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Asymptotic behavior for a viscoelastic Kirchhoff equation with distributed delay and Balakrishnan–Taylor damping

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Abstract

A nonlinear viscoelastic Kirchhoff-type equation with Balakrishnan–Taylor damping and distributed delay is studied. By the energy method we establish the general decay rate under suitable hypothesis.

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Keywords: Kirchhoff equation; Exponential decay; Distributed delay term; Viscoelastic term; Energy method

1 Introduction

Let $\mathcal{H} = \Omega \times (\tau_1, \tau_2) \times (0, \infty)$, in the present work, we consider the following Kirchhoff equation:

$$\begin{cases} |u_t|^p u_{tt} - (\zeta_0 + \zeta_1 \|\nabla u\|_2^2 + \sigma(\nabla u, \nabla u_t)_{L^2(\Omega)})\Delta u(t) - \Delta u_{tt}(t) \\ \quad + \alpha(t) \int_0^t h(t-\varrho)\Delta u(\varrho)\, d\varrho + \beta_1 |u_t(t)|^{m-2} u_t(t) \\ \quad + \int_{\tau_1}^{\tau_2} |\beta_2(s)| |u_t(t-s)|^{m-2} u_t(t-s)\, ds = 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad \text{in } \Omega, \\ u_t(x,-t) = f_0(x,t), \quad \text{in } \Omega \times (0,\tau_2), \\ u(x,t) = 0, \quad \text{in } \partial\Omega \times (0,\infty), \end{cases} \tag{1.1}$$

where $\Omega \in \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. $\zeta_0, \zeta_1, \sigma, \beta_1$ are positive constants, $p \geq 0$ for $N = 1, 2$, and $0 \leq p \leq \frac{4}{N-2}$ for $N \geq 3$, and $m \geq 1$ for $N = 1, 2$, and $1 < m \leq \frac{N+2}{N-2}$ for $N \geq 3$. $\tau_1 < \tau_2$ are nonnegative constants such that $\beta_2 : [\tau_1, \tau_2] \to \mathbb{R}$ represents distributive time delay, h, α are positive functions.

Physically, the relationship between the stress and strain history in the beam inspired by Boltzmann theory is called viscoelastic damping term, where the kernel of the term of memory is the function h. See [4–6, 9–11, 13–18, 22, 29, 31, 32, 34, 35]. It has been studied by many authors, especially in Kirchhoff's equations (see [8, 10, 19–21, 23–26, 30, 33]).

In [2], Balakrishnan and Taylor proposed a new model of damping called the Balakrishnan–Taylor damping, as it relates to the span problem and the plate equation. For more depth, here are some papers that focused on the study of this damping: [2, 3, 8, 10, 16, 18, 20, 27, 35].

The effect of the delay often appears in many applications and practical problems and turns a lot of systems into different problems worth studying. Recently, the stability and the asymptotic behavior of evolution systems with time delay, especially the distributed delay effect, have been studied by many authors. See [7, 10–12, 14, 28].

Based on all of the above, we believe that the combination of these terms of damping (memory term, Balakrishnan–Taylor damping, and the distributed delay) in one particular problem with the addition of $\alpha(t)$ to the term of memory and the distributed delay term $(\int_{\tau_1}^{\tau_2} |\beta_2(s)||u_t(t-s)|^{m-2}u_t(t-s)\,ds)$ constitutes a new problem worthy of study and research, different from the above that we will try to shed light on.

Our paper is divided into several sections: in the next section we lay down the hypotheses, concepts, and lemmas we need, and in the last section we prove our main result.

2 Preliminaries

For studying our problem, in this section we will need some materials.

Firstly, we introduce the following hypotheses for β_2, h, and α:

(A1) $h, \alpha : \mathbb{R}_+ \to \mathbb{R}_+$ are nonincreasing C^1 functions satisfying

$$\begin{aligned} &h(t) > 0, \qquad \alpha(t) > 0, \qquad l_0 = \int_0^\infty h(\varrho)\,d\varrho < \infty, \\ &\zeta_0 - 2\alpha(t)\int_0^t h(\varrho)\,d\varrho \geq l > 0, \end{aligned} \tag{2.1}$$

where $l = 1 - l_0$.

(A2) $\exists \vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing C^1 function satisfying

$$\vartheta(t)h(t) + h'(t) \leq 0, \quad t \geq 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{-\alpha'(t)}{\vartheta(t)\alpha(t)} = 0. \tag{2.2}$$

(A3) $\beta_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} \bigl|\beta_2(s)\bigr|\,ds < \beta_1. \tag{2.3}$$

Let us introduce

$$(h \circ \psi)(t) := \int_\Omega \int_0^t h(t-\varrho)\bigl|\psi(t) - \psi(\varrho)\bigr|^2\,d\varrho\,dx$$

and

$$M(t) := \bigl(\zeta_0 + \zeta_1\|\nabla u\|_2^2 + \sigma\bigl(\nabla u(t), \nabla u_t(t)\bigr)_{L^2(\Omega)}\bigr).$$

Lemma 2.1 (Sobolev–Poincare inequality [1]) *Let* $2 \leq q < \infty$ $(n = 1, 2)$ *or* $2 \leq q < \frac{2n}{n-2}$ $(n \geq 3)$. *Then* $\exists c_* = c(\Omega, q) > 0$ *such that*

$$\|u\|_q \leq c_*\|\nabla u\|_2, \quad \forall u \in H_0^1(\Omega).$$

As in [28], we take the following new variables:

$$y(x,\rho,s,t)=u_t(x,t-s\rho)$$

which satisfy

$$\begin{cases} sy_t(x,\rho,s,t)+y_\rho(x,\rho,s,t)=0,\\ y(x,0,s,t)=u_t(x,t). \end{cases} \tag{2.4}$$

So, problem (1.1) can be written as

$$\begin{cases} |u_t|^p u_{tt}-(\zeta_0+\zeta_1\|\nabla u\|_2^2+\sigma(\nabla u,\nabla u_t)_{L^2(\Omega)})\Delta u(t)+\alpha(t)\int_0^t h(t-\varrho)\Delta u(\varrho)\,d\varrho\\ \quad -\Delta u_{tt}(t)+\beta_1|u_t(t)|^{m-2}u_t(t)+\int_{\tau_1}^{\tau_2}|\beta_2(s)||y(x,1,s,t)|^{m-2}y(x,1,s,t)\,ds=0,\\ sy_t(x,\rho,s,t)+y_\rho(x,\rho,s,t)=0,\\ u(x,0)=u_0(x),\qquad u_t(x,0)=u_1(x),\quad \text{in }\Omega,\\ y(x,\rho,s,0)=f_0(x,\rho s),\quad \text{in }\Omega\times(0,1)\times(0,\tau_2),\\ u(x,t)=0,\quad \text{in }\partial\Omega\times(0,\infty), \end{cases} \tag{2.5}$$

where

$$(x,\rho,s,t)\in\Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Now, we give the energy functional.

