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Blow-up criterion for the density dependent inviscid Boussinesq equations

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Abstract

In this work, we consider the density-dependent incompressible inviscid Boussinesq equations in $\mathbb{R}^N$ ($N \geq 2$). By using the basic energy method, we first give the a priori estimates of smooth solutions and then get a blow-up criterion. This shows that the maximum norm of the gradient velocity field controls the breakdown of smooth solutions of the density-dependent inviscid Boussinesq equations. Our result extends the known blow-up criteria.

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1 Introduction

This paper is devoted to investigating the initial value problem associated to the following density-dependent inviscid incompressible Boussinesq equations in $(x,t) \in \mathbb{R}^N \times (0,+\infty)$ with $N \geq 2$:

$$\begin{cases} \rho_t + \mathbf{v} \cdot \nabla \rho = 0, \\ \rho(\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v}) + \nabla P = \rho \theta e_N, \qquad \operatorname{div} \mathbf{v} = 0, \\ \theta_t + \mathbf{v} \cdot \nabla \theta = 0, \\ (\rho, \mathbf{v}, \theta)|_{t=0} = (\rho_0, \mathbf{v}_0, \theta_0), \end{cases} \tag{1.1}$$

where e_N denotes the vertical unit vector $(0,\dots,0,1)$, and $\rho, \mathbf{v}, \theta$, and P denote the fluid density, velocity field, temperature, and pressure, respectively, while $\rho_0, \mathbf{v}_0$, and θ_0 are the given corresponding initial data with $\nabla \cdot \mathbf{v}_0 = 0$.

When $\theta \equiv 0$, system (1.1) reduces to the initial value problem associated to the incompressible density-dependent Euler equations. Chae and Lee [4] showed the local well-posedness of the incompressible density-dependent Euler equations in the L^2-type critical Besov space. Zhou et al. [18] generalized the result of [4] to the L^p-type critical Besov space and obtained the following blow-up criterion:

$$\lim_{T \to T^*} \int_0^T \|\nabla \times \mathbf{v}\|_{\dot{B}^{\frac{N}{p}}_{p,1}} \, dt = \infty \tag{1.2}$$

for $1 < p < \infty$. Very recently, Bae et al. [1] derived a refined blow-up criterion

$$\lim_{T\to T^*}\int_0^T \|\nabla \mathbf{v}\|_{L^\infty}\,dt = \infty. \tag{1.3}$$

When ρ is constant, system (1.1) becomes the initial value problem associated to the homogeneous inviscid Boussinesq equations. The local well-posedness and regularity criteria are well-established; see, for example, [2, 3, 5, 7, 9, 12, 16]. In particular, by using Littlewood–Paley method, the authors in [2] and [7] derived the blow-up criterion (1.3) in Besov–Morrey spaces (see Remark 1.3 in [2]) and Hölder spaces [7], respectively. Let us mention that the global regularity question of the inviscid Boussinesq system (1.1) is a rather challenging problem.

Compared with the homogeneous flow, fewer works are concerned with the nonhomogeneous system (1.1). Regarding the local existence and blow-up criteria results, one can refer to [14, 17]. Precisely, Qiu and Yao [14] developed the methods of [4] and [18] and got the blow-up criterion (1.2) in the Besov framework. Xu [17] obtained the blow-up criterion (1.3) for smooth solutions to the 2-dimensional compressible Boussinesq equations. In this paper, we are going to establish the local existence and blow-up criterion (1.3) for the N-dimensional ($N \geq 2$) system (1.1) by applying the standard energy method. We suppose that

$$0 < \underline{\rho} \leq \rho_0(x) \leq \overline{\rho} < \infty,$$

where $\underline{\rho}$ and $\overline{\rho}$ are positive constants and assume $\rho_0 \to \underline{\rho}$ as $|x| \to \infty$. Different from the homogeneous case, the classical energy method cannot be applied directly to the equation of $\mathbf{v}$ fulfilling

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla P + \theta e_N. \tag{1.4}$$

To obtain the H^s estimate of $\mathbf{v}$, we need the elaborate estimates of P. To this end, as in [1], we introduce the following two variables to deal with the term $\frac{1}{\rho}\nabla P$:

$$a \stackrel{\text{def}}{=} \rho - \underline{\rho}, \qquad b \stackrel{\text{def}}{=} \frac{1}{\rho} - \frac{1}{\underline{\rho}}.$$

