Rong and Bai *Advances in Difference Equations* (2015) 2015:82
DOI 10.1186/s13662-015-0430-x

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Lyapunov-type inequality for a fractional differential equation with fractional boundary conditions

Ji Rong[1,2] and Chuanzhi Bai[1*]

*Correspondence:
czbai8@sohu.com
[1]Department of Mathematics, Huaiyin Normal University, Huaian, Jiangsu 223300, P.R. China
Full list of author information is available at the end of the article

Abstract
In this paper, a Lyapunov-type inequality is obtained for a fractional differential equation under fractional boundary conditions. We then use this inequality to obtain an interval where a certain Mittag-Leffler function has no real zeros.

MSC: 34A08; 34A40; 26D10; 33E12

Keywords: Lyapunov inequality; Caputo fractional derivative; Mittag-Leffler function

1 Introduction

The Lyapunov inequality [1] has proved to be very useful in various problems related with differential equations; for example, see [2, 3] and the references therein. The Lyapunov inequality states that a necessary condition for the boundary value problem

$$\begin{cases} y''(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = 0 = y(b) \end{cases} \tag{1.1}$$

to have nontrivial solutions is that

$$\int_a^b |q(s)|\, ds > \frac{4}{b-a}, \tag{1.2}$$

where $q : [a, b] \to R$ is a continuous function, and the zeros a and b of every solution $y(t)$ are consecutive. Since then, many generalizations of the Lyapunov inequality have appeared in the literature (see [4–9] and the references therein).

Recently, the research of Lyapunov-type inequalities for fractional boundary value problem has begun. In [10], Ferreira investigated a Lyapunov-type inequality for the Caputo fractional boundary value problem

$$\begin{cases} {}_a^C D^\alpha y(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = 0 = y(b), \end{cases} \tag{1.3}$$

where ${}_a^C D^\alpha$ is the Caputo fractional derivative of order α, $1 < \alpha \le 2$, the zeros a and b are consecutive, and q is a real and continuous function. It was proved that if (1.3) has a nontrivial solution, then

$$\int_a^b |q(t)|\,ds \geq \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{1.4}$$

Obviously, if we set $\alpha = 2$ in (1.4), one can obtain the Lyapunov classical inequality (1.2). In [11], the same author studied a differential equation that depends on the Riemann-Liouville fractional derivative and gave a Lyapunov-type inequality. In both works [10, 11], some interesting applications to the localization of real zeros of certain Mittag-Leffler functions were presented.

In [12], Jleli and Samet considered the fractional differential equation

$$ {}_a^{\mathrm{C}}D^\alpha y(t) + q(t)y(t) = 0, \quad a < t < b, 1 < \alpha \leq 2,$$

with the mixed boundary conditions

$$y(a) = 0 = y'(b) \tag{1.5}$$

or

$$y'(a) = 0 = y(b). \tag{1.6}$$

For boundary conditions (1.5) and (1.6), two Lyapunov-type inequalities were established respectively as follows:

$$\int_a^b (b-s)^{\alpha-2}|q(s)|\,ds \geq \frac{\Gamma(\alpha)}{\max\{\alpha-1, 2-\alpha\}(b-a)} \tag{1.7}$$

and

$$\int_a^b (b-s)^{\alpha-1}|q(s)|\,ds \geq \Gamma(\alpha). \tag{1.8}$$

Motivated by the above works, we consider in this paper a Caputo fractional differential equation under boundary condition involving the Caputo fractional derivative. More precisely, we consider the boundary value problem

$$\begin{cases} {}_a^{\mathrm{C}}D^\alpha y(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = 0, & {}_a^{\mathrm{C}}D^\beta y(b) = 0, \end{cases} \tag{1.9}$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, and $q : [a, b] \to R$ is a continuous function. We write (1.9) as an equivalent integral equation and then, by using some properties of its Green function, we are able to get a corresponding Lyapunov-type inequality. After that, we show that this inequality can be used to obtain a real interval where a certain Mittag-Leffler function has no real zeros. Our results generalize the main results of Jleli and Samet [12].

2 Preliminaries

In this section, we introduce the definitions of the Riemann-Liouville fractional integral and the Caputo fractional derivative and give some lemmas which are used in this article.

