Qiu and Li *Advances in Difference Equations* (2015) 2015:29
DOI 10.1186/s13662-015-0365-2

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Two kinds of bifurcation phenomena in a quartic system

Jianlong Qiu and Feng Li*

*Correspondence:
lf0539@126.com.cn
School of Science, Linyi University,
Linyi, Shandong 276005, P.R. China

Abstract
In this paper, the center conditions and the conditions for bifurcations of limit cycles from a third-order nilpotent critical point in a class of quartic systems are investigated. Taking the system coefficients as parameters, explicit expressions for the first 11 quasi-Lyapunov constants are deduced. As a result, we prove that 11 or 12 small-amplitude limit cycles could be created from the third-order nilpotent critical point by two different perturbations.

MSC: 34C05; 34C07

Keywords: third-order nilpotent critical point; center-focus problem; bifurcation of limit cycles; quasi-Lyapunov constant

1 Introduction

For a given family of polynomial differential equations, the number of Lyapunov constants needed to solve the center-focus problem is related to the so-called *cyclicity* of the point. Many works have been devoted to study this problem; see [1–3].

As far as the maximum number of small-amplitude limit cycles is concerned, there have been many results. Bifurcating from an elementary center or an elementary focus, one of the best-known results is $M(2) = 3$, which was solved by Bautin in 1952 [4]. For $n = 3$, a number of results have been obtained. Around an elemental focus, James and Lloyd [5] considered a particular class of cubic systems to obtain eight limit cycles in 1991, and the systems were reinvestigated a couple of years later by Ning *et al.* [6] to find another solution of eight limit cycles. Yu and Corless [7] constructed a cubic system and combined symbolic and numerical computations to show nine limit cycles in 2009, which was confirmed by purely symbolic computation with all real solutions obtained in 2013 [8]. Another cubic system was also recently constructed by Lloyd and Pearson [9] to show nine limit cycles with purely symbolic computation. Recently, Yu and Tian showed that there could be 12 limit cycles around a singular point in a planar cubic-degree polynomial system [10]. For $n = 4$, Huang gave an example of a quartic system with eight limit cycles bifurcated from a fine focus [11]. In recent years, bifurcations of limit cycles from degenerate critical points were investigated intensively. Especially, for a nilpotent critical point, there were also many results as regards limit cycles; see [12, 13].

In this paper, we consider another quartic system,

$$\begin{aligned}\frac{dx}{dt} &= y + \lambda\left(-b_{12}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{31}x^3y + a_{22}x^2y^2 - 4b_{04}xy^3 + a_{04}y^4\right),\\ \frac{dy}{dt} &= -2x^3 + \lambda\left(b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + b_{40}x^4 - \frac{3}{2}a_{31}x^2y^2 + b_{13}xy^3 + b_{04}y^4\right).\end{aligned} \tag{1.1}$$

We will show that 11 or 12 limit cycles can be bifurcated from the origin by different perturbations.

The rest of this paper is organized as follows. In Section 2, some preliminary results in [14] which are needed in the following sections will be given. In Section 3, the linear recursive formulas in [14] are used to compute the first 11 quasi-Lyapunov constants and then obtain the sufficient and necessary conditions for a center. In Section 4, one kind of different bifurcation is discussed to confirm that 11 limit cycles can bifurcate from quartic systems. In Section 5, another kind of interesting bifurcation phenomenon is discussed to confirm that 12 limit cycles can bifurcate from quartic systems.

To perform the computations in this paper, we have used the computer algebra system MATHEMATICA 7.

2 Preliminary knowledge

For convenience, in this section we present some results taken from [15] for the center-focus problem of third-order nilpotent critical points in planar dynamical systems. We introduce some notions and results; for more details, see [15].

