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On hyper-order of solutions of higher order linear differential equations with meromorphic coefficients

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Abstract

In this paper, we investigate the growth of meromorphic solutions of the differential equations

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=0$$

and

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f=F(z),$$

where $A_0(z)\not\equiv 0, A_1(z),\ldots,A_{k-1}(z)$ and $F(z)\not\equiv 0$ are meromorphic functions. A precise estimation of the hyper-order of meromorphic solutions of the above equations is given provided that there exists one dominant coefficient, which improves and extends previous results given by Belaïdi, Chen, *etc.*

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1 Introduction and main results

For a meromorphic function f in the complex plane $\mathbb{C}$, the order of growth and the lower order of growth are defined as

$$\rho(f)=\limsup_{r\to\infty}\frac{\log^+T(r,f)}{\log r},\qquad \mu(f)=\liminf_{r\to\infty}\frac{\log^+T(r,f)}{\log r},$$

respectively. The exponent of convergence of the poles sequence of f is defined as

$$\lambda\left(\frac{1}{f}\right)=\limsup_{r\to\infty}\frac{\log^+N(r,f)}{\log r}.$$

For fast-growing meromorphic functions the growth is typically measured in terms of the hyper-order, defined as

$$\rho_2(f)=\limsup_{r\to\infty}\frac{\log^+\log^+T(r,f)}{\log r}.$$

Nevanlinna theory of meromorphic functions is a powerful tool in the field of complex differential equations. For an introduction to the theory of differential equations in the complex plane by using Nevanlinna theory; see, for example [1–3]. Active research in this field was started by Wittich [4] and his students in the 1950s and 1960s. The order of growth of solutions of the equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0 \tag{1.1}$$

is one of the aims in studying complex differential equations, where $A_0(z) \not\equiv 0, A_1(z), \ldots, A_k(z)$ are entire functions. For the case $k = 2$, from [5–7], we know that every nontrivial solution of the equation

$$f'' + A(z)f' + B(z)f = 0 \tag{1.2}$$

is of infinite order provided that (i) $\rho(A) < \rho(B)$; or (ii) $\rho(B) < \rho(A) \leq \frac{1}{2}$; or (iii) $A(z)$ is polynomial and $B(z)$ is transcendental with $\rho(B) = 0$. Motivated by the result above, it is obvious that every nontrivial solution of (1.1) is of infinite order provided that the coefficient $A_0(z)$ is dominant over others. For example, if $\max\{\rho(A_j), j = 1, 2, \ldots, k-1\} < \rho(A_0)$, then every nontrivial solution of (1.1) is of infinite order. Another generalized condition is found by Hellerstein *et al.* in [8].

Theorem 1.1 ([8]) *Let* $A_0(z), A_1(z), \ldots, A_{k-1}(z)$, $F(z)$ *be entire functions. Suppose there exists an integer* s, $0 \leq s \leq k-1$, *such that*

$$\max\Big\{\rho(F), \max_{\substack{0\leq j\leq k-1\\ j\neq s}} \rho(A_j)\Big\} < \rho(A_s) \leq \frac{1}{2}.$$

Then every solution of

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = F(z) \tag{1.3}$$

is either a polynomial or an entire function of infinite order.

In 2000, Chen and Yang studied the hyper-order of solutions of (1.1).

Theorem 1.2 ([9]) *Let* $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ *be entire function satisfying*

$$\max\big\{\rho(A_j), j = 1, 2, \ldots, k-1\big\} < \rho(A_0) < \infty.$$

Then every nontrivial solution f *of* (1.1) *satisfies* $\rho_2(f) = \rho(A_0)$.

In Theorems 1.1 and 1.2, the authors consider all coefficients are entire functions. When the coefficients $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ and $F(z)$ are meromorphic functions, many authors investigated the value distribution of solutions of (1.1) and (1.3); see, for example, [10–18]. Especially, we mention the following result given by Chen [14], in which a precise estimation of hyper-order of solutions of (1.1) is obtained.