Definition 2.1 Let $\alpha \geq 0$ and f be a real function defined on $[a,b]$. The Riemann-Liouville fractional integral of order α is defined by ${}_aI^0 f \equiv f$ and

$$\left({}_aI^\alpha f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,ds, \quad \alpha > 0, t \in [a,b].$$

Definition 2.2 The Caputo derivative of fractional order $\alpha \geq 0$ is defined by ${}_a^{\mathrm{C}}D^0 f \equiv f$ and

$$\left({}_a^{\mathrm{C}}D^\alpha f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,ds, \quad \alpha > 0, t \in [a,b],$$

where n is the smallest integer greater or equal to α.

The following results are standard within the fractional calculus theory involving the Caputo differential operator.

Lemma 2.1 ([13], Chapter 2) *Let $\gamma > \alpha > 0, f \in C[a,b]$, then*

$${}_a^{\mathrm{C}}D^\alpha \left({}_aI^\gamma f(t)\right) = {}_aI^{\gamma-\alpha} f(t), \quad t \in [a,b].$$

Lemma 2.2 ([14], Section 2) *Let $y \in C[a,b]$ and $1 < \alpha \leq 2$, then*

$${}_aI^\alpha \left({}_a^{\mathrm{C}}D^\alpha y\right)(t) = y(t) + c_0 + c_1(t-a)$$

for some real constants c_0 and c_1.

3 Main results

We begin by writing problem (1.9) in its equivalent integral form.

Lemma 3.1 *$y \in C[a,b]$ is a solution of the boundary value problem* (1.9) *if and only if y satisfies the integral equation*

$$y = \int_a^b G(t,s) q(s) y(s)\,ds,$$

where

$$G(t,s) = H(t,s)(b-s)^{\alpha-\beta-1}$$

and

$$H(t,s) = \begin{cases} \frac{\Gamma(2-\beta)(t-a)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}} - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}(b-s)^{1+\beta-\alpha}, & a \leq s \leq t \leq b, \\ \frac{\Gamma(2-\beta)(t-a)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}}, & a \leq t \leq s \leq b. \end{cases} \tag{3.1}$$

Proof From (1.9) and Lemma 2.2, we obtain

$$y(t) = c_0 + c_1(t-a) - \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} q(s) y(s)\,ds,$$

where c_0 and c_1 are some real constants. By the boundary condition $y(a) = 0$, we can obtain that $c_0 = 0$. Thus, we have

$$y(t) = c_1(t-a) - \left({}_aI^\alpha qy\right)(t). \tag{3.2}$$

By (3.2), we get

$${}_a^CD^\beta y(t) = \frac{c_1}{\Gamma(2-\beta)}(t-a)^{1-\beta} - \frac{1}{\Gamma(\alpha-\beta)}\int_a^b (t-s)^{\alpha-\beta-1}q(s)y(s)\,ds. \tag{3.3}$$

Since ${}_a^CD^\beta y(b) = 0$, we have from (3.3) that

$$c_1 = \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}}\int_a^b (b-s)^{\alpha-\beta-1}q(s)y(s)\,ds. \tag{3.4}$$

Substitute (3.4) into (3.2), we obtain

$$y(t) = \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}}\int_a^b (b-s)^{\alpha-\beta-1}(t-a)q(s)y(s)\,ds - \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}q(s)y(s)\,ds,$$

which concludes the proof. □

Lemma 3.2 *If* $1 < \alpha < 2$ *and* $0 < \beta < 1$, *then*

$$\Gamma(\alpha) < \frac{\Gamma(\alpha-\beta)}{\Gamma(2-\beta)} < \Gamma(\alpha-1). \tag{3.5}$$

Proof Consider the logarithmic derivative of the gamma function

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}. \tag{3.6}$$

We have by [15], p.264, that

$$\psi(x) = -\gamma - \sum_{k=0}^\infty\left(\frac{1}{x+k} - \frac{1}{k+1}\right), \tag{3.7}$$

where γ is an Euler constant. From (3.7) we obtain

$$\frac{d\psi(x)}{dx} = \sum_{k=0}^\infty \frac{1}{(x+k)^2} > 0. \tag{3.8}$$

Since $\alpha < 2$, we get by (3.6) and (3.8) that $\psi(\alpha-x) < \psi(2-x)$, that is,

$$\frac{\Gamma'(\alpha-x)}{\Gamma(\alpha-x)} < \frac{\Gamma'(2-x)}{\Gamma(2-x)}. \tag{3.9}$$

Let

$$f(x) = \frac{\Gamma(\alpha-x)}{\Gamma(2-x)}, \quad 0 < x < \alpha.$$

Then we have by (3.9) that

$$f'(x) = \frac{-\Gamma'(\alpha - x)\Gamma(2 - x) + \Gamma(\alpha - x)\Gamma'(2 - x)}{(\Gamma(2 - x))^2} > 0.$$