The origin of a system is a third-order monodromic critical point if and only if the system can be written in the following form of a real autonomous planar system:

$$\begin{aligned}\frac{dx}{dt} &= y + \mu x^2 + \sum_{i+2j=3}^{n} a_{ij}x^iy^j = X(x,y),\\ \frac{dy}{dt} &= -2x^3 + 2\mu xy + \sum_{i+2j=4}^{n} b_{ij}x^iy^j = Y(x,y).\end{aligned} \tag{2.1}$$

Theorem 2.1 *For any positive integer s and a given real number sequence,*

$$\{c_{0\beta}\}, \quad \beta \ge 3, \tag{2.2}$$

one can construct successively the terms with the coefficients $c_{\alpha\beta}$ satisfying $\alpha \neq 0$ of the formal series,

$$M(x,y) = y^2 + \sum_{\alpha+\beta=3}^{\infty} c_{\alpha\beta}x^\alpha y^\beta = \sum_{k=2}^{\infty} M_k(x,y), \tag{2.3}$$

such that

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)M - (s+1)\left(\frac{\partial M}{\partial x}X + \frac{\partial M}{\partial y}Y\right) = \sum_{m=3}^{\infty} \omega_m(s,\mu)x^m, \tag{2.4}$$

where $M_k(x,y)$ is a kth-degree homogeneous polynomial of x, y for all k and $s\mu = 0$, $c_{\alpha\beta}$ and $\omega_m(s,\mu)$ are constants which will be determined by (2.4).

Equation (2.4) is linear with respect to the function M, so that we can easily obtain the following recursive formulas for calculating $c_{\alpha\beta}$ and $\omega_m(s,\mu)$.

Theorem 2.2 *For $\alpha \geq 1$, $\alpha + \beta \geq 3$ in (2.3) and (2.4), $c_{\alpha\beta}$ can be uniquely determined by the recursive formula,*

$$c_{\alpha\beta} = \frac{1}{(s+1)\alpha}(A_{\alpha-1,\beta+1} + B_{\alpha-1,\beta+1}); \tag{2.5}$$

and for $m \geq 1$, $\omega_m(s,\mu)$ can be uniquely determined by the recursive formula,

$$\omega_m(s,\mu) = A_{m,0} + B_{m,0}, \tag{2.6}$$

where

$$\begin{aligned} A_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} \bigl[k - (s+1)(\alpha - k + 1)\bigr] a_{kj} c_{\alpha-k+1,\beta-j}, \\ B_{\alpha\beta} &= \sum_{k+j=2}^{\alpha+\beta-1} \bigl[j - (s+1)(\beta - j + 1)\bigr] b_{kj} c_{\alpha-k,\beta-j+1}. \end{aligned} \tag{2.7}$$

Note in (2.7) that the following coefficients:

$$\begin{aligned} &c_{00} = c_{10} = c_{01} = 0, \\ &c_{20} = c_{11} = 0, \qquad c_{02} = 1, \\ &c_{\alpha\beta} = 0, \quad \text{if } \alpha < 0 \text{ or } \beta < 0, \end{aligned} \tag{2.8}$$

have been set. The mth-order quasi-Lyapunov constant is defined as

$$\lambda_m = \frac{\omega_{2m+4}(s,\mu)}{2m - 4s - 1}. \tag{2.9}$$

Clearly, the recursive formulas in Theorem 2.2 are linear with respect to all $c_{\alpha\beta}$. Therefore, it is convenient to develop programs for computing quasi-Lyapunov constants by using computer algebraic system such as MATHEMATICA.

3 Quasi-Lyapunov constants and center conditions

According to Theorem 2.1, for system (1.1), we can find a positive integer s and a formal series $M(x,y) = x^4 + y^2 + o(r^4)$, such that (2.4) holds. Applying the recursive formulas in Theorem 2.2 to carry out the calculations, we have

$$\begin{aligned} &\omega_3 = \omega_4 = \omega_5 = 0, \\ &\omega_6 = -\frac{1}{3} b_{21}\lambda(-1+4s), \\ &\omega_7 \sim 3(s+1)c_{03}, \\ &\omega_8 \sim -\frac{2(a_{12}+3b_{03})\lambda}{5}(-3+4s), \\ &\omega_9 \sim -\frac{2(a_{22}+3b_{13})\lambda}{3}(-1+s). \end{aligned} \tag{3.1}$$

It follows from (2.9) and (3.1) that the first two quasi-Lyapunov constants of system (1.1) are given by

$$\mu_1 = \frac{\omega_6}{1-4s} = \frac{b_{21}\lambda}{3},$$
$$\mu_2 = \frac{\omega_8}{3-4s} = \frac{2(a_{12}+3b_{03})\lambda}{5}.$$

Setting $\omega_7 = \omega_9 = 0$ yields $c_{03} = 0$ and $s = 1$.