Theorem 1.3 ([14], Theorem 2) *Let $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ be meromorphic functions. Suppose there exists an integer s, $0 \leq s \leq k-1$, satisfying*

$$\max\left\{\rho(A_j), j \neq s, \lambda\left(\frac{1}{A_s}\right)\right\} < \mu(A_s) \leq \rho(A_s) < \frac{1}{2}.$$

If equation (1.1) *has a meromorphic solution, then every transcendental meromorphic solution f of* (1.1) *satisfies $\rho_2(f) = \rho(A_s)$.*

In 2005, Xiao and Chen considered the non-homogeneous equation (1.3), the following result is proved.

Theorem 1.4 ([19]) *Let $A_0(z), A_1(z), \ldots, A_{k-1}(z), F(z)$ be meromorphic functions. Suppose there exists an integer s, $0 \leq s \leq k-1$, satisfying*

$$\max\left\{\rho(A_j), j \neq s, \lambda\left(\frac{1}{A_s}\right), \rho(F)\right\} < \mu(A_s) \leq \rho(A_s) < \frac{1}{2}.$$

If equation (1.3) *has a meromorphic solution, then every transcendental meromorphic solution f of* (1.3) *satisfies $\rho_2(f) = \rho(A_s)$.*

Belaïdi studied equation (1.1), a precise estimation of hyper-order of solutions of (1.1) is also obtained by using different conditions from those mentioned above, in which the growths of the coefficients are limited in a set having positive densities.

Theorem 1.5 ([20]) *Let E be a set of complex numbers satisfying $\overline{\text{dens}}(\{|z| : z \in E\}) > 0$, and let $A_j(z)$, $j = 0, 1, \ldots, k-1$, be entire functions such that*

$$\max\{\rho(A_j), j = 1, \ldots, k-1\} \leq \rho(A_0) = \rho < \infty;$$

and for some constants $0 \leq \beta < \alpha$, we have, for all $\varepsilon > 0$ sufficiently small,

$$|A_0(z)| \geq \exp(\alpha |z|^{\rho-\varepsilon}), \qquad |A_j(z)| \leq \exp(\beta |z|^{\rho-\varepsilon}), \quad j = 1, 2, \ldots, k-1,$$

as $z \to \infty$ for $z \in E$. Then every nontrivial solution f of (1.1) *satisfies $\rho_2(f) = \rho(A_0)$.*

Theorem 1.5 and the remaining theorems involve the logarithmic measure and densities of set, which will be recalled in Section 2. In this paper, we study the growth of solutions of (1.1) and (1.3), and one of the goals is to extend Theorems 1.3 and 1.4 in which the condition $\rho(A_s) < \frac{1}{2}$ is deleted. On the other hand, we consider the case of a meromorphic coefficient in Theorem 1.5. The following results are proved by combining the methods of Theorems 1.3, 1.4, and 1.5.

Theorem 1.6 *Let E be a set of complex numbers satisfying $\mathrm{m_l}(\{|z| : z \in E\}) = \infty$, and let $A_j(z)$, $j = 0, 1, \ldots, k-1$, be meromorphic functions. Suppose there exists an integer s, $0 \leq s \leq k-1$, satisfying*

$$\max\left\{\rho(A_j), j \neq s, \lambda\left(\frac{1}{A_s}\right)\right\} < \mu(A_s) \leq \rho(A_s) = \rho < \infty;$$

and for some constants $0 \le \beta < \alpha$, *we have, for all* $\varepsilon > 0$ *sufficiently small,*

$$\left|A_j(z)\right| \le \exp\bigl(\beta |z|^{\rho-\varepsilon}\bigr), \quad j \ne s, \tag{1.4}$$

$$\left|A_s(z)\right| \ge \exp\bigl(\alpha |z|^{\rho-\varepsilon}\bigr), \tag{1.5}$$

as $z \to \infty$ *for* $z \in E$. *Then every nontrivial meromorphic solution* f *whose poles are of uniformly bounded multiplicities of equation* (1.1) *satisfies* $\rho_2(f) = \rho(A_s)$.

For the case of non-homogeneous equation, we get the following result.

