+1}^{0}y(i)-\frac{6}{h+1}\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)+\frac{12}{(h+1)(h+2)}\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\sum_{m=k}^{0}y(m)$, $\Theta_0=\sum_{k=-h+1}^{0}y(k)$.

Remark 2 Theorem 2 gives a new form of summation inequality and the idea which stimulates our interests in establishing a novel combinational summation inequality underlying quadrature rules. Based on Theorem 1 and Theorem 2, an improved summation inequality can be obtained as follows.

Theorem 3 *For a positive definite matrix $R>0$, any sequence of discrete-time variables $y:[-h,0]\cap Z\to R^n$, and any two sequences of discrete-time variables $p_i:[-h,0]\cap Z\to R$ satisfying $\sum_{k=-h+1}^{0}p_i(k)=0$, $i=1,2$, $\sum_{k=-h+1}^{0}p_1(k)p_2(k)=0$, then the following inequality holds*:

$$\begin{aligned}
&\sum_{i=-h+1}^{0}y(i)^TRy(i)\\
&\quad\geq\frac{1}{h}\Theta_0^TR\Theta_0+\frac{1}{\sum_{i=-h+1}^{0}p_1(i)^2}\Bigg[\sum_{i=-h+1}^{0}y(i)p_1(i)\Bigg]^TR\Bigg[\sum_{i=-h+1}^{0}y(i)p_1(i)\Bigg]\\
&\qquad+\frac{1}{\sum_{i=-h+1}^{0}p_2(i)^2}\Bigg[\sum_{i=-h+1}^{0}y(i)p_2(i)\Bigg]^TR\Bigg[\sum_{i=-h+1}^{0}y(i)p_2(i)\Bigg],
\end{aligned} \tag{22}$$

where $\Theta_0=\sum_{k=-h+1}^{0}y(k)$.

Proof Let $z(i) = y(i) - \frac{1}{h}\Theta_0 - \frac{p_1(i)}{\sum_{i=-h+1}^{0} p_1(i)^2}\sum_{i=-h+1}^{0} y(i)p_1(i)$. Based on the proof of Theorem 1, we have

$$\begin{aligned}
&\sum_{i=-h+1}^{0} y(i)^T Ry(i) \\
&\quad = \frac{1}{h}\Theta_0^T R\Theta_0 + \frac{1}{\sum_{i=-h+1}^{0} p_1(i)^2}\left[\sum_{i=-h+1}^{0} y(i)p_1(i)\right]^T R\left[\sum_{i=-h+1}^{0} y(i)p_1(i)\right] \\
&\qquad + \sum_{i=-h+1}^{0} z(i)^T Rz(i).
\end{aligned} \tag{23}$$

Let $x(i) = z(i) - \frac{p_2(i)}{\sum_{i=-h+1}^{0} p_2(i)^2}\sum_{i=-h+1}^{0} z(i)p_2(i)$. Similarly, we have

$$\begin{aligned}
&\sum_{i=-h+1}^{0} z(i)^T Rz(i) \\
&\quad = \frac{1}{\sum_{i=-h+1}^{0} p_2(i)^2}\left[\sum_{i=-h+1}^{0} z(i)p_2(i)\right]^T R\left[\sum_{i=-h+1}^{0} z(i)p_2(i)\right] + \sum_{i=-h+1}^{0} x(i)^T Rx(i).
\end{aligned} \tag{24}$$

So

$$\begin{aligned}
&\sum_{i=-h+1}^{0} y(i)^T Ry(i) \\
&\quad = \frac{1}{h}\Theta_0^T R\Theta_0 + \frac{1}{\sum_{i=-h+1}^{0} p_1(i)^2}\left[\sum_{i=-h+1}^{0} y(i)p_1(i)\right]^T R\left[\sum_{i=-h+1}^{0} y(i)p_1(i)\right] \\
&\qquad + \frac{1}{\sum_{i=-h+1}^{0} p_2(i)^2}\left[\sum_{i=-h+1}^{0} z(i)p_2(i)\right]^T R\left[\sum_{i=-h+1}^{0} z(i)p_2(i)\right] \\
&\qquad + \sum_{i=-h+1}^{0} x(i)^T Rx(i) \\
&\quad \geq \frac{1}{h}\Theta_0^T R\Theta_0 + \frac{1}{\sum_{i=-h+1}^{0} p_1(i)^2}\left[\sum_{i=-h+1}^{0} y(i)p_1(i)\right]^T R\left[\sum_{i=-h+1}^{0} y(i)p_1(i)\right] \\
&\qquad + \frac{1}{\sum_{i=-h+1}^{0} p_2(i)^2}\left[\sum_{i=-h+1}^{0} z(i)p_2(i)\right]^T R\left[\sum_{i=-h+1}^{0} z(i)p_2(i)\right].
\end{aligned} \tag{25}$$