Furthermore, with $s = 1$, we obtain the following results.

Proposition 3.1 *For system* (1.1), *one can determine successively the terms of the formal series* $M(x,y) = x^4 + y^2 + o(r^4)$, *such that*

$$\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)M - 2\left(\frac{\partial M}{\partial x}X + \frac{\partial M}{\partial y}Y\right) = \sum_{m=1}^{12} \mu_m\left[(2m-5)x^{2m+4} + o\left(r^{26}\right)\right], \tag{3.2}$$

where μ_m *is the mth-order quasi-Lyapunov constant at the origin of system* (1.1), $m = 1, 2, \ldots, 11$.

Theorem 3.1 *For system* (1.1), *the first* 11 *quasi-Lyapunov constants at the origin are given by*

$$\begin{aligned}
\mu_1 &= \frac{b_{21}\lambda}{3},\\
\mu_2 &= \frac{2(a_{12}+3b_{03})\lambda}{5},\\
\mu_3 &= \frac{b_{40}(2a_{22}+3b_{13})\lambda^2}{35},\\
\mu_4 &= -\frac{(2a_{22}+3b_{13})a_{31}\lambda^2}{15},\\
\mu_5 &= \frac{20b_{04}(2a_{22}+3b_{13})\lambda^2}{77},\\
\mu_6 &= \frac{-4b_{03}(172a_{22}-13b_{13})(2a_{22}+3b_{13})\lambda^3}{3{,}003},\\
\mu_7 &= \frac{8b_{03}(-4{,}563a_{04}+2{,}593a_{22}b_{12}\lambda)(2a_{22}+3b_{13})\lambda^3}{83{,}655},\\
\mu_8 &= \frac{56a_{22}b_{03}(3{,}061b_{12}^2\lambda+54{,}366a_{03})(2a_{22}+3b_{13})\lambda^4}{168{,}070{,}5},\\
\mu_9 &= \frac{256a_{22}b_{03}(2a_{22}+3b_{13})(-345{,}402{,}1b_{12}^3\lambda+258{,}836{,}526b_{03}^2)\lambda^5}{667{,}727{,}289{,}45},\\
\mu_{10} &= -128a_{22}b_{03}(2a_{22}+3b_{13})\\
&\quad\times\frac{(-478{,}112{,}236{,}791{,}537{,}5b_{12}^4\lambda^2+145{,}729{,}657{,}408{,}679{,}111{,}7a_{22}^2)\lambda^5}{432{,}080{,}529{,}791{,}586{,}126{,}75},\\
\mu_{11} &= -\frac{257{,}127{,}887{,}226{,}223{,}945{,}896{,}687{,}968a_{22}^3b_{03}b_{12}(2a_{22}+3b_{13})\lambda^8}{483{,}358{,}905{,}172{,}324{,}820{,}519{,}887{,}5}.
\end{aligned} \tag{3.3}$$

In the above expressions of μ_k, for each $k = 2,\ldots,11$, $\mu_1 = \mu_2 = \cdots = \mu_{k-1} = 0$ have been set.

Theorem 3.1 directly gives the following assertion.

