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On the existence of three solutions of Dirichlet fractional systems involving the *p*-Laplacian with Lipschitz nonlinearity

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On the occasion of the 80th birthday of the second author's mother, Mrs. Fatma Bint Al-Tayeb Zeghdoud.

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Abstract
A class of perturbed fractional nonlinear systems is studied. The dynamical system possesses two control parameters and a Lipschitz nonlinearity order of $p-1$. The multiplicity of the weak solutions is proved by means of the variational method and by Ricceri critical points theorems. An illustrative example is analyzed in order to highlight the obtained result.

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1 Introduction

Fractional differential equations have proved to be promising tools in the modeling of diverse phenomena in various fields, such as physics, chemistry, biology, engineering and economics. In recent years, there was a significant development in fractional differential equations due to the possibility of accounting for a larger class of memory properties. For instance, consider the studies of Miller and Ross [1], Boulaaras et al. [2, 3], Podlubny [4], Hilfer [5], Kilbas et al. [6], and the related papers [1, 7–16] and the references therein.

Critical point theory was very useful in determining the existence of solutions to complete differential equations with certain boundary conditions; see, for example, in the extensive literature on the subject, the classical books [17–19], and the references therein. However, so far, some problems were created for fractional boundary value problems (F-BVP) by exploiting this approach, where it is often very difficult to create suitable spaces and functions for fractional problems.

In [20], the authors investigated the following nonlinear fractional differential equation depending on two parameters:

$$\begin{cases} {}_tD_T^{\alpha_i}(a_i(t){}_0D_t^{\alpha_i}u_i(t)) \\ \quad = \lambda F_{u_i}(t,u_1(t),u_2(t),\dots,u_n(t)) \\ \qquad + \mu G_{u_i}(t,u_1(t),u_2(t),\dots,u_n(t)) + h_i(u_i) \quad \text{a.e } [0,T], \\ u_i(0) = u_i(T) = 0, \end{cases} \tag{1.1}$$

for $1 \leq i \leq n$, where $\alpha_i \in (0;1]$, ${}_0D_t^{\alpha_i}$ and ${}_tD_T^{\alpha_i}$ are the left and right Riemann–Liouville fractional derivatives of order α_i, respectively, with $a_i \in L^\infty([0,T])$ for

$$a_{i0} = \operatorname{ess}\inf_{[0,T]} a_i(t) > 0 \quad \text{for } 1 \leq i \leq n, \lambda, \mu$$

are positive parameters, $F, G : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ are measurable functions with respect to $t \in [0,T]$ for every $(x_1,\dots,x_n) \in \mathbb{R}^n$ and are C^1 with respect to $(x_1,\dots,x_n) \in \mathbb{R}^n$ for a.e. $t \in [0,T]$, F_{u_i} and G_{u_i} denote the partial derivative of F and G with respect to u_i, respectively, and $h_i : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions with the Lipschitz constants $L_i > 0$ for $1 \leq i \leq n$, i.e.,

$$\left|h_i(x_1) - h_i(x_2)\right| \leq L_i|x_1 - x_2|$$

for every $x_1, x_2 \in \mathbb{R}$, and $h_i(0) = 0$ for $1 \leq i \leq n$. Motivated by [21] and [22], using a three critical points theorem obtained in [23], which is recalled in the next section (Theorem 2), the existence of at least three solutions for this system is demonstrated.

For example, according to some assumptions, in [24], by using variational methods the authors obtained the existence of at least one weak solution for the following p-Laplacian fractional differential equation [24]:

$$\begin{cases} {}_tD_T^{\alpha}(\phi_p({}_0D_t^{\alpha}u(t))) = \lambda f(t,u(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.2}$$

where ${}_0D_t^{\alpha}$ and ${}_tD_T^{\alpha}$ are the left and right Riemann–Liouville fractional derivatives with $0 < \alpha \leq 1$, respectively, the function $\phi_p(s) = |s|^{p-2}s$, $p > 1$. Taking a class of fractional differential equation with p-Laplacian operator as a model, Li et al. investigated the following equation recently [25]:

$$\begin{cases} {}_tD_T^{\alpha}(\frac{1}{w(t)^{p-2}}\varphi_p({}_0D_t^{\alpha}u(t))) + \lambda u(t) = f(t,u,{}_0^cD_t^{\alpha}u(t)) + h(u(t)) \quad \text{a.e. } t \in [0,T], \\ u(0) = u(T) = 0, \end{cases} \tag{1.3}$$

with $\frac{1}{p} < \alpha \leq 1$, λ a non-negative real parameter.