Since $\sum_{k=-h+1}^{0} p_i(k) = 0$ $(i = 1, 2)$ and $\sum_{k=-h+1}^{0} p_1(k)p_2(k) = 0$, we obtain

$$\begin{aligned}
&\sum_{i=-h+1}^{0} z(i)p_2(i) \\
&\quad = \sum_{i=-h+1}^{0}\left[y(i) - \frac{1}{h}\Theta_0 - \frac{p_1(i)}{\sum_{j=-h+1}^{0} p_1(j)^2}\sum_{j=-h+1}^{0} y(j)p_1(j)\right]p_2(i)
\end{aligned}$$

$$= \sum_{i=-h+1}^{0} y(i)p_2(i) - \frac{\Theta_0}{h}\sum_{i=-h+1}^{0} p_2(i) - \frac{\sum_{i=-h+1}^{0} p_1(i)p_2(i)}{\sum_{i=-h+1}^{0} p_1(i)^2}\sum_{i=-h+1}^{0} y(i)p_1(i)$$
$$= \sum_{i=-h+1}^{0} y(i)p_2(i). \tag{26}$$

This completes the proof of Theorem 3. □

Noting that $\sum_{i=-h+1}^{0} p_m(i) = 0$ $(m = 1, 2)$ and $\sum_{i=-h+1}^{0} p_1(i)p_2(i) = 0$, combining Theorem 3 with Corollary 2 and Theorem 2 gives the following result.

Corollary 5 *For a positive definite matrix* $R > 0$ *and any sequence of discrete-time variables* $y : [-h, 0] \cap Z \to R^n$, *the following inequality holds*:

$$\sum_{i=-h+1}^{0} y(i)^T Ry(i) \geq \frac{1}{h}\Theta_0^T R\Theta_0 + \frac{3}{h}\frac{h+1}{h-1}\Omega_1^{*T} R\Omega_1^* + \frac{5(h+1)(h+2)}{(h-2)(h-1)h}\Omega_2^T R\Omega_2, \tag{27}$$

where $\Theta_0 = \sum_{k=-h+1}^{0} y(k)$, $\Omega_1^* = \sum_{s=-h+1}^{0} y(s) - \frac{2}{h+1}\sum_{k=-h+1}^{0}\sum_{s=k}^{0} y(s)$, $\Omega_2 = \sum_{i=-h+1}^{0} y(i) - \frac{6}{h+1}\sum_{i=-h+1}^{0}\sum_{k=i}^{0} y(k) + \frac{12}{(h+1)(h+2)}\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\sum_{m=k}^{0} y(m)$.

Corollary 6 *For a positive definite matrix* $R > 0$ *and any sequence of discrete-time variables* $y : [-h, 0] \cap Z \to R^n$, *the following inequality holds*:

$$\sum_{i=-h+1}^{0} \Delta x(i)^T R\Delta x(i) \geq \frac{1}{h}\Theta_0^T R\Theta_0 + \frac{3}{h}\frac{h+1}{h-1}\Omega_1{}^T R\Omega_1 + \frac{5(h+1)(h+2)}{(h-2)(h-1)h}\Omega_2^T R\Omega_2, \tag{28}$$

where $\Theta_0 = x(0) - x(-h)$, $\Omega_1 = x(0) + x(-h) - \frac{2}{h+1}\sum_{k=-h}^{0} x(k)$, $\Omega_2 = x(0) - x(-h) + \frac{6h}{(h+1)(h+2)} \times \sum_{i=-h}^{0} x(i) - \frac{12}{(h+1)(h+2)}\sum_{i=-h+1}^{0}\sum_{k=i}^{0} x(k)$.

Proof Let $y(i) = \Delta x(i) = x(i) - x(i-1)$, $\Omega_1 = -\Omega_1^*$ in Corollary 5. Then $\Theta_0 = \sum_{k=-h+1}^{0} y(k) = x(0) - x(-h)$. Simple computation leads to

$$\Omega_1 = \sum_{s=-h+1}^{0} y(s) - \frac{2}{h+1}\sum_{k=-h+1}^{0}\sum_{s=-h+1}^{k} y(s)$$
$$= x(0) - x(-h) - \frac{2}{h+1}\sum_{k=-h+1}^{0}\bigl[x(k) - x(-h)\bigr]$$
$$= x(0) + x(-h) - \frac{2}{h+1}\sum_{k=-h+1}^{0} x(k) - \frac{2}{h+1}x(-h)$$
$$= x(0) + x(-h) - \frac{2}{h+1}\sum_{k=-h}^{0} x(k) \tag{29}$$

and

$$\begin{aligned}
\Omega_2 &= \sum_{i=-h+1}^{0} y(i) - \frac{6}{h+1}\sum_{i=-h+1}^{0}\sum_{k=i}^{0} y(k) + \frac{12}{(h+1)(h+2)}\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\sum_{m=k}^{0} y(m) \\
&= \sum_{i=-h+1}^{0}\bigl[x(i)-x(i-1)\bigr] - \frac{6}{h+1}\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\bigl[x(k)-x(k-1)\bigr] \\
&\quad + \frac{12}{(h+1)(h+2)}\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\sum_{m=k}^{0}\bigl[x(m)-x(m-1)\bigr] \\
&= x(0)-x(-h) - \frac{6}{h+1}\sum_{i=-h+1}^{0}\bigl[x(0)-x(i-1)\bigr] \\
&\quad + \frac{12}{(h+1)(h+2)}\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\bigl[x(0)-x(k-1)\bigr] \\
&= x(0)-x(-h) + \frac{6}{h+1}\sum_{i=-h+1}^{0} x(i-1) - \frac{6h}{h+1}x(0) \\
&\quad + \frac{12}{(h+1)(h+2)}x(0)\sum_{i=-h+1}^{0}(-i+1) - \frac{12}{(h+1)(h+2)}\sum_{i=-h+1}^{0}\sum_{k=i}^{0} x(k-1) \\
&= x(0)-x(-h) + \frac{6}{h+1}\sum_{i=-h+1}^{0} x(i-1) \\
&\quad - \frac{6h}{(h+1)(h+2)}x(0) - \frac{12}{(h+1)(h+2)}\sum_{i=-h+1}^{0}\sum_{k=i}^{0} x(k-1).
\end{aligned} \tag{30}$$