Proposition 3.2 *The first* 11 *quasi-Lyapunov constants at the origin of system* (1.1) *are zero if and only if one of the following conditions is satisfied*:

$$b_{21} = a_{31} = b_{40} = b_{04} = a_{12} = b_{03} = 0, \tag{3.4}$$

$$b_{21} = 0, \qquad a_{12} = -3b_{03}, \qquad a_{22} = -\frac{3b_{13}}{2}, \tag{3.5}$$

$$b_{21} = a_{31} = b_{40} = b_{04} = b_{13} = a_{04} = a_{22} = 0, \qquad a_{12} = -3b_{03}. \tag{3.6}$$

When condition (3.4) is satisfied, system (1.1) can be brought into the form

$$\begin{aligned} \frac{dx}{dt} &= y + \lambda\left(-b_{12}x^2y + a_{03}y^3 + a_{22}x^2y^2 + a_{04}y^4\right), \\ \frac{dy}{dt} &= -2x^3 + \lambda\left(b_{13}xy^3 + b_{12}xy^2\right), \end{aligned} \tag{3.7}$$

whose vector field is symmetric with respect to the y-axis.

When condition (3.5) holds, system (1.1) can be rewritten as

$$\begin{aligned} \frac{dx}{dt} &= y + \lambda\left(-3b_{03}xy^2 + a_{03}y^3 - \frac{3}{2}b_{13}x^2y^2 + a_{04}y^4 - b_{12}x^2y + a_{31}x^3y\right), \\ \frac{dy}{dt} &= -2x^3 + \lambda\left(b_{03}y^3 + b_{13}xy^3 + b_{04}y^4 - \frac{3}{2}a_{31}x^2y^2 + b_{40}x^4 + b_{12}x^2y\right), \end{aligned} \tag{3.8}$$

which has an analytic first integral:

$$\begin{aligned} H(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^4 + \lambda\bigg(&\frac{a_{03}}{4}y^4 + \frac{a_{04}}{5}y^5 - \frac{b_{40}}{5}x^5 - \frac{b_{12}}{2}x^2y^2 \\ &- \frac{a_{31}}{2}x^3y^2 - b_{03}xy^3 - \frac{b_{13}}{2}x^2y^3 - b_{04}xy^4\bigg). \end{aligned} \tag{3.9}$$

When condition (3.6) holds, system (1.1) becomes

$$\begin{aligned} \frac{dx}{dt} &= y - b_{12}x^2y - 3b_{03}xy^2 + a_{03}y^3, \\ \frac{dy}{dt} &= -2x^3 + b_{12}xy^2 + b_{03}y^3, \end{aligned} \tag{3.10}$$

which has an analytic first integral

$$H(x,y) = \frac{1}{2}y^2 + \frac{1}{2}x^4 + \lambda\left(-\frac{b_{12}}{2}x^2y^2 - b_{03}xy^3\right).$$

From Proposition 3.2 we have the following theorem.

Theorem 3.2 *The origin of system* (1.1) *is a center if and only if the first* 11 *quasi-Lyapunov constants are zero, that is, one of the conditions in Proposition* 3.2 *is satisfied.*

4 The first kind of multiple bifurcation of limit cycles

Now, we will prove that the perturbed system of (1.1) can generate 11 limit cycles enclosing an elementary node at the origin of unperturbed system (1.1) when the third-order nilpotent critical point $O(0,0)$ is a 11th-order weak focus.

Using the fact $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = \mu_8 = \mu_9 = \mu_{10} = 0$, $\mu_{11} \neq 0$, we obtain the following.

Theorem 4.1 *The origin of system* (1.1) *is a* 11*th-order weak focus if and only if* $a_{22}b_{03}b_{12} \times (2a_{22} + 3b_{13}) \neq 0$ *and*

$$\begin{aligned}
&b_{21} = b_{40} = a_{31} = b_{04} = 0, \\
&a_{12} = -3b_{03}, \qquad b_{13} = \frac{171a_{22}}{13}, \\
&a_{04} = \frac{2{,}593a_{22}b_{12}\lambda}{4{,}563}, \\
&a_{03} = -\frac{3{,}061b_{12}^2}{54{,}366}, \\
&b_{03}^2 = \frac{345{,}402{,}1b_{12}^3\lambda}{258{,}836{,}526{,}0}, \\
&a_{22}^2 = \frac{478{,}112{,}236{,}791{,}537{,}5b_{12}^4\lambda^2}{145{,}729{,}657{,}408{,}679{,}111{,}7}.
\end{aligned} \tag{4.1}$$