Theorem 1.7 *Let* E *and* $A_j(z), j = 0, 1, \ldots, k-1$ *be defined as Theorem* 1.6, *and let* $F(z) \not\equiv 0$ *be meromorphic function. Suppose there exists an integer* s, $0 \le s \le k-1$, *satisfying*

$$\max\left\{\rho(A_j), j \ne s, \lambda\left(\frac{1}{A_s}\right), \rho(F)\right\} < \mu(A_s) \le \rho(A_s) = \rho < \infty;$$

and for some constants $0 \le \beta < \alpha$, *we have, for all* $\varepsilon > 0$ *sufficiently small, equations* (1.4) *and* (1.5) *hold as* $z \to \infty$ *for* $z \in E$. *Then every nontrivial meromorphic solution* f *whose poles are of uniformly bounded multiplicities of equation* (1.3) *satisfies* $\rho_2(f) = \rho(A_s)$.

Remark 1.8 From [12], Remark 3.1, we know that the condition that the multiplicity of poles of the meromorphic solution f is uniformly bounded is necessary. Hence the condition was missing in Theorems 1.3 and 1.4. Of course, the condition could also be changed by $\delta(\infty, f) > 0$.

2 Auxiliary results

The Lebesgue linear measure of a set $E \subset [0, \infty)$ is $\mathrm{m}(E) = \int_E dt$, and the logarithmic measure of a set $F \subset [1, \infty)$ is $\mathrm{m_l}(F) = \int_F \frac{dt}{t}$. The upper and lower densities of $E \subset [0, \infty)$ are given by

$$\overline{\mathrm{dens}}(E) = \limsup_{r \to \infty} \frac{\mathrm{m}(E \cap [0, r])}{r}$$

and

$$\underline{\mathrm{dens}}(E) = \liminf_{r \to \infty} \frac{\mathrm{m}(E \cap [0, r])}{r},$$

respectively.

A lemma on logarithmic derivatives due to Gundersen [21] plays an important role in proving our results.

Lemma 2.1 *Let* f *be a transcendental meromorphic function, and let* $\alpha > 1$ *be a given real constant. Let* k *and* j *be two integers such that* $k > j \ge 0$. *Then there exists a set* $E \subset (1, \infty)$ *with* $\mathrm{m_l}(E) < \infty$, *and a constant* $B > 0$ *depending only on* α *and* j, k, *such that, for all* z *satisfying* $|z| \notin (E \cup [0, 1])$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le B\left(\frac{T(\alpha r, f)}{r} \log^{\alpha} r \log T(\alpha r, f)\right)^{k-j}.$$

The following result was proved originally in [22]; see also [14], Lemma 3.

Lemma 2.2 *Let f be a meromorphic function of order* $\rho(f) = \beta < \infty$. *Then, for any given* $\varepsilon > 0$, *there exists a set* $E \subset (1,\infty)$ *with* $\mathrm{m_l}(E) < \infty$ *and* $\mathrm{m}(E) < \infty$, *such that, for all z satisfying* $|z| = r \notin ([0,1] \cup E)$,

$$\bigl|f(z)\bigr| \le \exp\bigl(r^{\beta+\varepsilon}\bigr).$$

The next lemma is related to the central index.

Lemma 2.3 ([14], Lemma 2) *Let* $f(z) = \frac{g(z)}{d(z)}$ *be a meromorphic function, where* $g(z)$ *and* $d(z)$ *are entire functions satisfying*

$$\mu(g) = \mu(f) = \mu \le \rho(g) = \rho(f) \le \infty, \qquad \lambda(d) = \rho(d) = \lambda\left(\frac{1}{f}\right) = \beta < \mu.$$

Then there exists a set $E \subset (1,\infty)$ *with* $\mathrm{m_l}(E) < \infty$, *such that, for all z satisfying* $|z| = r \notin ([0,1] \cup E)$ *and* $|g(z)| = M(r,g)$, $M(r,g) = \max_{|z|=r} |g(z)|$, *we have*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^n \bigl(1 + o(1)\bigr), \quad n \ge 1,$$

where $\nu_g(r)$ *denotes the central index of* $g(z)$.

Lemma 2.4 *Let* $f(z) = \frac{g(z)}{d(z)}$ *be a meromorphic function, where* $g(z)$ *and* $d(z)$ *are entire functions. If* $0 \le \rho(d) < \mu(f)$, *then* $\mu(g) = \mu(f)$, $\rho(g) = \rho(f)$. *Moreover, if* $\rho(f) = \infty$, *then* $\rho_2(f) = \rho_2(g)$.

Proof We divide the proof into the following three cases.