As a consequence, we use the usual energy method to deal with P, which satisfies

$$-\operatorname{div}\left(\frac{1}{\rho}\nabla P\right) = \operatorname{div}(\mathbf{v}\cdot\nabla\mathbf{v} - \theta e_N). \tag{1.5}$$

By virtue of $(1.1)_1$, we see that a and b satisfy

$$a_t + \mathbf{v}\cdot\nabla a = 0, \qquad b_t + \mathbf{v}\cdot\nabla b = 0, \tag{1.6}$$

with the initial data

$$a_0 = \rho_0 - \underline{\rho}, \qquad b_0 = \frac{1}{\rho_0} - \frac{1}{\underline{\rho}},$$

respectively.

The main result of this paper is stated as follows.

Theorem 1.1 *Let $N \geq 2$ and $a_0, b_0, \mathbf{v}_0, \theta_0 \in H^s$, where $s > 1 + \frac{N}{2}$ and $\operatorname{div}\mathbf{v}_0 = 0$. Then, there exists $T^* > 0$ such that system (1.1) has a unique solution $(a, b, \mathbf{v}, \theta)$ with $a, b, \mathbf{v}, \theta \in C([0, T^*); H^s)$. In addition, the solution $(a, b, \mathbf{v}, \theta)$ blows up at T^* if and only if*

$$\limsup_{t\to T^*} \|(a, b, \mathbf{v}, \theta)(t)\|_{H^s} = \infty \quad \Longleftrightarrow \quad \lim_{T\to T^*} \int_0^T \|\nabla \mathbf{v}(t)\|_{L^\infty}\, dt = \infty. \tag{1.7}$$

Remark 1.1 Our result (1.7) extends the criterion in [14], i.e., criterion (1.2). On the other hand, when $\theta \equiv 0$, system (1.1) becomes the classical inhomogeneous incompressible Euler system, and we recover the result in [1].

2 Proof of the main result

The proof of Theorem 1.1 is divided into two parts, i.e., the local existence and the blow-up criterion.

Proof (*Local existence*). We first recall some basic lemmas that will be applied to the proof of the local existence.

Lemma 2.1 (Picard theorem on a Banach space, [13]). *Let $O \subset B$ be an open subset of a Banach space B and $F : O \to B$ be a mapping that satisfies the following properties*:

- *$F(X)$ maps O to B*;
- *F is locally Lipschitz continuous, namely, for any $X \in O$ there exists $L > 0$ and an open neighborhood $U_X \subset O$ of X such that*

$$\|F(M) - F(N)\|_B \leq L\|M - N\|_B \quad \text{for all } M, N \in U_X.$$

Then for any $X_0 \in O$, there exists a time T such that the ODE

$$\frac{dX}{dt} = F(X), \qquad X|_{t=0} = X_0 \in O,$$

has a unique (*local*) *solution $X \in C^1([0, T]; O)$.*

Lemma 2.2 (Continuation of an autonomous ODE on a Banach space, [13]) *Let $O \subset B$ be an open subset of a Banach space B and let $F : O \to B$ be a locally Lipschitz continuous operator. Then the unique solution $X \in C^1([0, T]; O)$ to the autonomous ODE,*

$$\frac{dX}{dt} = F(X), \qquad X|_{t=0} = X_0 \in O,$$

either exists globally in time, or $T < \infty$ and $X(t)$ leaves the open set O as $t \to T$.

Lemma 2.3 (Compactness lemma, [15]) *Let X, B, Y be Banach spaces, and $X \subset B \subset Y$ with compact imbedding $X \hookrightarrow B$. Let F be bounded in $L^\infty(0, T; X)$ and $\frac{\partial F}{\partial t}$ be bounded in $L^r(0, T; Y)$ where $r > 1$. Then F is relatively compact in $C([0, T]; B)$.*

Let us first briefly explain the idea of the proof of the local well-posedness, see [13, Chap. 3], or [5] for details. As in [5], we regularize system (1.1) and then due to Lemmas 2.1 and 2.2, for any $\epsilon > 0$, we obtain the global solution $(a^\epsilon, b^\epsilon, \mathbf{v}^\epsilon, \theta^\epsilon)$ of the regularized Boussinesq equations in

$$C\big([0,\infty);\big(H^s\big)^4\big) \cap C^1\big([0,\infty);\big(H^{s-1}\big)^4\big), \quad \text{where } s > 1 + \frac{N}{2}.$$