An identical transformation leads to

$$\begin{aligned}
\Omega_2 &= x(0)-x(-h) + \frac{6}{h+1}\sum_{i=-h}^{0} x(i) - \frac{6}{h+1}x(0) \\
&\quad - \frac{6h}{(h+1)(h+2)}x(0) - \frac{12}{(h+1)(h+2)}\sum_{i=-h+1}^{0}\sum_{k=i}^{0} x(k-1) \\
&= x(0)-x(-h) + \frac{6}{h+1}\sum_{i=-h}^{0} x(i) \\
&\quad - \frac{12}{(h+1)(h+2)}\left[(h+1)x(0) + \sum_{i=-h+1}^{0}\sum_{k=i}^{0} x(k-1)\right] \\
&= x(0)-x(-h) + \frac{6h}{(h+1)(h+2)}\sum_{i=-h}^{0} x(i) - \frac{12}{(h+1)(h+2)}\sum_{i=-h+1}^{0}\sum_{k=i}^{0} x(k).
\end{aligned} \tag{31}$$

This completes the proof of Corollary 6. □

Remark 3 The right-hand side of summation inequality in Corollary 5 (or Corollary 6) contains a term $\frac{5(h+1)(h+2)}{(h-2)(h-1)h}\Omega_2^T R\Omega_2$. However, the summation inequality in Theorem A or

Theorem B neglects this term. If $h > 2$ and $\Omega_2 \neq 0$, then $\frac{5(h+1)(h+2)}{(h-2)(h-1)h}\Omega_2^T R\Omega_2 > 0$. Since a positive quantity is added in the right-hand side of the inequality, the summation inequality in Corollary 5 (or Corollary 6) can provide a sharper bound for $\sum_{i=-h+1}^{0} y^T(i)Ry(i)$ than the summation inequalities in [24, 25].

As we have mentioned before, Jensen's inequality has mostly been used as a powerful mathematical tool in dealing with the difference of Lyapunov-Krasovskii functionals, single or double. In the case of a double, just like $\sum_{i=-h+1}^{0}\sum_{k=i}^{0} y(k)^T Ry(k)$, Jensen's inequality may neglect some terms, which unavoidably introduces conservatism. Then we will give some improved double summation inequalities.

Theorem 4 *For a positive definite matrix $R > 0$, any sequence of discrete-time variables $y : [-h, 0] \cap Z \to R^n$, and any nonzero sequence of discrete-time variables $p : [-h, 0] \cap Z \to R$ satisfying $\sum_{i=-h+1}^{0}\sum_{k=i}^{0} p(k) = 0$, the following inequality holds*:

$$\sum_{i=-h+1}^{0}\sum_{k=i}^{0} y(k)^T Ry(k) \geq \frac{2}{h(h+1)} E_0^T RE_0 + \frac{1}{\sum_{i=-h+1}^{0}\sum_{k=i}^{0} p(k)^2} E_1^T RE_1, \tag{32}$$

where $E_0 = \sum_{i=-h+1}^{0}\sum_{k=i}^{0} y(k)$, $E_1 = \sum_{i=-h+1}^{0}\sum_{k=i}^{0} y(k)p(k)$.

Proof Define the energy function as $J(v) = \sum_{i=-h+1}^{0}\sum_{k=i}^{0} z(k)^T Rz(k)$ and $z(i) = y(i) - \frac{2}{h(h+1)}E_0 - p(i)v$. Similar to the proof of Theorem 1, we are now proceeding to find a vector $\hat{v}$ to minimize the energy function $J(v)$.

If $\sum_{i=-h+1}^{0}\sum_{k=i}^{0} p(k)^2 > 0$ and $\sum_{i=-h+1}^{0}\sum_{k=i}^{0} p(k) = 0$, then

$$\begin{aligned} J'(v) &= -2\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\left[y(k)p(k) - \frac{2}{h(h+1)}E_0 p(k) - p(k)^2 v\right]^T R \\ &= \left[-2\sum_{i=-h+1}^{0}\sum_{k=i}^{0} y(k)p(k) + 2v\sum_{i=-h+1}^{0}\sum_{k=i}^{0} p(k)^2\right]^T R + 2\frac{2}{h(h+1)}E_0^T \sum_{i=-h+1}^{0}\sum_{k=i}^{0} p(k)R \\ &= \left[-2\sum_{i=-h+1}^{0}\sum_{k=i}^{0} y(k)p(k) + 2v\sum_{i=-h+1}^{0}\sum_{k=i}^{0} p(k)^2\right]^T R. \end{aligned} \tag{33}$$