Case 1. $\rho(f) < \infty$. Since $T(r,f) \le T(r,g) + T(r,d) + O(1)$, for any given $\varepsilon \in (0, \frac{\rho(f)-\rho(d)}{2})$, there exists an increasing sequence (r_n) with $r_n \to \infty$ as $n \to \infty$, such that

$$r_n^{\rho(f)-\varepsilon} \le T(r_n,f), \qquad T(r_n,d) \le r_n^{\rho(d)+\varepsilon}$$

hold for all sufficiently large n. This implies that, for all sufficiently large n,

$$r_n^{\rho(f)-\varepsilon} \le T(r_n,g) + r_n^{\rho(d)+\varepsilon} + O(1).$$

Thus, for all sufficiently large n,

$$r_n^{\rho(f)-\varepsilon}\bigl(1 - o(1)\bigr) \le T(r_n,g) + O(1).$$

Hence, $\rho(f) \le \rho(g)$. On the other hand, since $T(r,g) \le T(r,f) + T(r,d)$, and $\rho(d) < \rho(f)$, we get

$$\rho(g) \le \rho(f).$$

Therefore, we get $\rho(g) = \rho(f)$.

In a similar way to above and the definition of the lower order of growth, we can prove

$$\mu(g) = \mu(f).$$

Case 2. $\mu(f) = \infty$. Suppose on the contrary to the assertion that $\mu(g) < \mu(f)$. We aim for a contradiction. By the definition of the lower order of growth, there exist a constant $M > 0$ and an increasing sequence (r_n) with $r_n \to \infty$ as $n \to \infty$, such that

$$T(r_n, g) < r_n^M, \qquad T(r_n, d) \le r_n^M$$

hold for all sufficiently large n. Then, for all sufficiently large n,

$$T(r_n, f) \le 2r_n^M + O(1).$$

Therefore, $\mu(f) \le M$. This is a contradiction with our assumption.

Case 3. $\mu(f) < \infty$ and $\rho(f) = \infty$. In a similar way to proving cases 1 and 2, we can prove case 3.

Finally, we will prove $\rho_2(f) = \rho_2(g)$. Suppose that $\rho(f) = \infty$. Then there exists an increasing sequence (r_n) with $r_n \to \infty$ as $n \to \infty$, such that

$$\rho_2(f) = \lim_{n\to\infty} \frac{\log\log T(r_n, f)}{\log r_n}.$$

Combining $\rho(d) < \mu(f)$ and the definitions of the order and the lower order, we get

$$\lim_{n\to\infty} \frac{T(r_n, d)}{T(r_n, f)} = 0.$$

Then there exists a positive integer N, such that, for $n > N$,

$$T(r_n, f) \le 2T(r_n, g) + O(1).$$

Hence, $\rho_2(f) \le \rho_2(g)$. In a similar way to proving case 1, we get $\rho_2(f) \ge \rho_2(g)$. Therefore, $\rho_2(f) = \rho_2(g)$. □

Lemma 2.5 ([23], Lemma 2) *Let f be an entire function of infinite order, with the hyper-order $\rho_2(f) < \infty$, and let $\nu(r)$ be the central index of f. Then*

$$\limsup_{r\to\infty} \frac{\log^+ \log^+ \nu(r)}{\log r} = \rho_2(f).$$

Lemma 2.6 ([5], Lemma 5) *Let $g : [0, \infty) \to \mathbf{R}$ and $h : [0, \infty) \to \mathbf{R}$ be monotonically non-decreasing functions such that $g(r) \le h(r)$ outside of an exceptional set E with $\mathrm{m_l}(E) < \infty$. Then, for any $\alpha > 1$, there exists an $r_0 > 1$ such that $g(r) \le h(\alpha r)$ for all $r > r_0$.*

Lemma 2.7 *Let f be a meromorphic solution of equation* (1.1), *where $A_0(z), A_1(z), \ldots, A_{k-1}(z)$ are meromorphic functions. If there exists an integer number $s \in \{0, 1, 2, \ldots, k-1\}$*

that satisfies

$$\max\{\rho(A_j), j \neq s\} < \mu(A_s),$$

then $\rho(f) \geq \rho(A_s)$, $\mu(f) \geq \mu(A_s)$.