Proof Solving $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = \mu_8 = \mu_9 = \mu_{10} = 0$, we obtain the following relations:

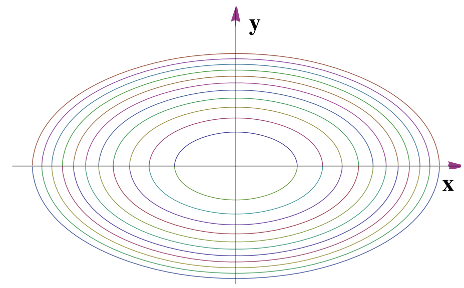
$$\begin{aligned}
&b_{21} = b_{40} = a_{31} = b_{04} = 0, \\
&a_{12} = -3b_{03}, \qquad b_{13} = \frac{171a_{22}}{13}, \\
&a_{04} = \frac{2{,}593a_{22}b_{12}\lambda}{4{,}563}, \\
&a_{03} = -\frac{3{,}061b_{12}^2}{54{,}366}, \\
&b_{03}^2 = \frac{345{,}402{,}1b_{12}^3\lambda}{258{,}836{,}526{,}0}, \\
&a_{22}^2 = \frac{478{,}112{,}236{,}791{,}537{,}5b_{12}^4\lambda^2}{145{,}729{,}657{,}408{,}679{,}111{,}7},
\end{aligned} \tag{4.2}$$

while $\mu_{11} \neq 0$ implies

$$a_{22}b_{03}b_{12} \neq 0.$$

So when condition in Theorem 4.1 holds, the origin of system (1.1) is a 11th-order weak focus. □

Figure 1 Phase portrait of system (4.3).



Next, we study the perturbed system of (1.1), given by

$$\begin{aligned}\frac{dx}{dt} &= \delta x + y + \lambda\bigl(-b_{12}(\varepsilon)x^2y + a_{12}(\varepsilon)xy^2 + a_{03}(\varepsilon)y^3 + a_{31}x^3y \\ &\quad + a_{22}(\varepsilon)x^2y^2 - 4b_{04}(\varepsilon)xy^3 + a_{04}(\varepsilon)y^4\bigr), \\ \frac{dy}{dt} &= \delta y - 2x^3 + \lambda\biggl(b_{21}(\varepsilon)x^2y + b_{12}(\varepsilon)xy^2 + b_{03}(\varepsilon)y^3 + b_{40}(\varepsilon)x^4 \\ &\quad - \frac{3}{2}a_{31}(\varepsilon)x^2y^2 + b_{13}(\varepsilon)xy^3 + b_{04}(\varepsilon)y^4\biggr).\end{aligned} \tag{4.3}$$

When the conditions in (4.2) hold, using the relationships $\mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = \mu_7 = \mu_8 = \mu_9 = \mu_{10} = 0$, we can determine the values of

$$b_{21}, \quad a_{12}, \quad b_{40}, \quad a_{31}, \quad b_{04}, \quad b_{13}, \quad a_{04}, \quad a_{03}, \quad b_{03}, \quad b_{12}.$$

Hence, when the conditions in Theorem 4.1 are satisfied, we have

$$\begin{aligned}&\frac{\partial(\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7,\mu_8,\mu_9,\mu_{10})}{\partial(b_{21},a_{12},b_{40},a_{31},b_{04},b_{13},a_{04},a_{03},b_{03},a_{22})} \\ &\quad = \frac{154{,}946{,}109{,}366{,}382{,}001{,}351{,}461{,}510{,}098{,}905{,}727{,}812{,}847{,}534{,}08a_{22}^{11}b_{03}^4b_{12}^6\lambda^6}{130{,}962{,}590{,}631{,}294{,}708{,}682{,}345{,}700{,}356{,}793{,}953{,}279{,}312{,}5} \\ &\quad \neq 0.\end{aligned} \tag{4.4}$$

Further, Theorem 3.1.3 in [15] shows that if the origin of system $(4.3)|_{\delta=\varepsilon=0}$ is a weak focus of order m, then, when $0 < \delta, \varepsilon \ll 1$, (4.3) has at most m limit cycles in a neighborhood of the origin. Namely the following theorem holds.