Thus, $f(0) < f(\beta) < f(1)$ $(0 < \beta < 1)$, which implies that (3.5) holds. □

Lemma 3.3 *Assume that* $0 < \beta \leq 1$ *and* $1 < \alpha \leq 1 + \beta$ *hold. Then*

$$|H(t,s)| \leq \begin{cases} \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}(b-a)^{\beta}, & a \leq t \leq s \leq b, \\ \max\{\frac{1}{\Gamma(\alpha)} - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{2-\alpha}{\alpha-1} \cdot \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\}(b-a)^{\beta}, & a \leq s \leq t \leq b. \end{cases} \tag{3.10}$$

Proof Throughout the proof we consider $\beta < 1$ since when $\beta = 1$ our study is reduced to the case in [12]. For $a \leq t \leq s \leq b$, we easily know that

$$|H(t,s)| \leq \frac{\Gamma(2-\beta)(s-a)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}} \leq \frac{\Gamma(2-\beta)(b-a)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}} = \frac{\Gamma(2-\beta)(b-a)^{\beta}}{\Gamma(\alpha-\beta)}.$$

For convenience, let

$$\psi(t,s) = \frac{\Gamma(2-\beta)(t-a)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}} - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}(b-s)^{1+\beta-\alpha}, \quad a \leq s \leq t \leq b.$$

Fixing arbitrary $s \in [a,b)$ and differentiating $\psi(t,s)$ with respect to t, we obtain

$$\psi_t(t,s) = \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}} - \frac{1}{\Gamma(\alpha-1)}(t-s)^{\alpha-2}(b-s)^{1+\beta-\alpha}, \quad s < t. \tag{3.11}$$

From (3.11) we easily know that $\psi_t(t^*,s) = 0$ if and only if

$$t^* = s + \left[\frac{\Gamma(2-\beta)\Gamma(\alpha-1)}{\Gamma(\alpha-\beta)} \cdot \frac{(b-s)^{\alpha-\beta-1}}{(b-a)^{1-\beta}}\right]^{\frac{1}{\alpha-2}} \tag{3.12}$$

provided $t^* \leq b$, *i.e.*, as long as $s \leq b - l$, where

$$l = \left[\frac{\Gamma(\alpha-\beta)}{\Gamma(2-\beta)\Gamma(\alpha-1)}\right]^{\frac{1}{1-\beta}}(b-a) < b - a \quad \text{(by (3.5))}.$$

Hence, if $s > b - l$, then

$$\psi_t(t,s) < 0, \quad t \in (s,b). \tag{3.13}$$

On the other hand, we have

$$\lim_{t \to s^+} \psi(t,s) = \frac{\Gamma(2-\beta)(s-a)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}} \quad \text{and} \quad \psi(b,s) = \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}(b-a)^{\beta} - \frac{1}{\Gamma(\alpha)}(b-s)^{\beta}.$$

Thus, we obtain

$$|\psi(s,s)| \leq \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}(b-a)^{\beta} \tag{3.14}$$

and

$$\begin{aligned}
\left|\psi(b,s)\right| &= \left|\left(\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)} - \frac{1}{\Gamma(\alpha)}\right)(b-s)^{\beta} + \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\left((b-a)^{\beta} - (b-s)^{\beta}\right)\right| \\
&\le \max\left\{\left(\frac{1}{\Gamma(\alpha)} - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\right)(b-s)^{\beta}, \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}(s-a)^{\beta}\right\}
\end{aligned} \tag{3.15}$$

by $(s-a)^{\beta} + (b-s)^{\beta} \ge (b-a)^{\beta}$ and (3.5). Thus, we have by (3.13)-(3.15) that

$$\begin{aligned}
\left|\psi(t,s)\right| &\le \max\left\{\left|\psi(s,s)\right|, \left|\psi(b,s)\right|\right\} \\
&\le \max\left\{\left(\frac{1}{\Gamma(\alpha)} - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\right), \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\right\}(b-a)^{\beta}, \\
&\quad a < b-l < s < t \le b.
\end{aligned} \tag{3.16}$$

It remains to verify the result when $s \le b-l$, *i.e.*, when $t^* \le b$. It is easy to check that

$$\psi_t(t,s) < 0 \quad \text{for } t < t^* \quad \text{and} \quad \psi_t(t,s) \ge 0 \quad \text{for } t \ge t^*.$$