Lemma 2.2 *The energy functional E, defined by*

$$\begin{aligned} E(t)={}&\frac{1}{p+2}\|u_t\|_{p+2}^{p+2}+\frac{1}{2}\left(\zeta_0-\alpha(t)\int_0^t h(\varrho)\,d\varrho\right)\|\nabla u(t)\|_2^2\\ &+\frac{1}{2}\|\nabla u_t(t)\|_2^2+\frac{\zeta_1}{4}\|\nabla u(t)\|_2^4+\frac{\alpha(t)}{2}(h\circ\nabla u)(t)\\ &+\frac{m-1}{m}\int_0^1\int_{\tau_1}^{\tau_2}s|\beta_2(s)|\|y(x,\rho,s,t)\|_m^m\,ds\,d\rho, \end{aligned} \tag{2.6}$$

satisfies

$$\begin{aligned} E'(t)\le{}&-\eta_0\|u_t(t)\|_m^m+\frac{\alpha(t)}{2}(h'\circ\nabla u)(t)\\ &-\frac{\alpha'(t)}{2}\left(\int_0^t h(\varrho)\,d\varrho\right)\|\nabla u(t)\|_2^2-\frac{\sigma}{4}\left(\frac{d}{dt}\{\|\nabla u(t)\|_2^2\}\right)^2, \end{aligned} \tag{2.7}$$

where $\eta_0=\beta_1-\int_{\tau_1}^{\tau_2}|\beta_2(s)|\,ds>0$.

Proof Taking the inner product of $(2.5)_1$ with u_t, then integrating over Ω, we find

$$\big(|u_t|^p u_{tt}(t),u_t(t)\big)_{L^2(\Omega)}-\big(M(t)\Delta u(t),u_t(t)\big)_{L^2(\Omega)}-\big(\Delta u_{tt}(t),u_t(t)\big)_{L^2(\Omega)}$$

$$+\left(\alpha(t)\int_0^t h(t-\varrho)\Delta u(\varrho)\,d\varrho, u_t(t)\right)_{L^2(\Omega)} + \beta_1\big(|u_t|^{m-2}u_t, u_t\big)_{L^2(\Omega)}$$
$$+\int_{\tau_1}^{\tau_2}\big|\beta_2(s)\big|\big(\big|y(x,1,s,t)\big|^{m-2}y(x,1,s,t), u_t(t)\big)_{L^2(\Omega)}\,ds = 0. \tag{2.8}$$

By computation, integration by parts, and the last condition in (2.5), we get

$$\begin{aligned}\big(|u_t|^p u_{tt}(t), u_t(t)\big)_{L^2(\Omega)} &= \int_\Omega u_t(t).|u_t|^p u_{tt}(t)\,dx\\ &= \frac{1}{p+2}\frac{d}{dt}\big(\|u_t(t)\|_{p+2}^{p+2}\big),\end{aligned} \tag{2.9}$$

$$\begin{aligned}-\big(\Delta u_{tt}(t), u_t(t)\big)_{L^2(\Omega)} &= \int_\Omega \nabla u_t(t)\nabla u_{tt}(t)\,dx\\ &= \frac{1}{2}\frac{d}{dt}\big(\|\nabla u_t(t)\|_2^2\big).\end{aligned} \tag{2.10}$$

By integration by parts, we find

$$\begin{aligned}&-\big(M(t)\Delta u(t), u_t(t)\big)_{L^2(\Omega)}\\ &\quad= -\Big(\big(\zeta_0+\zeta_1\|\nabla u\|_2^2+\sigma\big(\nabla u(t),\nabla u_t(t)\big)_{L^2(\Omega)}\big)\Delta u(t), u_t(t)\Big)_{L^2(\Omega)}\\ &\quad= \big(\zeta_0+\zeta_1\|\nabla u\|_2^2+\sigma\big(\nabla u(t),\nabla u_t(t)\big)_{L^2(\Omega)}\big)\int_\Omega \nabla u(t).\nabla u_t(t)\,dx\\ &\quad= \big(\zeta_0+\zeta_1\|\nabla u\|_2^2+\sigma\big(\nabla u(t),\nabla u_t(t)\big)_{L^2(\Omega)}\big)\frac{d}{dt}\left\{\int_\Omega \big|\nabla u(t)\big|^2\,dx\right\}\\ &\quad= \frac{d}{dt}\left\{\frac{1}{2}\left(\zeta_0+\frac{\zeta_1}{2}\|\nabla u\|_2^2\right)\big\|\nabla u(t)\big\|_2^2\right\} + \frac{\sigma}{4}\frac{d}{dt}\big\{\big\|\nabla u(t)\big\|_2^2\big\}^2,\end{aligned} \tag{2.11}$$

and we have

$$\begin{aligned}&\left(\int_0^t h(t-\varrho)\Delta u(\varrho)\,d\varrho, u_t(t)\right)_{L^2(\Omega)}\\ &\quad= \int_0^t h(t-\varrho)\big(\Delta u(\varrho), u_t(t)\big)_{L^2(\Omega)}\,d\varrho\\ &\quad= -\int_0^t h(t-\varrho)\left[\int_\Omega \nabla u(x,\varrho)\nabla u(x,t)\,dx\right]d\varrho,\end{aligned} \tag{2.12}$$

and

$$-\nabla u(x,\varrho).\nabla u(x,t) = \frac{1}{2}\frac{d}{dt}\big\{\big|\nabla u(x,\varrho)-\nabla u(x,t)(t)\big|^2\big\} - \frac{1}{2}\frac{d}{dt}\big\{\big|\nabla u(x,t)\big|^2\big\}, \tag{2.13}$$

then

$$\begin{aligned}&-\int_0^t h(t-\varrho)\big(\nabla u(\varrho),\nabla u_t(t)\big)_{L^2(\Omega)}\,d\varrho\\ &\quad= -\int_0^t h(t-\varrho)\int_\Omega\left[\frac{1}{2}\frac{d}{dt}\big\{\big|\nabla u(x,\varrho)-\nabla u(x,t)\big|^2\big\}\right]dx\,ds.\end{aligned}$$

$$\begin{aligned}
&-\int_0^t h(t-\varrho)\int_\Omega\left[\frac{1}{2}\frac{d}{dt}\left\{\left|\nabla u(x,t)\right|^2\right\}\right]dx\,d\varrho \\
&=\frac{1}{2}\int_0^t h(t-\varrho)\left[\frac{d}{dt}\left\{\int_\Omega\left|\nabla u(x,t)-\nabla u(x,\varrho)\right|^2dx\right\}\right]d\varrho \\
&\quad-\frac{1}{2}\int_0^t h(t-\varrho)\left[\frac{d}{dt}\left\{\left\|\nabla u(x,t)\right\|_2^2\right\}\right]dx\,d\varrho. \qquad (2.14)
\end{aligned}$$