Proof By equation (1.1),

$$-A_s = \frac{f^{(k)}}{f^{(s)}} + A_{k-1}\frac{f^{(k-1)}}{f^{(s)}} + \cdots + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}} + A_{s-1}\frac{f^{(s-1)}}{f^{(s)}} + \cdots + A_0\frac{f}{f^{(s)}}.$$

Combining the formula above and the first main theory in Nevanlinna theory, we get

$$T(r,A_s) \leq \sum_{j\neq s}^{k} T(r,f^{(j)}) + \sum_{j\neq s}^{k-1} T(r,A_j) + kT(r,f^{(s)}) + O(1).$$

In a similar way to proving $T(r,f') \leq 2T(r,f) + m(r,\frac{f'}{f})$, see [24], p.97, for every integer $j \in [1,k]$,

$$T(r,f^{(j)}) \leq m\left(r,\frac{f^{(j)}}{f^{(j-1)}}\right) + 2m\left(r,\frac{f^{(j-1)}}{f^{(j-2)}}\right) + \cdots + 2^{j-1}m\left(r,\frac{f'}{f}\right) + 2^{j}T(r,f).$$

Combining the two inequalities above,

$$T(r,A_s) \leq \sum_{j\neq s}^{k-1} T(r,A_j) + c_1T(r,f) + c_2\sum_{j=1}^{k} m\left(r,\frac{f^{(j)}}{f^{(j-1)}}\right) + O(1), \tag{2.1}$$

where c_1, c_2 are positive constant.

It follows from $\rho(A_j) < \rho(A_s)$, $j \neq s$, and the lemma on the logarithmic derivative that

$$\rho(A_s) \leq \rho(f).$$

Set $b = \max\{\rho(A_j), j \neq s\}$. Then for any given $\varepsilon \in (0, \frac{\mu(A_s)-b}{2})$, there exists a constant $R > 1$, such that, for all $r > R$,

$$T(r,A_s) \geq r^{\mu(A_s)-\varepsilon}, \qquad T(r,A_j) \leq r^{b+\varepsilon}, \quad j \neq s.$$

Combining the two inequalities above and (2.1), we get

$$r^{\mu(A_s)-\varepsilon} \leq c_1T(r,f) + c_2\sum_{j=1}^{k} m\left(r,\frac{f^{(j)}}{f^{(j-1)}}\right) + (k-1)r^{b+\varepsilon} + O(1).$$

By using the lemma on logarithmic derivative again, we get $\mu(f) \geq \mu(A_s)$. □

Lemma 2.8 *Let f be a meromorphic solution of equation* (1.3), *where* $A_0(z), A_1(z), \ldots,$ $A_{k-1}(z), F(z) \not\equiv 0$ *are meromorphic functions. If there exists an integer number* $s \in \{0, 1, 2,$

$\dots, k-1\}$ *that satisfies*

$$\max\{\rho(F), \rho(A_j), j \neq s\} < \mu(A_s),$$

then $\rho(f) \geq \rho(A_s)$, $\mu(f) \geq \mu(A_s)$.

Proof In a similar way to proving Lemma 2.7, we can get the proof of Lemma 2.8, here we omit the details. □

Lemma 2.9 *Let* $A_0(z), A_1(z), \dots, A_{k-1}(z)$ *be meromorphic functions of finite order. Suppose there exists an integer number* $s \in \{0, 1, 2, \dots, k-1\}$, *such that*

$$\max\left\{\rho(A_j), j \neq s, \lambda\left(\frac{1}{A_s}\right)\right\} < \mu(A_s).$$

Then every infinite order meromorphic solution f *whose poles are of uniformly bounded multiplicities of equation* (1.1) *satisfies* $\rho_2(f) \leq \rho(A_s)$.