Hence, we have

$$\left|\psi(t,s)\right| \le \max\left\{\left|\psi\left(t^*,s\right)\right|, \left|\psi(b,s)\right|, \left|\psi(s,s)\right|\right\}, \quad a < s \le b-l, s \le t \le b. \tag{3.17}$$

By (3.12) we have

$$\begin{aligned}
\left|\psi\left(t^*,s\right)\right| &= \left|\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}}\left[s + \left(\frac{\Gamma(2-\beta)\Gamma(\alpha-1)}{\Gamma(\alpha-\beta)} \cdot \frac{(b-s)^{\alpha-\beta-1}}{(b-a)^{1-\beta}}\right)^{\frac{1}{\alpha-2}} - a\right]\right. \\
&\quad \left. - \frac{1}{\Gamma(\alpha)}\left(\frac{\Gamma(2-\beta)\Gamma(\alpha-1)}{\Gamma(\alpha-\beta)} \cdot \frac{(b-s)^{\alpha-\beta-1}}{(b-a)^{1-\beta}}\right)^{\frac{\alpha-1}{\alpha-2}}(b-s)^{1+\beta-\alpha}\right| \\
&= \left|\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}}(s-a) + \left(\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}}\right)^{\frac{\alpha-1}{\alpha-2}}\right. \\
&\quad \cdot \left(\Gamma(\alpha-1)\right)^{\frac{1}{\alpha-2}}(b-s)^{\frac{\alpha-\beta-1}{\alpha-2}} \\
&\quad \left. - \frac{1}{\Gamma(\alpha)} \cdot \left(\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}}\right)^{\frac{\alpha-1}{\alpha-2}} \cdot \left(\Gamma(\alpha-1)\right)^{\frac{\alpha-1}{\alpha-2}}(b-s)^{\frac{\alpha-\beta-1}{\alpha-2}}\right| \\
&= \left|\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}}(s-a) + \frac{\alpha-2}{\alpha-1}\left(\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}}\right)^{\frac{\alpha-1}{\alpha-2}}\right. \\
&\quad \left. \cdot \left(\Gamma(\alpha-1)\right)^{\frac{1}{\alpha-2}}(b-s)^{\frac{\alpha-\beta-1}{\alpha-2}}\right| \\
&\le \max\left\{\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}}(s-a), \frac{2-\alpha}{\alpha-1}\left(\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}}\right)^{\frac{\alpha-1}{\alpha-2}}\right. \\
&\quad \left. \cdot \left(\Gamma(\alpha-1)\right)^{\frac{1}{\alpha-2}}(b-s)^{\frac{\alpha-\beta-1}{\alpha-2}}\right\}.
\end{aligned} \tag{3.18}$$

Obviously, we have

$$\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}}(s-a) \le \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}(b-a)^{\beta}, \tag{3.19}$$

and we obtain from Lemma 3.2 and condition $\alpha - \beta - 1 \le 0$ that

$$\begin{aligned}
&\frac{2-\alpha}{\alpha-1}\left(\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}}\right)^{\frac{\alpha-1}{\alpha-2}} \cdot \left(\Gamma(\alpha-1)\right)^{\frac{1}{\alpha-2}}(b-s)^{\frac{\alpha-\beta-1}{\alpha-2}} \\
&\quad \le \frac{2-\alpha}{\alpha-1}\left(\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)(b-a)^{1-\beta}}\right)^{\frac{\alpha-1}{\alpha-2}} \cdot \left(\Gamma(\alpha-1)\right)^{\frac{1}{\alpha-2}}(b-a)^{\frac{\alpha-\beta-1}{\alpha-2}} \\
&\quad = \frac{2-\alpha}{\alpha-1} \cdot \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)} \cdot \left(\frac{\Gamma(2-\beta)\Gamma(\alpha-1)}{\Gamma(\alpha-\beta)}\right)^{\frac{1}{\alpha-2}} \cdot (b-a)^{\beta} \\
&\quad < \frac{2-\alpha}{\alpha-1} \cdot \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}(b-a)^{\beta}.
\end{aligned} \tag{3.20}$$

Thus, from (3.18)-(3.20) we conclude that

$$\left|\psi\left(t^*,s\right)\right| \le \max\left\{1, \frac{2-\alpha}{\alpha-2}\right\}\frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}(b-a)^{\beta}, \quad s \in [a, b-l] \tag{3.21}$$

holds. From (3.14), (3.15), (3.21) and (3.17), we know that inequality (3.10) is true. The proof is complete. □