By (2.1), we get

$$\begin{aligned}
&\frac{1}{2}\int_0^t h(t-\varrho)\left[\frac{d}{dt}\left\{\int_\Omega\left|\nabla u(x,t)-\nabla u(x,\varrho)\right|^2dx\right\}\right]d\varrho \\
&=\frac{1}{2}\frac{d}{dt}\left\{\int_0^t h(t-\varrho)\left[\int_\Omega\left|\nabla u(x,t)-\nabla u(x,\varrho)\right|^2dx\right]\right\}d\varrho \\
&\quad-\frac{1}{2}\int_0^t h'(t-\varrho)\left[\int_\Omega\left|\nabla u(x,t)-\nabla u(x,\varrho)\right|^2dx\right]d\varrho \\
&=\frac{1}{2}\frac{d}{dt}(h\circ\nabla u)(t)-\frac{1}{2}\left(h'\circ\nabla u\right)(t), \qquad (2.15)
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{1}{2}\int_0^t h(t-\varrho)\left[\frac{d}{dt}\left\{\left\|\nabla u(t)\right\|_2^2\right\}\right]dx\,d\varrho \\
&=-\frac{1}{2}\left(\int_0^t h(t-\varrho)\,d\varrho\right)\left(\frac{d}{dt}\left\{\left\|\nabla u(t)\right\|_2^2\right\}\right)dx \\
&=-\frac{1}{2}\left(\int_0^t h(\varrho)\,d\varrho\right)\left(\frac{d}{dt}\left\{\left\|\nabla u(t)\right\|_2^2\right\}\right)dx \qquad (2.16)\\
&=-\frac{1}{2}\frac{d}{dt}\left\{\left(\int_0^t h(\varrho)\,d\varrho\right)\left\|\nabla u(t)\right\|_2^2\right\}+\frac{1}{2}h(t)\left\|\nabla u(t)\right\|_2^2.
\end{aligned}$$

Inserting (2.15) and (2.16) into (2.14) gives

$$\begin{aligned}
&\left(\alpha(t)\int_0^t h(t-\varrho)\Delta u(\varrho)\,d\varrho,u_t(t)\right)_{L^2(\Omega)} \\
&=\frac{d}{dt}\left\{\frac{\alpha(t)}{2}(h\circ\nabla u)(t)-\frac{\alpha(t)}{2}\left(\int_0^t h(\varrho)\,d\varrho\right)\left\|\nabla u(t)\right\|_2^2\right\} \\
&\quad-\frac{\alpha(t)}{2}\left(h'\circ\nabla u\right)(t)+\frac{\alpha(t)}{2}h(t)\left\|\nabla u(t)\right\|_2^2 \\
&\quad-\frac{\alpha'(t)}{2}(h\circ\nabla u)(t)+\frac{\alpha'(t)}{2}\left(\int_0^t h(\varrho)\,d\varrho\right)\left\|\nabla u(t)\right\|_2^2. \qquad (2.17)
\end{aligned}$$

Now, multiplying equation $(2.5)_2$ by $-y|\beta_2(s)|$, integrating over $\Omega\times(0,1)\times(\tau_1,\tau_2)$, and using $(2.4)_2$, we get

$$\begin{aligned}
&\frac{d}{dt}\frac{m-1}{m}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}s\left|\beta_2(s)\right|.\left|y(x,\rho,s,t)\right|^m ds\,d\rho\,dx \\
&=-(m-1)\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\left|\beta_2(s)\right|.|y|^{m-1}y_\rho\,ds\,d\rho\,dx
\end{aligned}$$

$$\begin{aligned}
&= -\frac{m-1}{m}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}\left|\beta_2(\varrho)\right|\frac{d}{d\rho}\left|y(x,\rho,s,t)\right|^m\,ds\,d\rho\,dx \\
&= \frac{m-1}{m}\int_\Omega\int_{\tau_1}^{\tau_2}\left|\beta_2(s)\right|\left(\left|y(x,0,s,t)\right|^m-\left|y(x,1,s,t)\right|^m\right)ds\,dx \\
&= \frac{m-1}{m}\left(\int_{\tau_1}^{\tau_2}\left|\beta_2(s)\right|ds\right)\int_\Omega\left|u_t(t)\right|^m dx \\
&\quad -\frac{m-1}{m}\int_\Omega\int_{\tau_1}^{\tau_2}\left|\beta_2(s)\right|.\left|y(x,1,s,t)\right|^m\,ds\,dx \\
&= \frac{m-1}{m}\left(\int_{\tau_1}^{\tau_2}\left|\beta_2(s)\right|ds\right)\left\|u_t(t)\right\|_m^m \\
&\quad -\frac{m-1}{m}\int_{\tau_1}^{\tau_2}\left|\beta_2(s)\right|\left\|y(x,1,s,t)\right\|_m^m\,ds. \qquad (2.18)
\end{aligned}$$

By Young's inequality, we have

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\left|\beta_2(s)\right|\left(\left|y(x,1,s,t)\right|^{m-2}y(x,1,s,t),u_t(t)\right)_{L^2(\Omega)}ds \qquad (2.19)\\
&\quad\le\frac{1}{m}\left(\int_{\tau_1}^{\tau_2}\left|\beta_2(s)\right|ds\right)\left\|u_t(t)\right\|_m^m+\frac{m-1}{m}\int_{\tau_1}^{\tau_2}\left|\beta_2(s)\right|\left\|y(x,1,s,t)\right\|_m^m\,ds.
\end{aligned}$$

By inserting (2.9)–(2.11) and (2.17)–(2.19) into (2.8), we find (2.6) and (2.7).

Hence, by (2.2), we get the function E is nonincreasing $\forall t\ge t_1$. This completes of the proof. □

Now we state the local existence of problem (2.5), whose proof can be found in [23, 24].