Let us mention that, for the proof of the above global existence of regularized solutions, one can refer to Theorem 3.2 in [13]. Next, noting that $H^{s-1} \subset L^\infty$ when $s > 1 + \frac{N}{2}$, we could show that there exists a $T = T(\|(a_0, b_0, \mathbf{v}_0, \theta_0)\|_{H^s})$, such that $(a^\epsilon, b^\epsilon, \mathbf{v}^\epsilon, \theta^\epsilon)$ is uniformly bounded in $L^\infty([0,T];(H^s)^4)$ and $(a^\epsilon_t, b^\epsilon_t, \mathbf{v}^\epsilon_t, \theta^\epsilon_t)$ is uniformly bounded in $L^\infty([0,T];(H^{s-1})^4)$. By virtue of Lemma 2.3, $\{(a^\epsilon, b^\epsilon, \mathbf{v}^\epsilon, \theta^\epsilon)\}$ is relatively compact in $C([0,T];(H^{s'})^4)$ for any $s' < s$. As a consequence, we can find a solution

$$(a, b, \mathbf{v}, \theta) \in C\big([0,T];\big(H^{s'}\big)^4\big) \cap L^\infty\big([0,T];\big(H^s\big)^4\big).$$

Then, we can prove

$$(a, b, \mathbf{v}, \theta) \in C\big([0,T];\big(H^s\big)^4\big) \cap C^1\big([0,T];\big(H^{s-1}\big)^4\big),$$

which is unique.

Moreover, there exist a maximal time of existence T^* (possibly infinite) and unique solution

$$(a, b, \mathbf{v}, \theta) \in C\big([0,T^*);\big(H^s\big)^4\big) \cap C^1\big([0,T^*);\big(H^{s-1}\big)^4\big).$$

If $T^* < \infty$, then

$$\limsup_{t \to T^*} \big\|(a, b, \mathbf{v}, \theta)(t)\big\|_{H^s} = \infty.$$

Through Sobolev imbedding, we have

$$(a, b, \mathbf{v}, \theta) \in C\big([0,T^*);\big(C^1\big)^4\big) \cap C^1\big([0,T^*);\big(C^0\big)^4\big),$$

which means that $(a, b, \mathbf{v}, \theta)$ is a classical solution of system (1.1).

Based on the above arguments, here we only present the key part, that is, the solution $(a^\epsilon, b^\epsilon, \mathbf{v}^\epsilon, \theta^\epsilon)$ of the regularized Boussinesq equations is uniformly bounded in $L^\infty([0,T];(H^s)^4)$ with respect to ϵ. The remaining parts such as the approximation to system (1.1), the process of taking limits, and that the solution is continuous in time in the highest norm H^s are omitted, which can be referred to [13] and [5] for details. To simplify the presentation, we also omit the superscript ϵ and denote $\Lambda \overset{\text{def}}{=} \sqrt{-\Delta}$ throughout the paper.

Step 1. H^s estimate of $(a, b, \mathbf{v}, \theta)$. Since $\operatorname{div}\mathbf{v} = 0$, it is easy to deduce (see [11, Theorem 2.1]) that

$$\big\|(\rho, a, b)(t)\big\|_{L^2 \cap L^\infty} \le C.$$

Applying the operator Λ^s to the first equation in (1.6) and taking the L^2 inner product with itself, we have

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s a\|_{L^2}^2 = -\int_{\mathbb{R}^N}\big[\big(\Lambda^s(\mathbf{v}\cdot\nabla a) - \mathbf{v}\cdot\nabla\Lambda^s a\big)\Lambda^s a\big]\,dx - \int_{\mathbb{R}^N}\mathbf{v}\Lambda^s\nabla a\Lambda^s a\,dx,$$

as div$\mathbf{v} = 0$, the last term is zero. One gets that

$$\frac{d}{dt}\|\Lambda^s a\|_{L^2} \le C\|\nabla\mathbf{v}\|_{L^\infty}\|\Lambda^s a\|_{L^2} + C\|\nabla a\|_{L^\infty}\|\Lambda^s\mathbf{v}\|_{L^2}. \tag{2.1}$$

Here and in what follows, we will frequently use the following two estimates for $s > 0$ (see [10]):