The solution $\hat{v}$ of $J'(v) = 0$ can be found as

$$\hat{v} = \left[\sum_{i=-h+1}^{0}\sum_{k=i}^{0} p(k)^2\right]^{-1} \sum_{i=-h+1}^{0}\sum_{k=i}^{0} y(k)p(k). \tag{34}$$

In this case, we have

$$\begin{aligned} J(\hat{v}) &= \sum_{i=-h+1}^{0}\sum_{k=i}^{0}\left[y(k) - \frac{2}{h(h+1)}E_0 - p(k)\hat{v}\right]^T R\left[y(k) - \frac{2}{h(h+1)}E_0 - p(k)\hat{v}\right] \\ &= \sum_{i=-h+1}^{0}\sum_{k=i}^{0}\left[y(k) - \frac{2}{h(h+1)}E_0\right]^T R\left[y(k) - \frac{2}{h(h+1)}E_0\right] \end{aligned}$$

$$
\begin{aligned}
&- 2 \sum_{i=-h+1}^{0} \sum_{k=i}^{0} \left[y(k) - \frac{2}{h(h+1)} E_0 \right]^T R p(k) \hat{\nu} + \sum_{i=-h+1}^{0} \sum_{k=i}^{0} p(k)^2 \hat{\nu}^T R \hat{\nu} \\
&= \sum_{i=-h+1}^{0} \sum_{k=i}^{0} \left[y(k) - \frac{2}{h(h+1)} E_0 \right]^T R \left[y(k) - \frac{2}{h(h+1)} E_0 \right] \\
&\quad + \sum_{i=-h+1}^{0} \sum_{k=i}^{0} p(k)^2 \hat{\nu}^T R \hat{\nu} - 2 \sum_{i=-h+1}^{0} \sum_{k=i}^{0} y(k) R p(k) \hat{\nu} \\
&\quad + 2 \sum_{i=-h+1}^{0} \sum_{k=i}^{0} \frac{2}{h(h+1)} E_0^T R p(k) \hat{\nu} \\
&= \sum_{i=-h+1}^{0} \sum_{k=i}^{0} \left[y(k) - \frac{2}{h(h+1)} E_0 \right]^T R \left[y(k) - \frac{2}{h(h+1)} E_0 \right] \\
&\quad - 2 \sum_{i=-h+1}^{0} \sum_{k=i}^{0} p(k)^2 \hat{\nu}^T R \hat{\nu} + \sum_{i=-h+1}^{0} \sum_{k=i}^{0} p(k)^2 \hat{\nu}^T R \hat{\nu} \\
&= \sum_{i=-h+1}^{0} \sum_{k=i}^{0} \left[y(k) - \frac{2}{h(h+1)} E_0 \right]^T R \left[y(k) - \frac{2}{h(h+1)} E_0 \right] \\
&\quad - \sum_{i=-h+1}^{0} \sum_{k=i}^{0} p(k)^2 \hat{\nu}^T R \hat{\nu} \\
&= \sum_{i=-h+1}^{0} \sum_{k=i}^{0} y(k)^T R y(k) - 2 \sum_{i=-h+1}^{0} \sum_{k=i}^{0} y(k)^T R \frac{2}{h(h+1)} E_0 \\
&\quad + \frac{4}{h^2(h+1)^2} \sum_{i=-h+1}^{0} \sum_{k=i}^{0} E_0^T R E_0 - \sum_{i=-h+1}^{0} \sum_{k=i}^{0} p(k)^2 \hat{\nu}^T R \hat{\nu} \\
&= \sum_{i=-h+1}^{0} \sum_{k=i}^{0} y(k)^T R y(k) - 2 E_0^T R \frac{2}{h(h+1)} E_0 \\
&\quad + \frac{4}{h^2(h+1)^2} \frac{h(h+1)}{2} E_0^T R E_0 - \sum_{i=-h+1}^{0} \sum_{k=i}^{0} p(k)^2 \hat{\nu}^T R \hat{\nu} \\
&\geq 0.
\end{aligned}
\tag{35}
$$

This completes the proof of Theorem 4. □

Specially, the choice of $p(k)$ in Theorem 4 as $p_3(k) = 3k + h - 1$ satisfying

$$
\sum_{i=-h+1}^{0} \sum_{k=i}^{0} p_3(k) = 0
$$

yields

$$
\sum_{i=-h+1}^{0} \sum_{k=i}^{0} p_3(k)^2 = \frac{(h-1)h(h+1)(h+2)}{4}
\tag{36}
$$

and

$$\sum_{i=-h+1}^{0}\sum_{k=i}^{0}p_3(k)y(k) = -2(h+2)\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k) + 6\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\sum_{m=k}^{0}y(m). \tag{37}$$

Let $\Omega_3 = \sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k) - \frac{3}{h+2}\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\sum_{m=k}^{0}y(m)$. Then the following inequality based on Theorem 4 holds:

$$\begin{aligned}
&\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)^TRy(k)\\
&\quad\geq \frac{2}{h(h+1)}\left(\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)\right)^T R\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)\\
&\qquad+\frac{1}{\sum_{i=-h+1}^{0}\sum_{k=i}^{0}p_3(k)^2}\left[\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)p_3(k)\right]^T R\left[\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)p_3(k)\right]\\
&\quad=\frac{2}{h(h+1)}\left(\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)\right)^T R\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)\\
&\qquad+\frac{4}{(h-1)h(h+1)(h+2)}\left[\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)p_3(k)\right]^T R\left[\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)p_3(k)\right]\\
&\quad=\frac{2}{h(h+1)}\left(\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)\right)^T R\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)+\frac{16(h+2)}{(h-1)h(h+1)}\Omega_3^TR\Omega_3.
\end{aligned} \tag{38}$$

Furthermore, we have the following corollary.