Theorem 2.3 *Suppose that* (2.1)–(2.3) *are satisfied. Then, for any* $u_0,u_1\in H_0^1(\Omega)\cap L^2(\Omega)$, *and* $f_0\in L^2(\Omega,(0,1),(\tau_1,\tau_2))$, *there exists a weak solution u of problem* (2.5) *such that*

$$\begin{aligned}
&u\in C\left(]0,T[,H_0^1(\Omega)\right)\cap C^1\left(]0,T[,L^2(\Omega)\right),\\
&u_t\in C\left(]0,T[,H_0^1(\Omega)\right)\cap L^2\left(]0,T[,L^2\left(\Omega,(0,1),(\tau_1,\tau_2)\right)\right).
\end{aligned}$$

3 General decay

In this section, we state and prove the asymptotic behavior of system (2.5). For this goal, we set

$$\Psi(t):=\frac{1}{p+1}\int_\Omega u(t)|u_t|^p u_t(t)\,dx+\frac{\sigma}{4}\left\|\nabla u(t)\right\|_2^4+\int_\Omega\nabla u(t)\nabla u_t(t)\,dx \qquad (3.1)$$

and

$$\Phi(t):=\int_\Omega\left(\Delta u_t-\frac{1}{p+1}|u_t|^p u_t\right)\int_0^t h(t-\varrho)\left(u(t)-u(\varrho)\right)d\varrho\,dx, \qquad (3.2)$$

and

$$\Theta(t):=\int_0^1\int_{\tau_1}^{\tau_2}se^{-\rho s}\left|\beta_2(s)\right|.\left\|y(x,\rho,s,t)\right\|_m^m\,ds\,d\rho. \qquad (3.3)$$

First, since the function h is positive and continuous, for all $t_0 > 0$, we have

$$\int_0^t h(\varrho)\,d\varrho \geq \int_0^{t_0} h(\varrho)\,d\varrho := h_0, \quad \forall t \geq t_0.$$

Lemma 3.1 *The functional* $\Psi(t)$ *defined in* (3.1) *satisfies, for any* $\varepsilon > 0$,

$$\Psi'(t) \leq \frac{1}{p+1}\|u_t\|_{p+2}^{p+2} - \big(l - \varepsilon(c_1 + c_2)\big)\|\nabla u\|_2^2 - \zeta_1\|\nabla u\|_2^4 + \frac{\alpha(t)}{4}(h \circ \nabla u)(t) + \|\nabla u_t\|_2^2 + c(\varepsilon)\left(\|u_t\|_m^m + \int_{\tau_1}^{\tau_2} |\beta_2(s)| \|y(x,1,s,t)\|_m^m\,ds\right). \tag{3.4}$$

Proof A differentiation of (3.1) and using $(2.5)_1$ give

$$\begin{aligned}
\Psi'(t) &= \frac{1}{p+1}\|u_t\|_{p+2}^{p+2} + \int_\Omega |u_t|^p u_{tt} u\,dx + \sigma\|\nabla u\|_2^2 \int_\Omega \nabla u_t \nabla u\,dx \\
&\quad + \int_\Omega \nabla u(t)\nabla u_{tt}(t)\,dx + \|\nabla u_t\|_2^2 \\
&= \frac{1}{p+1}\|u_t\|_{p+2}^{p+2} - \zeta_0\|\nabla u\|_2^2 - \zeta_1\|\nabla u\|_2^4 \underbrace{-\beta_1 \int_\Omega |u_t|^{m-2}u_t u\,dx}_{J_1} \\
&\quad \underbrace{+\alpha(t)\int_\Omega \nabla u(t)\int_0^t h(t-\varrho)\nabla u(\varrho)\,d\varrho\,dx}_{J_2} + \|\nabla u_t\|_2^2 \\
&\quad \underbrace{-\int_\Omega \int_{\tau_1}^{\tau_2} |\beta_2(s)| \big|y(x,1,s,t)\big|^{m-2} y(x,1,s,t).u\,ds\,dx}_{J_3}.
\end{aligned} \tag{3.5}$$

We estimate the last three terms of the RHS of (3.5). Applying Hölder's, Sobolev–Poincare, and Young's inequalities, (2.1) and (2.6), we find

$$\begin{aligned}
J_1 &\leq \varepsilon\beta_1^m\|u\|_m^m + c(\varepsilon)\|u_t\|_m^m \\
&\leq \varepsilon\beta_1^m c_p^m\|\nabla u\|_2^m + c(\varepsilon)\|u_t\|_m^m \\
&\leq \varepsilon\beta_1^m c_p^m\left(\frac{E(0)}{l}\right)^{(m-2)/2}\|\nabla u\|_2^2 + c(\varepsilon)\|u_t\|_m^m \\
&\leq \varepsilon c_1\|\nabla u\|_2^2 + c(\varepsilon)\|u_t\|_m^m
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
J_2 &\leq 2\alpha(t)\left(\int_0^t h(\varrho)\,d\varrho\right)\|\nabla u\|_2^2 + \frac{\alpha(t)}{4}(h \circ \nabla u)(t) \\
&\leq (\zeta_0 - l)\|\nabla u\|_2^2 + \frac{\alpha(t)}{4}(h \circ \nabla u)(t).
\end{aligned} \tag{3.7}$$

Similar to J_1, we have

$$J_3 \leq \varepsilon c_2\|\nabla u\|_2^2 + c(\varepsilon)\int_{\tau_1}^{\tau_2} |\beta_2(s)|.\|y(x,1,s,t)\|_m^m\,ds. \tag{3.8}$$

Combining (3.6)–(3.8) and (3.5), we get

$$\Psi'(t) \le \frac{1}{p+1}\|u_t\|_{p+2}^{p+2} - \big(l - \varepsilon(c_1+c_2)\big)\|\nabla u\|_2^2 - \zeta_1\|\nabla u\|_2^4 + \|\nabla u_t\|_2^2 + \frac{\alpha(t)}{4}(h\circ\nabla u)(t) + c(\varepsilon)\left(\|u_t\|_m^m + \int_{\tau_1}^{\tau_2}|\beta_2(s)|\,\|y(x,1,s,t)\|_m^m\,ds\right). \qquad \square$$

Lemma 3.2 *The functional* $\Phi(t)$ *defined in* (3.2) *satisfies, for any* $\delta > 0$,

$$\begin{aligned}\Phi'(t) \le{}& -\frac{1}{p+1}\left(\int_0^t h(\varrho)\,d\varrho\right)\|u_t\|_{p+2}^{p+2} + \delta\big(\zeta_0 + 2h_0^2\alpha(t)\big)\|\nabla u\|_2^2\\ &+ \zeta_1\delta\|\nabla u\|_2^4 + \delta\frac{\sigma E(0)}{l}\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2\\ &+ \left(c(\delta) + \left(2\delta + \frac{1}{4\delta}\right)c\alpha(t)\right)(h\circ\nabla u)(t)\\ &+ c(\delta)\left(\|u_t\|_m^m + \int_{\tau_1}^{\tau_2}|\beta_2(\varrho)|\,\|y(x,1,s,t)\|_m^m\,ds\right)\\ &+ \left(\delta_1\big(1 + c\big(E(0)\big)^p\big) - \int_0^t h(\varrho)\,d\varrho\right)\|\nabla u_t\|_2^2\\ &- \left(\frac{h(0)c_p^2}{4\delta_1} + c(\delta_1)\right)\big(h'\circ\nabla u\big)(t).\end{aligned} \tag{3.9}$$