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^2} \le C\big(\|\nabla f\|_{L^\infty}\|\Lambda^{s-1}g\|_{L^2} + \|\Lambda^s f\|_{L^2}\|g\|_{L^\infty}\big),$$
$$\|\Lambda^s(fg)\|_{L^2} \le C\|f\|_{L^\infty}\|\Lambda^s g\|_{L^2} + C\|g\|_{L^\infty}\|\Lambda^s f\|_{L^2}.$$

Similarly, for b and θ, we have

$$\frac{d}{dt}\|\Lambda^s b\|_{L^2} \le C\|\nabla\mathbf{v}\|_{L^\infty}\|\Lambda^s b\|_{L^2} + C\|\nabla b\|_{L^\infty}\|\Lambda^s\mathbf{v}\|_{L^2}, \tag{2.2}$$
$$\frac{d}{dt}\|\Lambda^s\theta\|_{L^2} \le C\|\nabla\mathbf{v}\|_{L^\infty}\|\Lambda^s\theta\|_{L^2} + C\|\nabla\theta\|_{L^\infty}\|\Lambda^s\mathbf{v}\|_{L^2}. \tag{2.3}$$

Next, we deal with $\mathbf{v}$. Multiplying $(1.1)_2$ by $\mathbf{v}$ and $(1.1)_3$ by θ, respectively, integrating in $\mathbb{R}^N$ and combining the resulting equations together, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}\big(\rho|\mathbf{v}|^2 + |\theta|^2\big)\,dx = \int_{\mathbb{R}^N}\rho\mathbf{v}\cdot\theta e_N\,dx \le C\|\sqrt{\rho}\|_{L^\infty}\int_{\mathbb{R}^N}\big(\rho|\mathbf{v}|^2 + |\theta|^2\big)\,dx,$$

which, together with Gronwall's inequality and the bound of ρ, yields

$$\|\mathbf{v}(t)\|_{L^2} + \|\theta(t)\|_{L^2} \le C. \tag{2.4}$$

Noting that $\mathbf{v}$ satisfies

$$\mathbf{v}_t + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla P + \theta e_N = -b\nabla P - \frac{1}{\underline{\rho}}\nabla P + \theta e_N,$$

we have

$$\frac{d}{dt}\|\Lambda^s\mathbf{v}\|_{L^2}^2 \le C\|\nabla\mathbf{v}\|_{L^\infty}\|\Lambda^s\mathbf{v}\|_{L^2}^2 + C\|\nabla P\|_{L^\infty}\|\Lambda^s b\|_{L^2}\|\Lambda^s\mathbf{v}\|_{L^2}$$
$$+ C\|b\|_{L^\infty}\|\Lambda^s(\nabla P)\|_{L^2}\|\Lambda^s\mathbf{v}\|_{L^2} + C\|\Lambda^s\theta\|_{L^2}\|\Lambda^s\mathbf{v}\|_{L^2},$$

which yields

$$\frac{d}{dt}\|\Lambda^s\mathbf{v}\|_{L^2} \le C\|\nabla\mathbf{v}\|_{L^\infty}\|\Lambda^s\mathbf{v}\|_{L^2} + C\|\nabla P\|_{L^\infty}\|\Lambda^s b\|_{L^2}$$
$$+ C\|\Lambda^s(\nabla P)\|_{L^2} + C\|\Lambda^s\theta\|_{L^2}. \tag{2.5}$$

Let $\mathcal{N} \stackrel{\text{def}}{=} \|a\|_{H^s} + \|b\|_{H^s} + \|\theta\|_{H^s} + \|\mathbf{v}\|_{H^s}$. Combining (2.1), (2.2), (2.3), and (2.5) gives

$$\frac{d}{dt}\mathcal{N} \leq C\big(1 + \big\|(\nabla a, \nabla b, \nabla\theta, \nabla\mathbf{v}, \nabla P)\big\|_{L^\infty}\big)\mathcal{N} + C\|\nabla P\|_{H^s}. \tag{2.6}$$

Step 2. H^s estimate of ∇P. We first give the L^2 bound of ∇P. Since $1/\rho \geq 1/\overline{\rho} > 0$, the classical L^2 theory used to (1.5) ensures that [8, Lemma 2]

$$\|\nabla P\|_{L^2} \leq \overline{\rho}\|\mathbf{v}\cdot\nabla\mathbf{v}\|_{L^2} + C\|\theta\|_{L^2},$$

which, together with (2.4), gives

$$\begin{aligned}\|\nabla P\|_{L^2} &\leq C\|\mathbf{v}\cdot\nabla\mathbf{v}\|_{L^2} + C\|\theta\|_{L^2}\\ &\leq C\|\mathbf{v}\|_{L^2}\|\nabla\mathbf{v}\|_{L^\infty} + C\|\theta\|_{L^2}\\ &\leq C\big(\|\nabla\mathbf{v}\|_{L^\infty} + 1\big).\end{aligned} \tag{2.7}$$