Corollary 7 *For a positive definite matrix $R > 0$ and any sequence of discrete-time variables $y : [-h, 0] \cap Z \to R^n$, the following inequality holds*:

$$\begin{aligned}
&\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)^TRy(k)\\
&\quad\geq \frac{2}{h(h+1)}\left(\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)\right)^T R\sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k)+\frac{16(h+2)}{(h-1)h(h+1)}\Omega_3^TR\Omega_3,
\end{aligned} \tag{39}$$

where $\Omega_3 = \sum_{i=-h+1}^{0}\sum_{k=i}^{0}y(k) - \frac{3}{h+2}\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\sum_{m=k}^{0}y(m)$.

Corollary 8 *For a positive definite matrix $R > 0$ and any sequence of discrete-time variables $y : [-h, 0] \cap Z \to R^n$, the following inequality holds*:

$$\begin{aligned}
&\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\Delta x(k)^TR\Delta x(k)\\
&\quad\geq \frac{2(h+1)}{h}\left[x(0)-\frac{1}{(h+1)}\sum_{i=-h}^{0}x(i)\right]^T R\left[x(0)-\frac{1}{(h+1)}\sum_{i=-h}^{0}x(i)\right]\\
&\qquad+\frac{4(h+1)(h+2)}{h(h-1)}\Omega_4^TR\Omega_4,
\end{aligned} \tag{40}$$

where $\Omega_4 = [x(0) + \frac{2}{(h+1)}\sum_{i=-h}^{0}x(i) - \frac{6}{(h+1)(h+2)}\sum_{i=-h}^{0}\sum_{k=i}^{0}x(k)]$.

Proof Let $y(i) = \Delta x(i) = x(i) - x(i-1)$ in Corollary 7, we have

$$\begin{aligned}
&\frac{2}{h(h+1)}\left(\sum_{i=-h+1}^{0}\sum_{k=i}^{0} y(k)\right)^T R \sum_{i=-h+1}^{0}\sum_{k=i}^{0} y(k) \\
&= \frac{2}{h(h+1)}\left(\sum_{i=-h+1}^{0}\left[x(0)-x(i-1)\right]\right)^T R \sum_{i=-h+1}^{0}\left[x(0)-x(i-1)\right] \\
&= \frac{2}{h(h+1)}\left[hx(0) - \sum_{i=-h+1}^{0} x(i-1)\right]^T R \left[hx(0) - \sum_{i=-h+1}^{0} x(i-1)\right] \\
&= \frac{2}{h(h+1)}\left[(h+1)x(0) - \sum_{i=-h}^{0} x(i)\right]^T R \left[(h+1)x(0) - \sum_{i=-h}^{0} x(i)\right] \\
&= \frac{2(h+1)}{h}\left[x(0) - \frac{1}{(h+1)}\sum_{i=-h}^{0} x(i)\right]^T R \left[x(0) - \frac{1}{(h+1)}\sum_{i=-h}^{0} x(i)\right] \qquad (41)
\end{aligned}$$

and

$$\begin{aligned}
\Omega_3 &= \sum_{i=-h+1}^{0}\sum_{k=i}^{0} y(k) - \frac{3}{h+2}\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\sum_{m=k}^{0} y(m) \\
&= (h+1)x(0) - \sum_{i=-h}^{0} x(i) - \frac{3}{h+2}\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\sum_{m=k}^{0}\left[x(m)-x(m-1)\right] \\
&= (h+1)x(0) - \sum_{i=-h}^{0} x(i) - \frac{3}{h+2}\sum_{i=-h+1}^{0}\sum_{k=i}^{0}\left[x(0)-x(k-1)\right] \\
&= (h+1)x(0) - \sum_{i=-h}^{0} x(i) - \frac{3}{h+2}\sum_{i=-h+1}^{0}\left[(-i+1)x(0) - \sum_{k=i}^{0} x(k-1)\right] \\
&= (h+1)x(0) - \sum_{i=-h}^{0} x(i) - \frac{3}{h+2}\frac{h(h+1)}{2}x(0) + \frac{3}{h+2}\sum_{i=-h+1}^{0}\sum_{k=i-1}^{-1} x(k) \\
&= (h+1)x(0) - \sum_{i=-h}^{0} x(i) - \frac{3}{h+2}\frac{h(h+1)}{2}x(0) \\
&\quad + \frac{3}{h+2}\sum_{i=-h+1}^{0}\left[-x(0) + x(i-1) + \sum_{k=i}^{0} x(k)\right] \\
&= (h+1)x(0) - \sum_{i=-h}^{0} x(i) - \frac{3h(h+1)}{2(h+2)}x(0) \\
&\quad - \frac{3h}{h+2}x(0) + \frac{3}{h+2}\sum_{i=-h+1}^{0} x(i-1) + \frac{3}{h+2}\sum_{i=-h+1}^{0}\sum_{k=i}^{0} x(k) \\
&= (h+1)x(0) - \sum_{i=-h}^{0} x(i) - \frac{3h(h+1)}{2(h+2)}x(0) \\
&\quad - \frac{3h}{h+2}x(0) - \frac{3}{h+2}x(0) + \frac{3}{h+2}\sum_{i=-h}^{0}\sum_{k=i}^{0} x(k). \qquad (42)
\end{aligned}$$