Proof A differentiation of (3.2) and using $(2.5)_1$ give

$$\begin{aligned}\Phi'(t) ={}& \int_\Omega\big(\Delta u_{tt} - u_{tt}|u_t|^p\big)\int_0^t h(t-\varrho)\big(u(t)-u(\varrho)\big)\,d\varrho\,dx\\ &+ \int_\Omega\left(\Delta u_t - \frac{1}{p+1}|u_t|^p u_t\right)\int_0^t h'(t-\varrho)\big(u(t)-u(\varrho)\big)\,d\varrho\,dx\\ &- \frac{1}{p+1}\left(\int_0^t h(\varrho)\,d\varrho\right)\|u_t\|_{p+2}^{p+2} - \left(\int_0^t h(\varrho)\,d\varrho\right)\|\nabla u_t\|_2^2\\ ={}& \underbrace{\big(\zeta_0 + \zeta_1\|\nabla u\|_2^2\big)\int_\Omega \nabla u\int_0^t h(t-\varrho)\big(\nabla u(t) - \nabla u(\varrho)\big)\,d\varrho\,dx}_{J_1}\\ &\underbrace{+\sigma\int_\Omega \nabla u\nabla u_t\,dx.\int_\Omega\nabla u\int_0^t h(t-\varrho)\big(\nabla u(t)-\nabla u(\varrho)\big)\,d\varrho\,dx}_{J_2}\\ &\underbrace{-\alpha(t)\int_\Omega\left(\int_0^t h(t-\varrho)\nabla u(\varrho)\,d\varrho\right).\left(\int_0^t h(t-\varrho)\big(\nabla u(t)-\nabla u(\varrho)\big)\,d\varrho\right)dx}_{J_3}\\ &\underbrace{-\beta_1\int_\Omega |u_t|^{m-2}u_t\left(\int_0^t h(t-\varrho)\big(u(t)-u(\varrho)\big)\,d\varrho\right)dx}_{J_4}\\ &-\int_\Omega\int_{\tau_1}^{\tau_2}|\beta_2(s)|\big|y(x,1,s,t)\big|^{m-2}y(x,1,s,t)\end{aligned}$$

$$\underbrace{\times \left(\int_0^t h(t-\varrho)\big(u(t)-u(\varrho)\big)\,d\varrho \right) ds\,dx}_{J_5}$$

$$\underbrace{-\frac{1}{p+1}\int_\Omega |u_t|^p u_t \int_0^t h'(t-\varrho)\big(u(t)-u(\varrho)\big)\,d\varrho\,dx}_{J_6}$$

$$\underbrace{-\int_\Omega \nabla u_t \int_0^t h'(t-\varrho)\big(\nabla u(t)-\nabla u(\varrho)\big)\,d\varrho\,dx}_{J_7}$$

$$-\frac{1}{p+1}\left(\int_0^t h(\varrho)\,d\varrho\right)\|u_t\|_{p+2}^{p+2} - \left(\int_0^t h(\varrho)\,d\varrho\right)\|\nabla u_t\|_2^2. \tag{3.10}$$

By estimating the terms J_i, $i = 1,\dots,7$, of the RHS of (3.10), exploiting Hölder's, Sobolev–Poincare, and Young's inequalities, (2.1) and (2.6), we find

$$\begin{aligned} |J_1| &\le \big(\zeta_0 + \zeta_1\|\nabla u\|_2^2\big)\left(\delta\|\nabla u\|_2^2 + \frac{c}{4\delta}(h\circ\nabla u)(t)\right) \\ &\le \delta\zeta_0\|\nabla u\|_2^2 + \delta\zeta_1\|\nabla u\|_2^4 + \left(\frac{\zeta_0 c}{4\delta} + \frac{\zeta_1 cE(0)}{4l\delta}\right)(h\circ\nabla u)(t) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} J_2 &\le \delta\sigma\left(\int_\Omega \nabla u\nabla u_t\,dx\right)^2\|\nabla u\|_2^2 + \frac{\sigma c}{4\delta}(h\circ\nabla u)(t) \\ &\le \delta\frac{\sigma E(0)}{l}\left(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\right)^2 + \frac{\sigma c}{4\delta}(h\circ\nabla u)(t), \end{aligned} \tag{3.12}$$

$$\begin{aligned} |J_3| &\le \delta\alpha(t)\int_\Omega\left(\int_0^t h(t-\varrho)\big(\big|\nabla u(t)-\nabla u(\varrho)\big| - \nabla\big|u(t)\big|\big)\,d\varrho\right)^2 dx \\ &\quad + \frac{1}{4\delta}\alpha(t)\int_\Omega\left(\int_0^t h(t-\varrho)\big(\nabla u(t)-\nabla u(\varrho)\big)\,d\varrho\right)^2 dx \\ &\le 2\delta h_0^2\alpha(t)\|\nabla u\|_2^2 + \left(2\delta + \frac{1}{4\delta}\right)c\alpha(t)(h\circ\nabla u)(t), \end{aligned} \tag{3.13}$$

$$\begin{aligned} |J_4| &\le c(\delta)\|u_t\|_m^m + \delta\beta_1^m\int_\Omega\left(\int_0^t h(t-\varrho)\big(u(t)-u(\varrho)\big)\,d\varrho\right)^m dx \\ &\le c(\delta)\|u_t\|_m^m + \delta\beta_1^m c_p^m\int_0^t h(t-\varrho)\big\|\nabla u(t)-\nabla u(\varrho)\big\|_2^m\,d\varrho \\ &\le c(\delta)\|u_t\|_m^m + \delta\left(\beta_1^m c_p^m\left(\frac{E(0)}{l}\right)^{(m-2)/2}\right)(h\circ\nabla u)(t) \\ &\le c(\delta)\|u_t\|_m^m + \delta c_3(h\circ\nabla u)(t). \end{aligned} \tag{3.14}$$

Similarly, we have

$$|J_5| \le c(\delta)\big\|y(x,1,s,t)\big\|_m^m + \delta c_4(h\circ\nabla u)(t). \tag{3.15}$$

By exploiting the Sobolev embedding, we have

$$\begin{aligned} |J_6| &\le \frac{1}{p+1}\left(\delta_1\|u_t\|_{2(p+1)}^{2(p+1)} + \frac{c}{\delta_1}\int_\Omega\int_0^t \left(-h'(t-\varrho)\right)\left|u(t)-u(\varrho)\right|^2 d\varrho\, dx\right) \\ &\le c\delta_1\left(E(0)\right)^p\|\nabla u_t\|_2^2 - c(\delta_1)\left(h'\circ\nabla u\right)(t) \end{aligned} \tag{3.16}$$

and

$$|J_7| \le \delta_1\|\nabla u_t\|_2^2 - \frac{h(0)}{4\delta_1}\left(h'\circ\nabla u\right)(t). \tag{3.17}$$