Theorem 3.4 *Let* $0 < \beta \le 1$ *and* $1 < \alpha \le 1 + \beta$. *If a nontrivial continuous solution of the fractional boundary value problem* (1.9) *exists, then*

$$\int_a^b (b-s)^{\alpha-\beta-1}\left|q(s)\right| ds \ge \frac{(b-a)^{-\beta}}{\max\{\frac{1}{\Gamma(\alpha)} - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{2-\alpha}{\alpha-1} \cdot \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\}}. \tag{3.22}$$

Proof Let $B = C[a,b]$ be the Banach space endowed with the norm $\|y\|_\infty = \sup_{t\in[a,b]} |y(t)|$. According to Lemma 3.1, the solution of (1.9) can be written as

$$y(t) = \int_a^b (b-s)^{\alpha-\beta-1} H(t,s) q(s) y(s)\, ds.$$

Now, an application of Lemma 3.3 yields

$$\begin{aligned}
\|y\|_\infty &\le \max\left\{\frac{1}{\Gamma(\alpha)} - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{2-\alpha}{\alpha-1} \cdot \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\right\} \\
&\quad \cdot (b-a)^{\beta}\|y\|_\infty \int_a^b (b-s)^{\alpha-\beta-1}\left|q(s)\right| ds,
\end{aligned}$$

which implies that (3.22) holds. □

Remark 3.1 If $\beta = 1$, then (3.22) reduces to the following Lyapunov-type inequality [12]:

$$\int_a^b (b-s)^{\alpha-2}\left|q(s)\right| ds \ge \frac{\Gamma(\alpha)}{\max\{\alpha-1, 2-\alpha\}(b-a)}.$$

Remark 3.2 If $\alpha = 2$ and $0 < \beta < 1$, then we have by Lemma 3.1 that

$$G(t,s) = \begin{cases} (t-a)(\frac{b-s}{b-a})^{1-\beta} - (t-s), & a \le s \le t \le b, \\ (t-a)(\frac{b-s}{b-a})^{1-\beta}, & a \le t \le s \le b. \end{cases}$$

Similar to the proof of Theorem 3.4, it is easy to obtain that the following Lyapunov-type inequality holds:

$$\int_a^b |q(s)|\, ds \geq \frac{1}{(b-a)^\beta}.$$

In the following, we will use Lyapunov-type inequalities (3.22) to obtain intervals where certain Mittag-Leffler functions have no real zeros. Let $z \in \mathbb{R}$ and consider the real zeros of the Mittag-Leffler functions $E_{\alpha,\gamma}(z)$, where

$$E_{\alpha,\gamma}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \gamma)}, \quad \alpha > 0, \gamma > 0 \text{ and } z \in \mathbb{C}.$$

Obviously, $E_{\alpha,\gamma}(z) > 0$ for all $z \geq 0$. Hence, the real zeros of $E_{\alpha,\gamma}(z)$, if they exist, must be negative real numbers.

Theorem 3.5 *Assume that* $0 < \beta \leq 1$ *and* $1 < \alpha \leq 1 + \beta$ *hold. Then the Mittag-Leffler function* $E_{\alpha,2-\beta}(x)$ *has no real zeros for*

$$x \in \left(-\frac{\alpha-\beta}{\max\{\frac{1}{\Gamma(\alpha)} - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{2-\alpha}{\alpha-1} \cdot \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\}}, 0\right].$$

Proof Let $a = 0$ and $b = 1$. Consider the following fractional Sturm-Liouville eigenvalue problem:

$$ {}_0^C D^\alpha y(t) + \lambda y(t) = 0, \quad t \in (0,1), \tag{3.23}$$

$$ y(0) = {}_0^C D^\beta y(1) = 0. \tag{3.24}$$

By the Laplace transform method as in [13, 16, 17], the general solution of the fractional differential equation (3.23) can be given as follows:

$$y(t) = AE_{\alpha,1}(-\lambda t^\alpha) + BtE_{\alpha,2}(-\lambda t^\alpha). \tag{3.25}$$

In the following discussion we will use the general solution (3.25) and its fractional Caputo derivative

$$ {}_0^C D^\beta y(t) = At^{-\beta} E_{\alpha,1-\beta}(-\lambda t^\alpha) + Bt^{1-\beta} E_{\alpha,2-\beta}(-\lambda t^\alpha). \tag{3.26}$$