Proof By equation (1.1),

$$-\frac{f^{(k)}}{f} = A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_s\frac{f^{(s)}}{f} + \cdots + A_1\frac{f'}{f} + A_0. \tag{2.2}$$

By Lemma 2.2, for any given $\varepsilon > 0$, there exists a set $E_1 \subset (1, \infty)$ with $\mathrm{m_l}(E_1) < \infty$, such that, for all z satisfying $|z| = r \notin ([0,1] \cup E_1)$,

$$\left|A_j(z)\right| \leq \exp\left(r^{\rho(A_s)+\varepsilon}\right), \quad j = 0, 1, \dots, k-1. \tag{2.3}$$

Since the poles of f come from the poles of $A_j(z)$, $j = 0, 1, \dots, k-1$, and the multiplicities of poles of f are uniformly bounded, we have $\lambda(\frac{1}{f}) < \mu(A_s)$. Set $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ is an entire function, $d(z)$ is the classic product of poles sequence of f. It follows from Lemmas 2.4 and 2.7 that

$$\lambda(d) = \rho(d) = \lambda\left(\frac{1}{f}\right) < \mu(A_s) \leq \mu(f) = \mu(g) \leq \rho(g) = \rho(f).$$

By Lemma 2.3, there exists a set $E_2 \subset (1, \infty)$ with $\mathrm{m_l}(E_2) < \infty$, such that, for all z satisfying $|z| = r \notin ([0,1] \cup E_2)$, $|g(z)| = M(r, g)$,

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^j \left(1 + o(1)\right), \quad j = 1, \dots, k-1. \tag{2.4}$$

It follows from (2.2), (2.3) and (2.4) that there exists a constant $R > 1$, such that, for all z satisfying $|z| = r \notin ([0,R] \cup E_1 \cup E_2)$, $|g(z)| = M(r, g)$, and $\nu_g(r) > 1$,

$$\begin{aligned}\left|\left(\frac{\nu_g(r)}{z}\right)^k \left(1 + o(1)\right)\right| \leq{}& \left\{\left|\left(\frac{\nu_g(r)}{z}\right)^{k-1} \left(1 + o(1)\right)\right| + \cdots + \left|\left(\frac{\nu_g(r)}{z}\right)\left(1 + o(1)\right)\right| + 1\right\} \\ &\times \exp\left(r^{\rho(A_s)+\varepsilon}\right).\end{aligned}$$

Then

$$\nu_g^k(r)\big|1+o(1)\big| \le k\exp\big(r^{\rho(A_s)+\varepsilon}\big)|z|^k \nu_g^{k-1}(r)\big|1+o(1)\big|.$$

By Lemmas 2.5 and 2.6, we get

$$\rho_2(g) \le \rho(A_s).$$

Combining Lemma 2.4 and the inequality above, we have

$$\rho_2(f) \le \rho(A_s).$$

This completes the proof. □

Lemma 2.10 *Let $A_0(z), A_1(z), \dots, A_{k-1}(z)$ and $F(z) \not\equiv 0$ be meromorphic functions of finite order. Suppose there exists an integer number $s \in \{0, 1, 2, \dots, k-1\}$, such that*

$$\max\left\{\rho(A_j), j \ne s, \lambda\left(\frac{1}{A_s}\right), \rho(F)\right\} < \mu(A_s).$$

Then every infinite order meromorphic solution f whose poles are of uniformly bounded multiplicities of equation (1.3) *satisfies $\rho_2(f) \le \rho(A_s)$.*

Proof By equation (1.3),

$$-\frac{f^{(k)}}{f} = A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_s\frac{f^{(s)}}{f} + \cdots + A_1\frac{f'}{f} + A_0 - \frac{F}{f}. \tag{2.5}$$

By Lemma 2.2, for any given $\varepsilon > 0$, there exists a set $E_3 \subset (1, \infty)$ with $m_l(E_3) < \infty$, such that, for all z satisfying $|z| = r \notin ([0,1] \cup E_3)$, we have (2.3) and

$$\big|F(z)\big| \le \exp\big(r^{\rho(A_s)+\varepsilon}\big). \tag{2.6}$$

Since the poles of f come from the poles of $A_j(z)$, $j = 0, 1, \dots, k-1$, $F(z)$, and the multiplicities of poles of f are uniformly bounded, we have $\lambda(\frac{1}{f}) < \mu(A_s)$. Set $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ is an entire function, $d(z)$ is a classic product of poles sequence of f. It follows from Lemmas 2.3, 2.4 and 2.8 that there exists a set $E_4 \subset (1, \infty)$ with $m_l(E_4) < \infty$, such that, for all z satisfying $|z| = r \notin ([0,1] \cup E_4)$, $|g(z)| = M(r,g)$, equation (2.4) holds.