Thanks to (1.5) again, one infers

$$-\mathrm{div}\left(\frac{1}{\rho}\Lambda^s\nabla P\right) = \Lambda^s\mathrm{div}(\mathbf{v}\cdot\nabla\mathbf{v} - \theta e_N) + \mathrm{div}\big[\Lambda^s(b\nabla P) - b\Lambda^s\nabla P\big]. \tag{2.8}$$

Taking the L^2 inner product with $\Lambda^s P$ in (2.8) yields that

$$\begin{aligned}&\int_{\mathbb{R}^N}\left(\frac{1}{\rho}\Lambda^s\nabla P\right)\cdot\Lambda^s\nabla P\,dx\\ &\quad= \int_{\mathbb{R}^N}\Lambda^{s-1}\mathrm{div}(\mathbf{v}\cdot\nabla\mathbf{v})\Lambda^{s+1}P\,dx - \int_{\mathbb{R}^N}\Lambda^{s-1}\mathrm{div}(\theta e_N)\Lambda^{s+1}P\,dx\\ &\qquad- \int_{\mathbb{R}^N}\big[\Lambda^s(b\nabla P) - b\Lambda^s\nabla P\big]\Lambda^s\nabla P\,dx.\end{aligned} \tag{2.9}$$

Based on that $1/\rho \geq 1/\overline{\rho} > 0$, we derive

$$\begin{aligned}\|\nabla P\|_{H^s}^2 &\leq C\big\|\mathrm{div}(\mathbf{v}\cdot\nabla\mathbf{v})\big\|_{H^{s-1}}\|\nabla P\|_{H^s}\\ &\qquad+ C\big(\|\nabla b\|_{L^\infty}\|\nabla P\|_{H^{s-1}} + \|b\|_{H^s}\|\nabla P\|_{L^\infty} + \|\theta\|_{H^s}\big)\|\nabla P\|_{H^s}\\ &\leq C\|\nabla\mathbf{v}\|_{L^\infty}\|\mathbf{v}\|_{H^s}\|\nabla P\|_{H^s}\\ &\qquad+ C\big(\|\nabla b\|_{L^\infty}\|\nabla P\|_{H^{s-1}} + \|b\|_{H^s}\|\nabla P\|_{L^\infty} + \|\theta\|_{H^s}\big)\|\nabla P\|_{H^s}.\end{aligned}$$

That is,

$$\begin{aligned}&\|\nabla P\|_{H^s}\\ &\quad\leq C\|\nabla\mathbf{v}\|_{L^\infty}\|\mathbf{v}\|_{H^s} + C\|\nabla b\|_{L^\infty}\|\nabla P\|_{H^{s-1}} + C\|b\|_{H^s}\|\nabla P\|_{L^\infty} + C\|\theta\|_{H^s}\\ &\quad\leq C\|\nabla\mathbf{v}\|_{L^\infty}\|\mathbf{v}\|_{H^s} + C\|\nabla b\|_{L^\infty}\|\nabla P\|_{H^s}^{\frac{s-1}{s}}\|\nabla P\|_{L^2}^{\frac{1}{s}} + C\|b\|_{H^s}\|\nabla P\|_{L^\infty} + C\|\theta\|_{H^s}\\ &\quad\leq \frac{1}{2}\|\nabla P\|_{H^s} + C\|\nabla b\|_{L^\infty}^s\|\nabla P\|_{L^2} + C\|\nabla\mathbf{v}\|_{L^\infty}\|\mathbf{v}\|_{H^s}\end{aligned}$$

$$+ C\|b\|_{H^s}\|\nabla P\|_{L^\infty} + C\|\theta\|_{H^s}, \tag{2.10}$$

which, combined with (2.7), implies

$$\|\nabla P\|_{H^s} \le C\big(1 + \|\nabla P\|_{L^\infty} + \|\nabla \mathbf{v}\|_{L^\infty}\big)\mathcal{N} + C\|\nabla b\|_{L^\infty}^s\big(\|\nabla \mathbf{v}\|_{L^\infty} + 1\big). \tag{2.11}$$