So

$$\begin{aligned}
\Omega_3 &= (h+1)x(0) - \sum_{i=-h}^{0} x(i) - \frac{3h(h+1)}{2(h+2)}x(0) \\
&\quad - \frac{3(h+1)}{h+2}x(0) + \frac{3}{h+2}\sum_{i=-h}^{0}\sum_{k=i}^{0} x(k) \\
&= -\frac{(h+1)}{2}x(0) - \sum_{i=-h}^{0} x(i) + \frac{3}{h+2}\sum_{i=-h}^{0}\sum_{k=i}^{0} x(k) \\
&= -\frac{(h+1)}{2}\left[x(0) + \frac{2}{(h+1)}\sum_{i=-h}^{0} x(i) - \frac{6}{(h+1)(h+2)}\sum_{i=-h}^{0}\sum_{k=i}^{0} x(k)\right].
\end{aligned} \tag{43}$$

Replacing $y(i)$ by $\Delta x(i)$ in Corollary 7 leads to

$$\begin{aligned}
&\sum_{i=-h+1}^{0}\sum_{k=i}^{0} \Delta x(k)^T R \Delta x(k) \\
&\quad \geq \frac{2}{h(h+1)}\left(\sum_{i=-h+1}^{0}\sum_{k=i}^{0} \Delta x(k)\right)^T R \sum_{i=-h+1}^{0}\sum_{k=i}^{0} \Delta x(k) \\
&\qquad + \frac{16(h+2)}{h(h+1)(h-1)}\Omega_3^T R \Omega_3 \\
&\quad = \frac{2(h+1)}{h}\left[x(0) - \frac{1}{(h+1)}\sum_{i=-h}^{0} x(i)\right]^T R \left[x(0) - \frac{1}{(h+1)}\sum_{i=-h}^{0} x(i)\right] \\
&\qquad + \frac{4(h+1)(h+2)}{h(h-1)}\left[x(0) + \frac{2}{(h+1)}\sum_{i=-h}^{0} x(i) - \frac{6}{(h+1)(h+2)}\sum_{i=-h}^{0}\sum_{k=i}^{0} x(k)\right]^T R \\
&\qquad \times \left[x(0) + \frac{2}{(h+1)}\sum_{i=-h}^{0} x(i) - \frac{6}{(h+1)(h+2)}\sum_{i=-h}^{0}\sum_{k=i}^{0} x(k)\right] \\
&\quad = \frac{2(h+1)}{h}\left[x(0) - \frac{1}{(h+1)}\sum_{i=-h}^{0} x(i)\right]^T R \left[x(0) - \frac{1}{(h+1)}\sum_{i=-h}^{0} x(i)\right] \\
&\qquad + \frac{4(h+1)(h+2)}{h(h-1)}\Omega_4^T R \Omega_4,
\end{aligned} \tag{44}$$

where $\Omega_4 = [x(0) + \frac{2}{(h+1)}\sum_{i=-h}^{0} x(i) - \frac{6}{(h+1)(h+2)}\sum_{i=-h}^{0}\sum_{k=i}^{0} x(k)]$.

This completes the proof of Corollary 8. □

Remark 4 The double Jensen inequality is often used to estimate a upper bound of $-\sum_{i=-h+1}^{0}\sum_{k=i}^{0} y(k)^T R y(k)$ in the difference of Lyapunov-Krasovskii functionals. In this paper, we have extended the double Jensen inequality. Some improved double summation inequalities are presented in Corollary 7 (or Corollary 8). Since these improved double summation inequalities contain $\frac{16(h+2)}{(h-1)h(h+1)}\Omega_3^T R \Omega_3$, they can provide a tighter bound for $\sum_{i=-h+1}^{0}\sum_{k=i}^{0} y(k)^T R y(k)$. Therefore, these improved double summation inequalities can be used to establish less conservative stability conditions for the discrete-time systems with variable delays.

3 Application in stability analysis

In this section, we will consider the following linear discrete system with time-varying delay:

$$\begin{cases} x(k+1) = Ax(k) + Bx(k-h(k)), & k \geq 0, \\ x(k) = \varphi(k), & k \in [-h_2, 0], \end{cases} \tag{45}$$

where $x(k) \in R^n$ is the state vector, φ is the initial value, A and B are $n \times n$ constant matrices. The delay $h(k)$ is assumed to be a positive integer-valued function, for some integers $h_2 \geq h_1 > 1$, $h(k) \in [h_1, h_2]$, $\forall k \geq 0$.

Based on the above summation inequalities, we will establish a new criterion on asymptotical stability for system (45).