Theorem 4.2 *If the origin of system* (1.1) *is a* 11*th-order weak focus, for* $0 < \delta, \varepsilon \ll 1$, *then, for system* (4.3), *in a small neighborhood of the origin, there exist exactly* 11 *small-amplitude limit cycles enclosing the origin* $O(0,0)$, *which is an elementary node*; *see Figure* 1.

5 The second kind of multiple bifurcation of limit cycles

In this section, we consider an interesting bifurcation of limit cycles which is different from the first kind of bifurcation discussed in the previous section. Consider the following perturbed system of (1.1):

$$\frac{dx}{dt} = y + \lambda\bigl(-b_{12}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{31}x^3y + a_{22}x^2y^2 - 4b_{04}xy^3 + a_{04}y^4\bigr),$$

$$\begin{aligned}\frac{dy}{dt} &= 4\delta\varepsilon y - (x^2-\varepsilon^2)\left(2x\left(1-\frac{\lambda b_{40}}{2}x\right)-\lambda b_{21}y\right)\\ &\quad + \lambda\left(b_{12}xy^2 + b_{03}y^3 - \frac{3}{2}a_{31}x^2y^2 + b_{13}xy^3 + b_{04}y^4\right).\end{aligned} \tag{5.1}$$

System (5.1) is called a double perturbed system of system (1.1). When $0 < |\varepsilon| \ll 1$, system (5.1) has three real singular points in the neighborhood of the origin, namely $O(0,0)$ and $P_{1,2}(\pm\varepsilon;0)$.

By using the following transformation:

$$\begin{aligned}&x = \varepsilon(u\pm 1), \qquad y = 2\varepsilon^2\frac{\delta u - \rho v}{1\pm\varepsilon\lambda(\mp b_{21}\varepsilon + \lambda a_{31}\varepsilon^2)}, \qquad t = \frac{\tau}{2\rho\varepsilon},\\ &\rho = \sqrt{\left(1\pm\varepsilon(\mp b_{21}\varepsilon + \lambda a_{31}\varepsilon^2)\right)\left(1\pm\frac{1}{2}b_{40}\varepsilon\right) - \delta^2},\end{aligned}$$

we can shift $P_{1,2}(\pm\varepsilon,0)$ of system (5.1) to the origin and obtain a new system in the form of

$$\begin{aligned}\frac{d\xi}{d\tau} &= \Phi(\xi,\eta,\varepsilon,\delta) = \frac{\delta\xi}{\rho} - \eta + \sum_{k+j=2}^{\infty} A_{kj}(\varepsilon,\delta)\xi^k\eta^j,\\ \frac{d\eta}{d\tau} &= \Psi(\xi,\eta,\varepsilon,\delta) = \xi + \frac{\delta\eta}{\rho} + \sum_{k+j=2}^{\infty} B_{kj}(\varepsilon,\delta)\xi^k\eta^j,\end{aligned} \tag{5.2}$$

where $\Phi(\xi,\eta,\varepsilon,\delta)$ and $\Psi(\xi,\eta,\varepsilon,\delta)$ are power series in (u,v,ε,δ) with nonzero convergence radius. So $P_{1,2}(\pm\varepsilon,0)$ of (5.1) are fine foci when $\delta\neq 0$, and weak foci or centers when $\delta = 0$. Especially for $\delta = 0$, corresponding to $P_{1,2}(\pm\varepsilon,0)$, system (5.1) are changed into the following system:

$$\begin{aligned}\frac{du}{dt} &= -v + \frac{1}{2\varepsilon^2(2+b_{40}\varepsilon\lambda)}\left(-a_{22}\lambda u^2v^2 + (b_{12}+3a_{31}\varepsilon)\lambda u^2 v\right)\\ &\quad + \frac{1}{-1+b_{12}\varepsilon^2\lambda + a_{31}\varepsilon^3\lambda}\left(a_{04}\lambda v^4 - \varepsilon(a_{12}-a_{22}\varepsilon)\lambda v^2 + (a_{03}+4b_{04}\varepsilon)\lambda v^3\right)\\ &\quad + \frac{1}{\sqrt{2}(A)^{\frac{1}{2}}(-1+b_{12}\varepsilon^2\lambda + a_{31}\varepsilon^3\lambda)}\\ &\quad \times\left(-4b_{04}\lambda uv^3 - (-a_{12}+a_{22}\varepsilon)\lambda uv^2 + 2\varepsilon(2b_{12}+3a_{31}\varepsilon)\lambda uv\right)\\ &\quad + \frac{a_{31}\lambda}{2\sqrt{2}(A)^{\frac{3}{2}}(-1+b_{12}\varepsilon^2\lambda + a_{31}\varepsilon^3\lambda)}u^3v,\\ \frac{dv}{dt} &= u + \frac{1}{2\varepsilon^2(2+b_{40}\varepsilon\lambda)}\left(-b_{13}\lambda uv^3 + 2b_{21}\lambda\varepsilon uv - (b_{12}+3a_{31}\varepsilon)\lambda uv^2\right)\\ &\quad + \frac{1}{2\sqrt{2}\varepsilon^2(2+b_{40}\varepsilon\lambda)A^{\frac{1}{2}}}\left(-b_{21}u^2v\lambda + \frac{3}{2}a_{31}\lambda u^2v^2 + \varepsilon(6+5b_{40}\varepsilon\lambda)u^2\right)\\ &\quad + \frac{1}{2\sqrt{2}\sqrt{A}(-1+b_{12}\varepsilon^2\lambda + a_{31}\varepsilon^3\lambda)}\\ &\quad \times\left(2b_{04}\lambda v^4 - \varepsilon(2b_{12}+3a_{31}\varepsilon)\lambda v^2 - 2(-b_{03}+b_{13}\varepsilon)\lambda v^3\right)\\ &\quad - \frac{(1+2b_{40}\varepsilon\lambda)(-1+b_{12}\varepsilon^2\lambda + a_{31}\varepsilon^3\lambda)}{2\varepsilon^4(2+b_{40}\varepsilon\lambda)^2}u^3 + \frac{b_{40}\lambda(-1+b_{12}\varepsilon^2\lambda + a_{31}\varepsilon^3\lambda)}{4\sqrt{2}\varepsilon^4(2+b_{40}\varepsilon\lambda)^2A^{\frac{1}{2}}}u^4.\end{aligned} \tag{5.3}$$

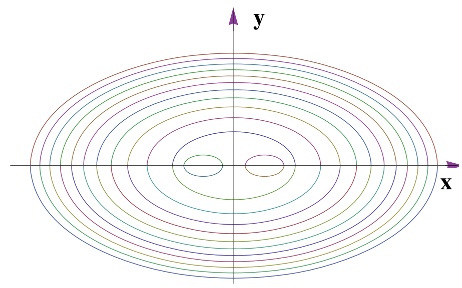


Figure 2 Phase portrait of system (5.1).

The first Lyapunov constant at the origin for system (5.3) is given by

$$\begin{aligned}V_1 &= -i\lambda\bigl(-2\varepsilon^2(a_{12} + 3b_{03} - 2a_{22}\varepsilon - 3b_{13}\varepsilon)(2 + b_{40}\varepsilon\lambda)^2 \\ &\quad + b_{21}\bigl(4 - \varepsilon\lambda\bigl(2\varepsilon(4b_{12} + 5a_{31}\varepsilon) + b_{40}\bigl(-4 + \varepsilon^2(6b_{12} + 7a_{31}\varepsilon)\lambda\bigr)\bigr)\bigr)\bigr).\end{aligned}$$

When the origin of system (1.1) is a 11th-order weak focus, the first Lyapunov constant of system (5.3) at the origin is

$$V_1 = 2(2a_{22} + 3b_{13})\varepsilon^3(2 + b40\varepsilon\lambda)^2 \neq 0,$$

when $\varepsilon \to 0$.