Step 3. L^∞ estimate of ∇P. Firstly, by interpolation inequality, we have for $N < p < \infty$ that

$$\|\nabla P\|_{L^\infty} \le C\|\Delta P\|_{L^p}^{\frac{pN}{pN-2N+2p}}\|\nabla P\|_{L^2}^{\frac{2p-2N}{pN-2N+2p}} \le C\|\Delta P\|_{L^p} + C\|\nabla P\|_{L^2}. \tag{2.12}$$

In order to estimate $\|\Delta P\|_{L^p}$, we have from (1.5) that

$$\Delta P = -\rho \operatorname{div}(\mathbf{v}\cdot\nabla\mathbf{v}) - \rho\nabla b\cdot\nabla P + \rho\partial_N\theta.$$

Then, by the interpolation inequality and Young's inequality again, one deduces

$$\begin{aligned}
\|\Delta P\|_{L^p} &\le \|\rho\|_{L^\infty}\|\nabla\mathbf{v}\|_{L^\infty}\|\nabla\mathbf{v}\|_{L^p} + \|\rho\|_{L^\infty}\|\nabla b\|_{L^\infty}\|\nabla P\|_{L^p} + \|\rho\|_{L^\infty}\|\nabla\theta\|_{L^p}\\
&\le C\|\nabla\mathbf{v}\|_{L^\infty}\|\nabla\mathbf{v}\|_{L^p} + C\|\nabla b\|_{L^\infty}\|\nabla P\|_{L^p} + C\|\nabla\theta\|_{L^p}\\
&\le C\|\nabla\mathbf{v}\|_{L^\infty}\|\nabla\mathbf{v}\|_{L^p} + C\|\nabla b\|_{L^\infty}\|\Delta P\|_{L^p}^{\frac{pN-2N}{pN-2N+2p}}\|\nabla P\|_{L^2}^{\frac{2p}{pN-2N+2p}} + C\|\nabla\theta\|_{L^p}\\
&\le \frac{1}{2}\|\Delta P\|_{L^p} + C\|\nabla b\|_{L^\infty}^{\frac{pN-2N+2p}{2p}}\|\nabla P\|_{L^2} + C\|\nabla\mathbf{v}\|_{L^\infty}\|\nabla\mathbf{v}\|_{L^p} + C\|\nabla\theta\|_{L^p}
\end{aligned}$$

for $N < p < \infty$, which implies

$$\|\Delta P\|_{L^p} \le C\|\nabla b\|_{L^\infty}^{\frac{pN-2N+2p}{2p}}\|\nabla P\|_{L^2} + C\|\nabla\mathbf{v}\|_{L^\infty}\|\nabla\mathbf{v}\|_{L^p} + C\|\nabla\theta\|_{L^p}. \tag{2.13}$$

This, together with (2.12) and (2.7), gives

$$\|\nabla P\|_{L^\infty} \le C\big(\|\nabla b\|_{L^\infty}^{\frac{pN-2N+2p}{2p}} + 1\big)\big(\|\nabla\mathbf{v}\|_{L^\infty} + 1\big) + C\|\nabla\mathbf{v}\|_{L^\infty}\|\nabla\mathbf{v}\|_{L^p} + C\|\nabla\theta\|_{L^p}. \tag{2.14}$$

Step 4. A priori estimates. Combining (2.6), (2.11), and (2.14) together, we end up with

$$\begin{aligned}
\frac{d}{dt}\mathcal{N} &\le C\big[1 + \big\|(\nabla a,\nabla b,\nabla\theta,\nabla\mathbf{v})\big\|_{L^\infty} + \big(\|\nabla b\|_{L^\infty}^{\frac{pN-2N+2p}{2p}} + 1\big)\big(\|\nabla\mathbf{v}\|_{L^\infty} + 1\big)\\
&\quad + \|\nabla\mathbf{v}\|_{L^\infty}\|\nabla\mathbf{v}\|_{L^p} + \|\nabla\theta\|_{L^p}\big]\mathcal{N} + C\big(\|\nabla\mathbf{v}\|_{L^\infty} + 1\big)\|\nabla b\|_{L^\infty}^s.
\end{aligned} \tag{2.15}$$

By Sobolev embedding $H^s \hookrightarrow W^{1,p}\cap W^{1,\infty}$ for $s > 1 + \frac{N}{2}$ and $N < p < \infty$, we have

$$\frac{d}{dt}\mathcal{N} \le C\mathcal{N}^{s+1}.$$

This completes the proof of local well-posedness for system (1.1) in H^s.