It follows from (2.3), (2.4), and (2.5) that there exists a constant $R > 1$, such that, for all z satisfying $|z| = r \notin ([0,R] \cup E_3 \cup E_4)$, $|g(z)| = M(r,g)$,

$$\begin{aligned}\left|\left(\frac{\nu_g(r)}{z}\right)^k\big(1+o(1)\big)\right| \le{}& \left\{\left|\left(\frac{\nu_g(r)}{z}\right)^{k-1}\big(1+o(1)\big)\right| + \cdots + \left|\left(\frac{\nu_g(r)}{z}\right)\big(1+o(1)\big)\right| + 1\right\}\\ &\times \exp\big(r^{\rho(A_s)+\varepsilon}\big) + \left|\frac{F(z)}{f(z)}\right|.\end{aligned} \tag{2.7}$$

Since $\rho(d) < \rho(A_s)$, for all $r > R$,

$$\big|d(z)\big| \le \exp\big(r^{\rho(A_s)+\varepsilon}\big). \tag{2.8}$$

It follows from (2.6) and (2.8) that

$$\left|\frac{F}{f}\right| = \left|\frac{F}{g}d\right| = \left|\frac{F}{M(r,g)}d\right| \le \exp\bigl(2r^{\rho(A_s)+\varepsilon}\bigr).$$

Combining (2.7) and the inequality above, we get

$$\bigl(\nu_g(r)\bigr)^k \bigl|1+o(1)\bigr| \le (k+1)\exp\bigl(2r^{\rho(A_s)+\varepsilon}\bigr)|z|^k \bigl(\nu_g(r)\bigr)^{k-1}\bigl|1+o(1)\bigr|.$$

Combining Lemmas 2.5, 2.6, and the inequality above, we get

$$\rho_2(g) \le \rho(A_s).$$

Combining Lemma 2.4 and the inequality above, we have

$$\rho_2(f) \le \rho(A_s).$$

This completes the proof. □

3 Proof of Theorems 1.6 and 1.7

Proof of Theorem 1.6 By Lemma 2.7, it is easy to see that equation (1.1) cannot have any nonzero rational solution. By (1.1),

$$\begin{aligned} -A_s &= \frac{f^{(k)}}{f^{(s)}} + A_{k-1}\frac{f^{(k-1)}}{f^{(s)}} + \cdots + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}} \\ &\quad + \frac{f}{f^{(s)}}\left(A_{s-1}\frac{f^{(s-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\right). \end{aligned} \tag{3.1}$$

By Lemma 2.1, for $\alpha = 2$, there exists a set $E_1 \subset (1,\infty)$ with $\mathrm{m_l}(E_1) < \infty$ and constant $B > 0$, such that, for all z satisfying $|z| = r \notin ([0,R_1] \cup E_1)$, where $R_1 > 1$ is a constant,

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \le Br\bigl(T(2r,f)\bigr)^{j-s+1}, \quad j = s+1, s+2, \ldots, k, \tag{3.2}$$

and

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le Br\bigl(T(2r,f)\bigr)^{j+1}, \quad j = 1, 2, \ldots, s-1. \tag{3.3}$$

By Lemma 2.3, there exists a set $E_2 \subset (1,\infty)$ with $\mathrm{m_l}(E_2) < \infty$, such that, for all z satisfying $|z| = r \notin ([0,1] \cup E_2)$ and $|g(z)| = M(r,g)$,

$$\frac{f^{(s)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^s \bigl(1+o(1)\bigr). \tag{3.4}$$

Considering the equation above, there exists a constant R_2 $(> R_1)$, such that, for all z satisfying $|z| = r > R_2$, $\nu_g(r) > 1$, $|1+o(1)| > \frac{1}{2}$, and $|g(z)| = M(r,g)$, $M(r,g) > 1$,

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \le 2r^s. \tag{3.5}$$

Set $H_1 = \{|z| : z \in E\} \backslash ([0, R_2] \cup E_1 \cup E_2)$. Then $m_l(H_1) = \infty$. It follows from (3.1), (3.2), (3.3), (3.5) and conditions (1.4), (1.5) that, for all z satisfying $|z| = r \in H_1$ and $|g(z)| = M(r, g)$,

$$\exp(\alpha|z|^{\rho-\varepsilon}) \le 2kBr^{s+1}\big(T(2r, f)\big)^{k+1}\exp(\beta|z|^{\rho-\varepsilon}).$$