First, the following notations are needed:

$$\begin{aligned}
&h_{12} = h_2 - h_1, \\
&e_i = [\underbrace{0, 0, \ldots, \overbrace{I}^{i}, \ldots, 0}_{8}]^T_{8n\times n}, \quad i = 1, 2, \ldots, 8, \\
&y(k) = x(k) - x(k-1), \\
&\xi(k) = \Bigg[x^T(k), x^T(k-h_1), x^T\big(k-h(k)\big), x^T(k-h_2), \\
&\qquad \frac{1}{h_1+1}\sum_{i=k-h_1}^{k} x^T(i), \frac{1}{h(k)-h_1+1}\sum_{i=k-h(k)}^{k-h_1} x^T(i), \\
&\qquad \frac{1}{h_2-h(k)+1}\sum_{i=k-h_2}^{k-h(k)} x^T(i), \sum_{i=-h_1+1}^{0}\sum_{j=k+i}^{k} x^T(j) \Bigg]^T, \\
&\alpha(k) = \Bigg[x^T(k), \sum_{i=k-h_1}^{k-1} x^T(i), \sum_{i=k-h_2}^{k-h_1-1} x^T(i), \sum_{i=-h_1+1}^{0}\sum_{j=k+i}^{k} x^T(j) \Bigg]^T, \\
&Z_{10} = \operatorname{diag}\left\{ Z_1, \frac{3(h_1+1)}{h_1-1} Z_1, \frac{5(h_1+1)(h_1+2)}{(h_1-2)(h_1-1)} Z_1 \right\}, \\
&Z_2^* = \begin{pmatrix} Z_2 & 0 \\ 0 & 3Z_2 \end{pmatrix}, \qquad Z_{20} = \begin{pmatrix} Z_2^* & X \\ * & Z_2^* \end{pmatrix}, \\
&\Pi_0 = [A, 0, B, 0, 0, 0, 0, 0]^T, \\
&\Pi_1 = \big[\Pi_0, (h_1+1)e_5 - e_2, \big(h(k)-h_1+1\big)e_6 + \big(h_2-h(k)+1\big)e_7 - e_3 - e_4, \\
&\qquad e_8 + h_1\Pi_0 - (h_1+1)e_5 + e_2\big], \\
&\Pi_2 = \big[e_1, (h_1+1)e_5 - e_1, \big(h(k)-h_1+1\big)e_6 + \big(h_2-h(k)+1\big)e_7 - e_3 - e_2, e_8\big], \\
&\Pi_3 = [A - I, 0, B, 0, 0, 0, 0, 0]^T, \\
&\Pi_4 = \left[e_1 - e_2, e_1 + e_2 - 2e_5, e_1 - e_2 + \frac{6h_1}{h_1+2} e_5 - \frac{12}{(h_1+1)(h_1+2)} e_8 \right], \\
&\Pi_5 = [e_3 - e_4, e_3 + e_4 - 2e_7, e_2 - e_3, e_2 + e_3 - 2e_6],
\end{aligned} \tag{46}$$

$$
\begin{aligned}
&\Pi_6 = e_1 - e_5,\\
&\Pi_7 = e_1 + \left(2 - \frac{6}{(h_1+2)}\right)e_5 - \frac{6}{(h_1+1)(h_1+2)}e_8,\\
&\Xi_1 = \Pi_1 P \Pi_1^T - \Pi_2 P \Pi_2^T,\\
&\Xi_2 = e_1 Q_1 e_1^T - e_2 Q_1 e_2^T + e_2 Q_2 e_2^T - e_4 Q_2 e_4^T,\\
&\Xi_3 = \Pi_3\bigl(h_1^2 Z_1 + h_{12}^2 Z_2\bigr)\Pi_3^T - \Pi_4 Z_{10}\Pi_4^T - \Pi_5 Z_{20}\Pi_5^T,\\
&\Xi_4 = \frac{h_1(h_1+1)}{2}\Pi_3 Z_3 \Pi_3^T - \frac{2(h_1+1)}{h_1}\Pi_6 Z_3 \Pi_6^T - \frac{4(h_1^2-1)}{h_1(h_1+2)}\Pi_7 Z_3 \Pi_7^T,\\
&\Xi = \sum_{i=1}^{4}\Xi_i.
\end{aligned}
$$

Theorem 5 *For given integers h_1, h_2 satisfying $1 < h_1 \le h_2$, system* (45) *is asymptotically stable for $h_1 \le h(k) \le h_2$, if there are positive define matrices $P \in R^{4n\times 4n}$, $Z_1 \in R^{n\times n}$, $Z_2 \in R^{n\times n}$, $Z_3 \in R^{n\times n}$, $Q_1 \in R^{n\times n}$, $Q_2 \in R^{n\times n}$, and any matrix $X \in R^{2n\times 2n}$ such that the following LMIs are satisfied*:

$$
\Xi < 0, \qquad Z_{20} \ge 0. \tag{47}
$$

Proof Choose a Lyapunov functional candidate as follows:

$$
V(k) = \sum_{j=1}^{4} V_j(k), \tag{48}
$$

where

$$
\begin{aligned}
&V_1(k) = \alpha^T(k) P \alpha(k),\\
&V_2(k) = \sum_{i=k-h_1}^{k-1} x^T(i) Q_1 x(i) + \sum_{i=k-h_2}^{k-h_1-1} x^T(i) Q_2 x(i),\\
&V_3(k) = h_1 \sum_{i=-h_1+1}^{0}\sum_{j=k+i}^{k} y^T(j) Z_1 y(j) + h_{12}\sum_{i=-h_2+1}^{-h_1}\sum_{j=k+i}^{k} y^T(j) Z_2 y(j),\\
&V_4(k) = \sum_{i=-h_1+1}^{0}\sum_{j=i}^{0}\sum_{u=k+j}^{k} y^T(u) Z_3 y(u).
\end{aligned} \tag{49}
$$