Next, we present the proof of the second part in Theorem 1.1, namely, the blow-up criterion.

(Blow-up criterion). We first show the "$\Rightarrow$" part in (1.7). From the equations of a, b, and θ, we obtain

$$\begin{aligned} &\big\|\big(\nabla a(t), \nabla b(t)\big)\big\|_{L^\infty} \le \big\|(\nabla a_0, \nabla b_0)\big\|_{L^\infty} \exp\biggl(\int_0^t \big\|\nabla \mathbf{v}(\tau)\big\|_{L^\infty}\,d\tau\biggr), \\ &\big\|\nabla\theta(t)\big\|_{L^p} \le \|\nabla\theta_0\|_{L^p} \exp\biggl(\int_0^t \big\|\nabla \mathbf{v}(\tau)\big\|_{L^\infty}\,d\tau\biggr). \end{aligned} \tag{2.16}$$

To deal with $\|\nabla\mathbf{v}\|_{L^p}$, we define the vorticity as $w \overset{\text{def}}{=} \nabla \times \mathbf{v}$ when $N = 2, 3$ or $w = w_{ij} \overset{\text{def}}{=} \partial_j v^i - \partial_i v^j$ when $N \ge 4$. Then we turn to consider the following equations:

$$\begin{aligned} &N = 2: \quad w_t + \mathbf{v}\cdot\nabla w = -\nabla b \cdot \nabla^\perp P + \partial_1\theta, \\ &N = 3: \quad w_t + \mathbf{v}\cdot\nabla w = w\nabla\mathbf{v} - \nabla b \times \nabla P + \nabla \times (\theta e_3), \\ &N \ge 4: \quad w_t + \mathbf{v}\cdot\nabla w = -w\nabla\mathbf{v} - \nabla b \wedge \nabla P + \nabla \wedge (\theta e_N), \end{aligned} \tag{2.17}$$

where $\nabla^\perp = (-\partial_2, \partial_1)$ and $\wedge$ represents the wedge product. Next we only estimate the case $N = 3$ since the other two cases could be handled similarly.

From $(2.17)_2$, applying (2.13) and the fact that (see [6])

$$\|\nabla\mathbf{v}\|_{L^p} \le C_p\|w\|_{L^p} \quad (1 < p < \infty),$$

we have for $N < p < \infty$ that

$$\begin{aligned} \frac{d}{dt}\|w\|_{L^p} &\le C\|\nabla\mathbf{v}\|_{L^\infty}\|w\|_{L^p} + C\|\nabla b\|_{L^\infty}\|\nabla P\|_{L^p} + C\|\nabla\theta\|_{L^p} \\ &\le C\|\nabla\mathbf{v}\|_{L^\infty}\|w\|_{L^p} + C\|\nabla b\|_{L^\infty}\|\Delta P\|_{L^p}^{\frac{pN-2N}{pN-2N+2p}}\|\nabla P\|_{L^2}^{\frac{2p}{pN-2N+2p}} + C\|\nabla\theta\|_{L^p} \\ &\le C\|\nabla\mathbf{v}\|_{L^\infty}\|w\|_{L^p} + C\|\nabla\theta\|_{L^p} + C\|\nabla b\|_{L^\infty} \\ &\quad \times \Bigl[\|\nabla b\|_{L^\infty}^{\frac{pN-2N+2p}{2p}}\|\nabla P\|_{L^2} + \|\nabla\mathbf{v}\|_{L^\infty}\|w\|_{L^p} + \|\nabla\theta\|_{L^p}\Bigr]^{\frac{pN-2N}{pN-2N+2p}} \\ &\quad \times \|\nabla P\|_{L^2}^{\frac{2p}{pN-2N+2p}}, \end{aligned}$$