By (3.25), (3.26) and the boundary conditions (3.24), we obtain that

$$A = 0, \qquad BE_{\alpha,2-\beta}(-\lambda) = 0.$$

Thus, the eigenvalues $\lambda \in \mathbb{R}$ of (3.23) and (3.24) are the solutions of

$$E_{\alpha,2-\beta}(-\lambda) = 0, \tag{3.27}$$

and the corresponding eigenfunctions are given by

$$y(t) = tE_{\alpha,2-\beta}\bigl(-\lambda t^{\alpha}\bigr), \quad t \in [0,1].$$

By Theorem 3.4, if a real eigenvalue λ of (3.23) and (3.24) exists, *i.e.*, $-\lambda$ is a zero of (3.27), then

$$\lambda \int_0^1 (1-s)^{\alpha-\beta-1}\,ds \geq \frac{1}{\max\{\frac{1}{\Gamma(\alpha)} - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{2-\alpha}{\alpha-1} \cdot \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\}},$$

that is,

$$\lambda \geq \frac{\alpha-\beta}{\max\{\frac{1}{\Gamma(\alpha)} - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{2-\alpha}{\alpha-1} \cdot \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}\}},$$

which concludes the proof. □

Remark 3.3 If $\beta = 1$, then Theorem 3.5 reduces to Theorem 3 in [12].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in drafting this manuscript and giving the main proofs. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Huaiyin Normal University, Huaian, Jiangsu 223300, P.R. China. [2]Department of Mathematics, Jiangsu Normal University, Xuzhou, Jiangsu 221116, P.R. China.

Acknowledgements
The authors thank the referees for their important and valuable comments. This work is supported by the Natural Science Foundation of Jiangsu Province (BK2011407) and Natural Science Foundation of China (11271364 and 10771212).

Received: 26 November 2014 Accepted: 2 March 2015 Published online: 07 March 2015

References
1. Lyapunov, AM: Problème général de la stabilité du mouvement. Ann. Fac. Sci. Univ. Toulouse **2**, 203-407 (1907)
2. Brown, RC, Hinton, DB: Lyapunov inequalities and their applications. In: Rassias, TM (ed.) Survey on Classical Inequalities. Math. Appl., vol. 517, pp. 1-25. Kluwer Academic, Dordrecht (2000)
3. Pachpatte, BG: Mathematical Inequalities. North Holland Mathematical Library. Elsevier, Amsterdam (2005)
4. Tiryaki, A: Recent developments of Lyapunov-type inequalities. Adv. Dyn. Syst. Appl. **5**(2), 231-248 (2010)
5. Cakmak, D, Tiryaki, A: Lyapunov-type inequality for a class of Dirichlet quasilinear systems involving the $(p_1, p_2, \ldots, p_n)$-Laplacian. J. Math. Anal. Appl. **369**, 76-81 (2010)
6. Tang, XH, He, X: Lower bounds for generalized eigenvalues of the quasilinear systems. J. Math. Anal. Appl. **385**, 72-85 (2012)
7. Yang, X, Kim, Y, Lo, K: Lyapunov-type inequality for a class of linear differential systems. Appl. Math. Comput. **219**, 1805-1812 (2012)
8. Yang, X, Kim, Y, Lo, K: Lyapunov-type inequality for a class of odd-order differential equations. J. Comput. Appl. Math. **234**, 2962-2968 (2010)
9. Sanchez, J, Vergara, V: A Lyapunov-type inequality for a ψ-Laplacian operator. Nonlinear Anal. **74**, 7071-7077 (2011)
10. Ferreira, RAC: On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function. J. Math. Anal. Appl. **412**, 1058-1063 (2014)
11. Ferreira, RAC: A Lyapunov-type inequality for a fractional boundary value problem. Fract. Calc. Appl. Anal. **16**(4), 978-984 (2013)
12. Jleli, M, Samet, B: Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions. Math. Inequal. Appl. **18**(2), 443-451 (2015)
13. Podlubny, I: Fractional Differential Equations. Academic Press San Diego (1999)
14. Zhang, S: Positive solutions for boundary-value problems of nonlinear fractional differential equations. Electron. J. Differ. Equ. **2006**, 36 (2006)
15. Abramowitz, M, Stegun, I: Handbook of Mathematical Functions. Dover, New York (1972)

16. Miller, KS, Ross, B: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
17. Duan, JS, Wang, Z, Liu, YL, Qiu, X: Eigenvalue problems for fractional ordinary differential equations. Chaos Solitons Fractals **46**, 46-53 (2013)