Next, we calculate the difference of $V(k)$. For $V_1(k)$ and $V_2(k)$, we have

$$
\Delta V_1(k) = \xi^T(k)\Xi_1\xi(k) \tag{50}
$$

and

$$
\Delta V_2(k) = \xi^T(k)\Xi_2\xi(k). \tag{51}
$$

Calculating $\Delta V_3(k)$ gives

$$\begin{aligned}\Delta V_3(k) &= h_1^2 y_{k+1}^T Z_1 y_{k+1} + h_{12}^2 y_{k+1}^T Z_2 y_{k+1} \\ &\quad - h_1 \sum_{i=k-h_1+1}^{k} y^T(i) Z_1 y(i) - h_{12} \sum_{i=k-h_2+1}^{k-h_1} y^T(i) Z_2 y(i). \end{aligned} \tag{52}$$

By Corollary 6, we get

$$-h_1 \sum_{i=k-h_1+1}^{k} y^T(i) Z_1 y(i) \leq -\xi^T(k) \Pi_4 Z_{10} \Pi_4^T \xi(k). \tag{53}$$

Under the condition of $Z_{20} > 0$, by Corollary 6 and the lower bounded lemma, we get

$$-h_{12} \sum_{i=k-h_2+1}^{k-h_1} y^T(i) Z_2 y(i) \leq -\xi^T(k) \Pi_5 Z_{20} \Pi_5^T \xi(k). \tag{54}$$

Then we have

$$\Delta V_3(k) \leq \xi^T(k) \Xi_3 \xi(k). \tag{55}$$

Calculating $\Delta V_4(k)$ gives

$$\Delta V_4(k) = \frac{h_1(h_1+1)}{2} y_{k+1}^T Z_3 y_{k+1} - \sum_{i=-h_1+1}^{0} \sum_{j=k+i}^{k} y^T(j) Z_3 y(j). \tag{56}$$

By Corollary 8, we have

$$\begin{aligned} &- \sum_{i=-h_1+1}^{0} \sum_{j=k+i}^{k} y^T(j) Z_3 y(j) \\ &\quad \leq \xi^T(k) \left(-\frac{2(h_1+1)}{h_1} \Pi_6 Z_3 \Pi_6^T - \frac{4(h_1+1)(h_1+2)}{h_1(h_1-1)} \Pi_7 Z_3 \Pi_7^T \right) \xi(k). \end{aligned} \tag{57}$$

Then we have

$$\Delta V_4(k) \leq \xi^T(k) \Xi_4 \xi(k). \tag{58}$$

Hence

$$\Delta V(k) \leq \xi^T(k) \sum_{i=1}^{4} \Xi_i \xi(k) = \xi^T(k) \Xi \xi(k). \tag{59}$$

If $\Xi < 0$, then $\Delta V(k) < 0$.

This completes the proof of Theorem 5. □

Table 1 Maximum bound h_2 with different h_1 (Example 1)

Method	3	5	7	11	13
[18]	10	11	12	13	13
[19]	13	14	15	17	19
[20]	17	17	18	20	22
[21]	17	18	18	20	23
[22]	18	19	21	25	25
[23]	22	22	22	23	24
[25]	21	21	22	23	24
Theorem 5	21	21	22	23	24

Remark 5 Theorem 5 gives a sufficient condition for asymptotical stability criterion for discrete-time system (45) with variable delay. The free-weighting matrix method was developed and was applied to the stability analysis of systems with time-varying delays [18]. However, the computational burden will increase because of the introduction of free-weighting matrices. Different from the free-weighting matrix method, some new sharper summation inequalities are developed via auxiliary functions. By employing these improved inequalities and the reciprocally convex combination inequality method, a less conservative result is derived. The conditions in Theorem 5 are described in terms of two matrix inequalities, which can be realized by using the linear matrix inequality algorithm proposed in [28].

4 Numerical example

In this section, to demonstrate the effectiveness of our proposed method, we consider the following example, which is widely used in the delay-dependent stability analysis of discrete-time systems with time delay.

Example 1 Consider the discrete-time system

$$x(k+1) = \begin{pmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{pmatrix} x(k) + \begin{pmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{pmatrix} x\bigl(k - h(k)\bigr).$$

Since the system addressed in [24] is a discrete-time system with constant delay, the stability criterion obtained cannot be applied to this system. For different h_1, the maximum allowable upper bounds of $h(k)$ guaranteeing this system to be asymptotically stable are given in Table 1 [18–23, 25]. From Table 1, Theorem 5 in our paper can provide larger feasible region than those of [18–21]. For the same h_1, the maximum allowable upper bound of $h(k)$ obtained in this paper is the same as that in [25]. Although more decision variables are needed in our stability criterion, the new summation inequality in Corollary 6 is sharper than that in [25].

5 Conclusions

In this paper, by the construction of an appropriate auxiliary function, some new summation inequalities are established. As an application of the summation inequality, an asymptotic stability analysis of discrete linear systems with time delay is carried out. Finally, a numerical example is provided to illustrate the usefulness of the theoretical results.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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