Therefore, we get

$$\rho_2(f) \ge \rho = \rho(A_s).$$

It follows from the inequality above and Lemma 2.9 that

$$\rho_2(f) = \rho(A_s).$$

This completes the proof. □

Proof of Theorem 1.7 By Lemma 2.8, we know that if equation (1.3) has solutions, then the solution must be transcendental. By equation (1.3),

$$\begin{aligned} -A_s = \frac{F}{f} \cdot \frac{f}{f^{(s)}} - \bigg\{ & \frac{f^{(k)}}{f^{(s)}} + A_{k-1}\frac{f^{(k-1)}}{f^{(s)}} + \cdots + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}} \\ & + \frac{f}{f^{(s)}}\bigg(A_{s-1}\frac{f^{(s-1)}}{f} + \cdots + A_1\frac{f'}{f} + A_0\bigg)\bigg\}. \end{aligned} \tag{3.6}$$

By Lemma 2.1, for $\alpha = 2$, there exists a set $E_3 \subset (1, \infty)$ with $m_l(E_3) < \infty$ and constant $B > 0$, such that, for all z satisfying $|z| = r \notin ([0, R_3] \cup E_3)$, where $R_3 > 1$ is a constant, equations (3.2) and (3.3) hold.

Set $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ is an entire function, $d(z)$ is a classic product of a poles sequence of f. Since the poles of f come from the poles of $A_j(z)$, $j = 0, 1, \ldots, k-1$, $F(z)$, and the multiplicities of the poles of f are uniformly bounded, we have $\lambda(\frac{1}{f}) = \lambda(d) = \rho(d) \le b$, where

$$b = \max\bigg\{\rho(A_j), j \ne s, \lambda\bigg(\frac{1}{A_s}\bigg), \rho(F)\bigg\}.$$

Let η be constant satisfying $b < \eta < \mu(A_s)$. By Lemma 2.2, there exists a set $E_4 \subset (1, \infty)$ with $m_l(E_4) < \infty$, such that, for all z satisfying $|z| = r \notin ([0, 1] \cup E_4)$,

$$\big|F(z)d(z)\big| \le \exp(r^\eta). \tag{3.7}$$

By Lemma 2.3, there exists a set $E_5 \subset (1, \infty)$ with $m_l(E_5) < \infty$, such that, for all z satisfying $|z| = r \notin ([0, 1] \cup E_5)$ and $|g(z)| = M(r, g)$, equation (3.4) holds. Therefore, there exists a constant R_4 ($> R_3$), such that, for all z satisfying $|z| = r > R_4$, $\nu_g(r) > 1$, $|1 + o(1)| > \frac{1}{2}$, and $|g(z)| = M(r, g)$, $M(r, g) > 1$, equation (3.5) holds.

Set $H_2 = \{|z| : z \in E\} \backslash ([0, R_4] \cup E_3 \cup E_4 \cup E_5)$. Then $m_l(H_2) = \infty$. It follows from (3.7) that, for all z satisfying $|z| = r \in H_2$, $r > R_4$, and $|g(z)| = M(r, g)$,

$$\left|\frac{F(z)}{f(z)}\right| = \left|\frac{F(z)}{g(z)}d(z)\right| = \left|\frac{F(z)}{M(r, g)}\right| \cdot \big|d(z)\big| \le \exp(r^\eta). \tag{3.8}$$

It follows from (3.2), (3.3), (3.5), (3.6), (3.8), and conditions (1.4), (1.5) that, for all z satisfying $|z| = r \in H_2$ and $|g(z)| = M(r,g)$,

$$\exp\bigl(\alpha r^{\rho-\varepsilon}\bigr) \leq 2(k+1)Br^{s+1}\bigl(T(2r,f)\bigr)^{k+1}\exp\bigl(\beta r^{\rho-\varepsilon}\bigr)\exp\bigl(r^{\eta}\bigr).$$

By $\eta < \mu(A_s) \leq \rho$ and for any given $\varepsilon \in (0, \frac{\mu(A_s)-\eta}{2})$, we get

$$\rho_2(f) \geq \rho = \rho(A_s).$$

It follows from the inequality above and Lemma 2.10 that

$$\rho_2(f) = \rho(A_s).$$

This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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