According to (3.11)–(3.17) and (3.10), we get (3.9). □

Lemma 3.3 *The functional* $\Theta(t)$ *defined in* (3.3) *satisfies*

$$\begin{aligned} \Theta'(t) \le{}& -\eta_1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\beta_2(s)\right|.\left\|y(x,\rho,s,t)\right\|_m^m ds\, d\rho \\ &- \eta_1\int_{\tau_1}^{\tau_2}\left|\beta_2(s)\right|.\left\|y(x,1,s,t)\right\|_m^m ds + \beta_1\left\|u_t(t)\right\|_m^m. \end{aligned} \tag{3.18}$$

Proof Differentiating $\Theta(t)$ and using $(2.5)_2$ give

$$\begin{aligned} \Theta'(t) &= -m\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}\left|\beta_2(s)\right|.|y|^{m-1}y_\rho(x,\rho,s,t)\, ds\, d\rho\, dx \\ &= -\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\left|\beta_2(s)\right|.\left|y(x,\rho,s,t)\right|^m ds\, d\rho\, dx \\ &\quad - \int_\Omega\int_{\tau_1}^{\tau_2}\left|\beta_2(s)\right|\left[e^{-s}\left|y(x,1,s,t)\right|^m - \left|y(x,0,s,t)\right|^m\right] ds\, dx. \end{aligned}$$

Applying $y(x,0,s,t) = u_t(x,t)$ and $e^{-s} \le e^{-s\rho} \le 1$ for any $0 < \rho < 1$ and setting $\eta_1 = e^{-\tau_2}$, we obtain

$$\begin{aligned} \Theta'(t) \le{}& -\eta_1\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\left|\beta_2(s)\right|.\left|y(x,\rho,s,t)\right|^m ds\, d\rho\, dx \\ &- \eta_1\int_\Omega\int_{\tau_1}^{\tau_2}\left|\beta_2(s)\right|\left|y(x,1,s,t)\right|^m ds\, dx + \int_{\tau_1}^{\tau_2}\left|\beta_2(s)\right| ds\int_\Omega |u_t|^m(t)\, dx, \end{aligned}$$

using (2.3), we find (3.18). □

Now, we introduce the functional

$$\mathcal{G}(t) := E(t) + \varepsilon_1\alpha(t)\Psi(t) + \varepsilon_2\alpha(t)\Phi(t) + \varepsilon_3\alpha(t)\Theta(t) \tag{3.19}$$

for some positive constants ε_i, $i = 1,2,3$, to be determined.

Lemma 3.4 *There exist* $\mu_1, \mu_2 > 0$ *such that*

$$\mu_1 E(t) \le \mathcal{G}(t) \le \mu_2 E(t). \tag{3.20}$$

Proof From (3.1), by using Hölder's inequality (for $q_1 = \frac{p+2}{p+1}$, $q_2 = p+2$), Young's, and Poincare inequalities (for $\kappa > 0$), and $\|u_t\|_{p+2}^p \le [(p+2)E(0)]^{\frac{p}{(p+2)}}$, we find

$$\begin{aligned}
\Psi(t) &\le \frac{1}{p+1}\|u_t(t)\|_{p+2}^{p+1}\|u(t)\|_{p+2} + \frac{1}{2}\left(\|\nabla u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2\right) \\
&\le \frac{\kappa}{2(p+1)^2}\|u_t(t)\|_{p+2}^{2(p+1)} + \frac{1}{2\kappa}\|u(t)\|_{p+2}^2 \\
&\quad + \frac{1}{2}\left(\|\nabla u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2\right) \\
&\le \frac{\kappa}{2(p+1)^2}\|u_t(t)\|_{p+2}^{p}\|u_t(t)\|_{p+2}^{p+2} + \frac{1}{2\kappa}\|u(t)\|_{p+2}^2 \\
&\quad + \frac{1}{2}\left(\|\nabla u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2\right) \\
&\le \frac{\kappa[(p+2)E(0)]^{\frac{p}{(p+2)}}}{2(p+1)^2}\|u_t(t)\|_{p+2}^{p+2} + c(\kappa)\|\nabla u(t)\|_2^2 + \frac{1}{2}\|\nabla u_t(t)\|_2^2,
\end{aligned} \tag{3.21}$$

where $c(\kappa) = (\frac{C_0}{2\kappa} + \frac{1}{2})$.

According to (3.21) and from (3.2)–(3.3), we get

$$\begin{aligned}
\left|\mathcal{G}(t) - E(t)\right| &\le \varepsilon_1|\alpha(t)|\left(\frac{\kappa[(p+2)E(0)]^{\frac{p}{(p+2)}}}{2(p+1)^2}\|u_t(t)\|_{p+2}^{p+2} + c(\kappa)\|\nabla u(t)\|_2^2\right) \\
&\quad + (\varepsilon_1+\varepsilon_2)\frac{|\alpha(t)|}{2}\|\nabla u_t(t)\|_2^2 + \varepsilon_1\sigma\frac{|\alpha(t)|}{4}\|\nabla u(t)\|_2^4 \\
&\quad + \varepsilon_2\frac{|\alpha(t)|}{2(p+1)}\|u_t(t)\|_{2(p+1)}^{2(p+1)} + \varepsilon_2\frac{|\alpha(t)|(\zeta_0 - l)c(p)}{2}(h\circ\nabla u)(t) \\
&\quad + \varepsilon_3|\alpha(t)|\int_0^1\int_{\tau_1}^{\tau_2} se^{-\rho s}|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m\,ds\,d\rho,
\end{aligned} \tag{3.22}$$

where $c(p) = (\frac{c_p}{p+1} + 1)$.

Using the fact that $0 < \alpha(t) \le \alpha(0)$ and $e^{-\rho s} < 1$, we find

$$\begin{aligned}
\left|\mathcal{G}(t) - E(t)\right| &\le \varepsilon_1\alpha(0)\left(\frac{\kappa[(p+2)E(0)]^{\frac{p}{(p+2)}}}{2(p+1)^2}\|u_t(t)\|_{p+2}^{p+2} + c(\kappa)\|\nabla u(t)\|_2^2\right) \\
&\quad + (\varepsilon_1+\varepsilon_2)\frac{|\alpha(0)|}{2}\|\nabla u_t(t)\|_2^2 + \varepsilon_1\sigma\frac{|\alpha(0)|}{4}\|\nabla u(t)\|_2^4 \\
&\quad + \varepsilon_2\frac{|\alpha(0)|c(E(0))^p}{2(p+1)}\|\nabla u_t(t)\|_2^2 + \varepsilon_2\frac{|\alpha(0)|(\zeta_0 - l)c(p)}{2}(h\circ\nabla u)(t) \\
&\quad + \varepsilon_3\alpha(0)\int_0^1\int_{\tau_1}^{\tau_2} se^{-\rho s}|\beta_2(s)|.\|y(x,\rho,s,t)\|_m^m\,ds\,d\rho \\
&\le C(\varepsilon_1,\varepsilon_2,\varepsilon_3,\kappa)E(t).
\end{aligned} \tag{3.23}$$