which, together with (2.7), implies that

$$\begin{aligned} &\frac{d}{dt}\|w\|_{L^p} \\ &\quad\le C\|\nabla\mathbf{v}\|_{L^\infty}\|w\|_{L^p} + C\|\nabla\theta\|_{L^p} + C\|\nabla b\|_{L^\infty}^{\frac{2p+pN-2N}{2p}}\bigl(\|\nabla\mathbf{v}\|_{L^\infty} + 1\bigr) \\ &\qquad + C\|\nabla b\|_{L^\infty}\bigl(\|\nabla\mathbf{v}\|_{L^\infty} + 1\bigr)\|w\|_{L^p}^{\frac{pN-2N}{pN-2N+2p}} \\ &\qquad + C\|\nabla b\|_{L^\infty}\|\nabla\theta\|_{L^p}^{\frac{pN-2N}{pN-2N+2p}}\bigl(\|\nabla\mathbf{v}\|_{L^\infty} + 1\bigr)^{\frac{2p}{pN-2N+2p}} \\ &\quad\le C\|\nabla\mathbf{v}\|_{L^\infty}\|w\|_{L^p} + C\|\nabla\theta\|_{L^p} + C\|\nabla b\|_{L^\infty}^{\frac{2p+pN-2N}{2p}}\bigl(\|\nabla\mathbf{v}\|_{L^\infty} + 1\bigr) \\ &\qquad + C\|\nabla b\|_{L^\infty}\bigl(\|\nabla\mathbf{v}\|_{L^\infty} + 1\bigr) \\ &\qquad + C\|\nabla b\|_{L^\infty}\bigl(\|\nabla\mathbf{v}\|_{L^\infty} + 1\bigr)\|w\|_{L^p} \end{aligned}$$

$$+ C\|\nabla b\|_{L^\infty}\|\nabla\theta\|_{L^p}^{\frac{pN-2N}{pN-2N+2p}}\big(\|\nabla \mathbf{v}\|_{L^\infty}+1\big)^{\frac{2p}{pN-2N+2p}}.$$

It follows by Gronwall's inequality and (2.16) that

$$\begin{aligned}
&\|w(t)\|_{L^p}\\
&\quad\le \exp\left[C\int_0^t\big(\|\nabla \mathbf{v}(\tau)\|_{L^\infty}+\|\nabla b(\tau)\|_{L^\infty}\big(\|\nabla \mathbf{v}(\tau)\|_\infty+1\big)\big)\,d\tau\right]\\
&\qquad\times\left[\|w_0\|_{L^p}+C\int_0^t\Big(\|\nabla b(\tau)\|_{L^\infty}^{\frac{2p+pN-2N}{2p}}\big(\|\nabla \mathbf{v}\|_{L^\infty}+1\big)+\|\nabla b(\tau)\|_{L^\infty}\big(\|\nabla \mathbf{v}\|_{L^\infty}+1\big)\right.\\
&\qquad\left.+\|\nabla b(\tau)\|_{L^\infty}\|\nabla\theta(\tau)\|_{L^p}^{\frac{pN-2N}{pN-2N+2p}}\big(\|\nabla \mathbf{v}\|_{L^\infty}+1\big)^{\frac{2p}{pN-2N+2p}}+\|\nabla\theta(\tau)\|_{L^p}\Big)\,d\tau\right]\\
&\quad\le C\big(\|w_0\|_{L^p},\|\nabla b_0\|_{L^\infty},\|\nabla\theta_0\|_{L^p}\big)\exp\exp\left(C\int_0^t\|\nabla \mathbf{v}\|_{L^\infty}\,d\tau\right).
\end{aligned}\tag{2.18}$$

Integrating (2.15) in time and exploiting (2.16) and (2.18), we finally deduce

$$\mathcal{N}(t)\le Ce^{Ct}\exp\exp\exp\left[C\int_0^t\|\nabla \mathbf{v}(\tau)\|_{L^\infty}\,d\tau\right],$$

which ends the proof of the "$\Rightarrow$" part in Theorem 1.1.

Finally, we show the "$\Leftarrow$" part in (1.7). Assume $a, b, \mathbf{v}$, and θ remain smooth on the time interval $[0, T^*]$, i.e.,

$$\sup_{0\le t\le T}\big(\|(a,b,\mathbf{v},\theta)(\cdot,t)\|_{H^s}\big)\le C_{T^*}<\infty.$$

Since $s > 1 + \frac{N}{2}$, by the Sobolev inequality,

$$\|\nabla \mathbf{v}(\cdot,t)\|_{L^\infty}\le\|\mathbf{v}(\cdot,t)\|_{H^s}\le C_{T^*},\quad 0\le t\le T^*,$$

which yields that

$$\int_0^{T^*}\|\nabla \mathbf{v}(\cdot,\tau)\|_{L^\infty}\,d\tau\le M_{T^*}<\infty.$$

This finishes the proof of Theorem 1.1. □

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