Similarly, summarizing the above results yields the following theorem.

Theorem 5.1 *If the origin of system* (1.1) *is a* 11*-order weak focus, choosing proper coefficients in system* (1.1), *when* $0 < |\varepsilon| \ll 1$, *there exist* 12 *limit cycles with the distribution of one limit cycle enclosing each of* $P_{1,2}(\pm\varepsilon, 0)$, *and ten limit cycles enclosing both* $(\varepsilon, 0)$ *and* $(-\varepsilon, 0)$ *in the neighborhood of origin*; *see Figure* 2.

The following result is easy to obtain from the above discussion.

Theorem 5.2 *If* $\delta = 0$, $b_{21} = 0$, $a_{12} = -3b_{03}$, $a_{22} = -\frac{3b_{13}}{2}$, *there exist three centers* $(0, 0)$ *and* $P(\pm\varepsilon, 0)$ *in* (5.1).

We have studied an interesting bifurcation which, different from the first kind of bifurcation, can generate 12 limit cycles by perturbing the quartic system with a nilpotent critical point.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgements
Jianlong Qiu and Feng Li are supported by the National Natural Science Foundation of China (Grant Nos. 11201211, 61273012, 11371373) and AMEP of Linyi University.

Received: 29 August 2014 Accepted: 7 January 2015 Published online: 31 January 2015

References

1. Gasull, A, Torregrosa, J: A new algorithm for the computation of the Lyapunov constans for some degenerated critical points. Proceedings of the Third World Congress of Nonlinear Analysts. Nonlinear Anal. **47**, 4479-4490 (2001)
2. Álvarez, MJ, Gasull, A: Monodromy and stability for nilpotent critical points. Int. J. Bifurc. Chaos **15**, 1253-1265 (2005)
3. Chavarriga, J, García, I, Giné, J: Integrability of centers perturbed by quasi-homogeneous polynomials. J. Math. Anal. Appl. **211**, 268-278 (1997)
4. Bautin, N: On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type. Mat. Sb. (N.S.) **30**, 181-196 (1952)
5. James, EM, Lloyd, NG: A cubic system with eight small-amplitude limit cycles. IMA J. Appl. Math. **47**, 163-171 (1991)
6. Ning, S, Ma, S, Kwek, KH, Zheng, Z: A cubic system with eight small-amplitude limit cycles. Appl. Math. Lett. **7**, 23-27 (1994)
7. Yu, P, Corless, R: Symbolic computation of limit cycles associated with Hilbert's 16th problem. Commun. Nonlinear Sci. Numer. Simul. **14**, 4041-4056 (2009)
8. Chen, C, Corless, R, Maza, M, Yu, P, Zhang, Y: A modular regular chains method and its application to dynamical system. Int. J. Bifurc. Chaos **23**, 1350154 (2013)
9. Lloyd, N, Pearson, J: A cubic differential system with nine limit cycles. J. Appl. Anal. Comput. **2**, 293-304 (2012)
10. Yu, P, Tian, Y: Twelve limit cycles around a singular point in a planar cubic-degree polynomial system. Commun. Nonlinear Sci. Numer. Simul. **19**, 2690-2705 (2014)
11. Huang, WT, Chen, AY: Bifurcation of limit cycles and isochronous centers for a quartic system. Int. J. Bifurc. Chaos **23**, 1350172 (2013)
12. Li, F, Liu, YR, Li, HW: Center conditions and bifurcation of limit cycles at three-order nilpotent critical point in a septic Lyapunov system. Math. Comput. Simul. **12**, 2595-2607 (2011)
13. Li, F, Wang, M: Bifurcation of limit cycles in a quintic system with ten parameters. Nonlinear Dyn. **71**, 213-222 (2013)
14. Liu, Y, Li, J: On third-order nilpotent critical points: integral factor method. Int. J. Bifurc. Chaos **21**, 1293-1309 (2011)
15. Liu, Y, Li, J: Some Classical Problems About Planar Vector Fields. Science Press, Beijing (2010) (in Chinese)