We pick $\kappa = 1$ and choose ε_1, ε_2, and ε_3 sufficiently small, then (3.20) follows from (3.23). □

Lemma 3.5 *There exist $k_7, k_8, t_0 > 0$ satisfying*

$$\mathcal{G}'(t) \le -k_7\alpha(t)E(t) + k_8\alpha(t)(h\circ\nabla u)(t), \quad t > t_0. \tag{3.24}$$

Proof A differentiation of (3.19), using (2.7), Lemmas 3.1, 3.2, and 3.3 lead to

$$\begin{aligned}\mathcal{G}'(t) :=\ & E'(t) + \varepsilon_1\alpha'(t)\Psi(t) + \varepsilon_2\alpha'(t)\Phi(t) + \varepsilon_3\alpha'(t)\Theta(t)\\ & + \varepsilon_1\alpha(t)\Psi'(t) + \varepsilon_2\alpha(t)\Phi'(t) + \varepsilon_3\alpha(t)\Theta'(t).\end{aligned} \tag{3.25}$$

By using the fact that $e^{-\rho s} < 1$, Young's and Sobolev–Poincare inequalities, we find

$$\begin{aligned}&\alpha'(t)\big(\varepsilon_1\Psi(t) + \varepsilon_2\Phi(t) + \varepsilon_3\Theta(t)\big)\\ &\quad\le -\alpha'(t)\bigg\{\varepsilon_1 C_1\|u_t\|_{p+2}^{p+2} + \varepsilon_1 c(\kappa)\|\nabla u\|_2^2 + \frac{1}{2}(\varepsilon_1 + \varepsilon_2 C_2)\|\nabla u_t\|_2^2\\ &\qquad + \varepsilon_2 C_3 h\circ\nabla u)(t) + \varepsilon_3\int_0^1\int_{\tau_1}^{\tau_2} s\big|\beta_2(s)\big|.\big\|y(x,\rho,s,t)\big\|_m^m\,ds\,d\rho\bigg\},\end{aligned} \tag{3.26}$$

where $C_1 = \frac{\kappa[(p+2)E(0)]^{p/(p+2)}}{2(p+1)^2} > 0$, $C_2 = 1 + \frac{c(E(0))^p}{p+1} > 0$, and $C_3 = \frac{(\zeta_0 - l)c(p)}{2} > 0$.

Hence, by using (2.7), Lemmas 3.1, 3.2, 3.3, and (3.26), we get

$$\begin{aligned}\mathcal{G}'(t) \le\ & \alpha(t)\bigg\{\frac{1}{p+1}(\varepsilon_1 - \varepsilon_2 h_0) - \varepsilon_1\frac{\alpha'(t)}{\alpha(t)}C_1\bigg\}\|u_t\|_{p+2}^{p+2}\\ & + \alpha(t)\bigg\{\varepsilon_2\delta\big(\zeta_0 + 2h_0^2\alpha(t)\big) - \varepsilon_1\big(l - \varepsilon(c_1 + c_2)\big)\\ & - \frac{\alpha'(t)}{2\alpha(t)}\bigg(\int_0^t h(\varrho)\,d\varrho\bigg) - \varepsilon_1\frac{\alpha'(t)c(\kappa)}{\alpha(t)}\bigg\}\|\nabla u\|_2^2\\ & + \alpha(t)\bigg\{\varepsilon_1 + \varepsilon_2\big[\delta_1\big(1 + c\big(E(0)\big)^p\alpha(t)\big) - h_0\big] - \frac{\alpha'(t)}{2\alpha(t)}(\varepsilon_1 + \varepsilon_2 C_2)\bigg\}\|\nabla u_t\|_2^2\\ & + \alpha(t)\{\varepsilon_2\zeta_1\delta - \varepsilon_1\zeta_1\}\|\nabla u\|_2^4\\ & + \alpha(t)\bigg\{\varepsilon_2\delta\frac{\sigma E(0)}{l} - \frac{\sigma}{4\alpha(0)}\bigg\}\bigg(\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2\bigg)^2\\ & + \alpha(t)\bigg\{\varepsilon_1\frac{\alpha(t)}{4} + \varepsilon_2\bigg(c(\delta) + \bigg(2\delta + \frac{1}{4\delta}\bigg)\bigg)c\alpha(t)) - \varepsilon_2\frac{\alpha'(t)C_3}{\alpha(t)}\bigg\}(h\circ\nabla u)(t)\\ & + \alpha(t)\bigg\{\frac{1}{2} - \varepsilon_2\bigg(\frac{h(0)c_p^2}{4\delta_1} + c(\delta_1)\bigg)\bigg\}\big(h'\circ\nabla u\big)(t)\\ & + \alpha(t)\bigg\{\varepsilon_1 c(\varepsilon) + \varepsilon_2 c(\delta) + \varepsilon_3\beta_1 - \frac{\eta_0}{\alpha(0)}\bigg\}\|u_t\|_m^m\\ & + \alpha(t)\{\varepsilon_1 c(\varepsilon) + \varepsilon_2 c(\delta) - \eta_1\varepsilon_3\}\int_{\tau_1}^{\tau_2}\big|\beta_2(s)\big|\big\|y(x,1,s,t)\big\|_m^m\,ds\\ & + \alpha(t)\varepsilon_3\bigg\{-\eta_1 - \frac{\alpha'(t)}{\alpha(t)}\bigg\}\int_0^1\int_{\tau_1}^{\tau_2} s\big|\beta_2(s)\big|.\big\|y(x,\rho,s,t)\big\|_m^m\,ds\,d\rho.\end{aligned} \tag{3.27}$$

Next, we carefully choose our constants.

Choose δ, δ_1, and ε small enough such that

$$h_0 - \delta_1\big(1 - c\big(E(0)\big)^p\big) > 0, \qquad \delta < \frac{h_0}{4}, \qquad \frac{\delta}{(l - \varepsilon(c_1 + c_2))}\big(\zeta_0 + 2h_0^2\big)\alpha(0) < \frac{1}{4}h_0.$$

For any fixed δ, δ_1, ε, we select ε_1, ε_2, and ε_3 so small satisfying

$$\frac{h_0}{4}\varepsilon_2 < \varepsilon_1 < \frac{h_0}{2}\varepsilon_2$$

and

$$\varepsilon_2 h_0 - \varepsilon_1 > 0,$$
$$\varepsilon_2\big[h_0 - \delta_1\big(1 - c\big(E(0)\big)^p\big)\big] - \varepsilon_1 > 0.$$

Then, we select ε_1, ε_2, and ε_3 so small that (3.20) and (3.27) remain valid, and further

$$\zeta_1(\varepsilon_1 - \varepsilon_2\delta) > 0, \qquad \frac{\sigma}{4\alpha(0)} - \varepsilon_2\delta\frac{\sigma E(0)}{l} > 0, \qquad \frac{1}{2} - \varepsilon_2\left(\frac{h(0)c_p^2}{4\delta} + c(\delta_1)\right) > 0,$$
$$\frac{\eta_0}{\alpha(0)} - \varepsilon_1 c(\varepsilon) - \varepsilon_2 c(\delta) - \varepsilon_3\beta_1 > 0, \qquad \eta_1\varepsilon_3 - \varepsilon_1 c(\varepsilon) - \varepsilon_2 c(\delta) > 0,$$

where $\eta_0 = \beta_1 - \int_{\tau_1}^{\tau_2}|\beta_2(s)|\,ds > 0$.

Therefore, (3.27) becomes, for positive constants k_i, $i = 1,\dots,6$,

$$\begin{aligned}\mathcal{G}'(t) \le{}& -\alpha(t)\left(k_1 + \varepsilon_1\frac{\alpha'(t)}{\alpha(t)}C_1\right)\|u_t\|_{p+2}^{p+2} - \alpha(t)k_2\|\nabla u\|_2^4 \\ &- \alpha(t)\left(k_3 + \frac{\alpha'(t)}{\alpha(t)}\left(\int_0^t h(\varrho)\,d\varrho\right) + \varepsilon_1\frac{\alpha'(t)c(\kappa)}{\alpha(t)}\right)\|\nabla u\|_2^2 \\ &- \alpha(t)\left(k_4 + \frac{\alpha'(t)}{2\alpha(t)}(\varepsilon_1 + \varepsilon_2 C_2)\right)\|\nabla u_t\|_2^2 \\ &+ \alpha(t)\left(k_5 - \varepsilon_2\frac{C_3\alpha'(t)}{\alpha(t)}\right)(h\circ\nabla u)(t) \\ &- \alpha(t)\left(k_6 + \varepsilon_3\frac{\alpha'(t)}{\alpha(t)}\right)\int_0^1\int_{\tau_1}^{\tau_2} s\big|\beta_2(s)\big|.\big\|y(x,\rho,s,t)\big\|_m^m\,ds\,d\rho. \end{aligned} \tag{3.28}$$

According to (2.2), $\lim_{t\to\infty}\frac{\alpha'(t)}{\alpha(t)} = 0$, we can choose $t_1 > t_0$ so that (3.28) can be written as

$$\begin{aligned}\mathcal{G}'(t) \le{}& -\alpha(t)\bigg(k_1\|u_t\|_{p+2}^{p+2} + k_2\|\nabla u\|_2^4 + k_3\|\nabla u\|_2^2 + k_4\|\nabla u_t\|_2^2 - k_5(h\circ\nabla u)(t) \\ &+ k_6\int_0^1\int_{\tau_1}^{\tau_2} s\big|\beta_2(s)\big|.\big\|y(x,\rho,s,t)\big\|_m^m\,ds\,d\rho\bigg) \\ \le{}& -\alpha(t)k_7E(t) + \alpha(t)k_8(h\circ\nabla u)(t), \quad \forall t \ge t_1. \end{aligned} \tag{3.29}$$

Theorem 3.6 *Suppose that* (2.1)–(2.3) *for any* (u_0, u_1, f_0) *satisfy* $E(0) > 0$. *Then the energy* $E(t)$ *of* (2.5) *decays to zero exponentially. That is,* $\exists\lambda_1,\lambda_2 > 0$ *such that*

$$E(t) \le \lambda_1 e^{-\lambda_2\int_{t_1}^{t}\alpha(\varrho)\vartheta(\varrho)\,d\varrho}, \quad \forall t \ge t_1. \tag{3.30}$$

Proof Multiplying (3.24) by $\vartheta(t)$, using (2.1) and (2.7), we find

$$\vartheta(t)\mathcal{G}'(t) \le -k_7\vartheta(t)\alpha(t)E(t) + k_8\alpha(t)\vartheta(t)(h\circ\nabla u)(t)$$

$$\begin{aligned}&\leq -k_7\vartheta(t)\alpha(t)E(t)-k_8\alpha(t)\bigl(h'\circ\nabla u\bigr)(t)\\&\leq -k_7\vartheta(t)\alpha(t)E(t)-k_8\left\{2E'(t)-\alpha'(t)\left(\int_0^t h(\varrho)\,d\varrho\right)\bigl\|\nabla u(t)\bigr\|_2^2\right\}.\end{aligned}\tag{3.31}$$

Since $\vartheta(t)$ is a nonincreasing function, we have

$$\frac{d}{dt}\bigl(\vartheta(t)\mathcal{G}(t)+2k_8E(t)\bigr)\leq -k_7\vartheta(t)\alpha(t)E(t)-k_8\alpha'(t)\left(\int_0^t h(\varrho)\,d\varrho\right)\bigl\|\nabla u(t)\bigr\|_2^2.\tag{3.32}$$

From (2.6) and (2.2) that $l\|\nabla u(t)\|_2^2\leq E(t)$, we find

$$\begin{aligned}\frac{d}{dt}\bigl(\vartheta(t)\mathcal{G}(t)+2k_8E(t)\bigr)&\leq -k_7\alpha(t)\vartheta(t)E(t)-k_8\alpha'(t)\left(\int_0^t h(\varrho)\,d\varrho\right)\bigl\|\nabla u(t)\bigr\|_2^2\\&\leq -k_7\alpha(t)\vartheta(t)E(t)-\frac{2k_8\alpha'(t)}{l}E(t)\\&\leq -\alpha(t)\vartheta(t)\left(k_7+\frac{2k_8l_0\alpha'(t)}{l\vartheta(t)\alpha(t)}\right)E(t).\end{aligned}\tag{3.33}$$

Since $\lim_{t\to\infty}\frac{\alpha'(t)}{\vartheta(t)\alpha(t)}=0$, we can choose $t_1\geq t_0$ such that $k_7+\frac{2k_8l_0\alpha'(t)}{l\alpha(t)\vartheta(t)}>0$ for $t\geq t_1$.

Finally, let

$$\mathcal{R}(t):=\mathcal{G}(t)\vartheta(t)+2k_8E(t)\sim E(t).\tag{3.34}$$

Hence, for some $\lambda_2>0$, we find

$$\mathcal{R}'(t)\leq -\lambda_2\alpha(t)\vartheta(t)\mathcal{R}(t),\quad \forall t\geq t_1.\tag{3.35}$$

Integrating of (3.35) over (t_1,t) gives

$$\mathcal{R}(t)\leq \mathcal{R}(t_1)e^{-\lambda_2\int_{t_1}^t\alpha(\varrho)\vartheta(\varrho)\,d\varrho},\quad \forall t\geq t_1.\tag{3.36}$$

Hence, (3.30) is established by virtue of (3.34) and (3.36). The proof is complete. □

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