The function

$$\varphi_p(s) = |s|^{p-2}s, \quad p \geq 2, f : [0;T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

is continuous and $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function.

By using the mountain pass theorem combined with iterative technique, the authors obtained the existence of at least one solution for problem (1.3). In addition, in [20], three weak solutions for a new class of fractional p-Laplacian for boundary value Systems were established by using variational methods and critical point theory. In contrast, motivated by [21] and [22], the existence of at least three solutions for system (1.4) is demonstrated in the present paper, by means of the three critical points theorem obtained in [23], which is recalled in the next section (Theorem 2). This theorem has been successfully employed to establish the existence of at least three solutions for the case of perturbed boundary value problems; see [7, 26–28] and [29].

Consider the following fractional nonlinear system:

$$\begin{cases} {}_tD_T^{\alpha_i}\phi_p({}_0D_t^{\alpha_i}u_i(t)) \\ \quad = \lambda F_{u_i}(t,u_1(t),u_2(t),\dots,u_n(t)) \\ \qquad + \mu G_{u_i}(t,u_1(t),u_2(t),\dots,u_n(t)) \\ \qquad + h_i(u_i(t)) \quad \text{a.e. } t\in[0,T],\ \text{for } 1\le i\le n \\ u_i(0)=u_i(T)=0, \end{cases} \tag{1.4}$$

where $\alpha_i \in (0;1]$, $\phi_p(t)=|t|^{p-2}t$, $t\neq 0$, $\phi_p(0)=0$, $p>1$,${}_0D_t^{\alpha_i}$ and ${}_tD_T^{\alpha_i}$ are the left and right Riemann–Liouville fractional derivatives of order α_i, respectively, for $1\le i\le n$, λ and μ are positive parameters, and $F,G:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ are measurable functions with respect to $t\in[0,T]$ for every $(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$ and are C^1 with respect to $(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$.for a.e. $t\in[0,T]$, F_{u_i}, G_{u_i} denote the partial derivative of F and G with respect to u_i, respectively, and $h_i:\mathbb{R}\to\mathbb{R}$ are Lipschitz continuous functions of order $(p-1)$ with Lipschizian constants $L_i>0$ for $1\le i\le n$, i.e.,

$$\left|h_i(x_1)-h_i(x_2)\right| \le L_i|x_1-x_2|^{p-1} \tag{1.5}$$

for every $x_1,x_2\in\mathbb{R}$, and $h_i(0)=0$ for $1\le i\le n$.

In this paper, the following conditions are assumed:

(H_0) $\alpha_i\in(\frac{1}{p};1]$ for $1\le i\le n$.

$(F1)$ for every $M>0$ and every $1\le i\le n$,

$$\sup_{|(x_1,x_2,\dots,x_n)|\le M}\left|F_{u_i}(t,x_1,x_2,\dots,x_n)\right|\in L^1\big([0,T]\big).$$

$(F2)$ $F(t,0,0,\dots,0)=0$ for a.e. $t\in[0,T]$.

(G) for every $M>0$ and every $1\le i\le n$,

$$\sup_{|(x_1,x_2,\dots,x_n)|\le M}\left|G_{u_i}(t,x_1,x_2,\dots,x_n)\right|\in L^1\big([0,T]\big).$$

This rest of this paper is organized as follows. The next section presents the necessary preliminary to develop the main contribution of this paper. In Sect. 3, the main result (Theorem 2) is derived, and meaningful consequences (Corollaries 1 and 2 and Example 1) are presented.

2 Preliminaries

For the sake of clarity, the necessary definitions and properties of fractional calculus are presented below.

Definition 1 ([6]) Let u be a function defined on $[a,b]$. The left and right Riemann–Liouville fractional derivatives of order $\alpha > 0$ for a function u are defined by

$$ {}_aD_t^{\alpha}u(t) := \frac{d^n}{dt^n}{}_aD_t^{\alpha-n}u(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-\alpha-1}u(s)\,ds $$

and

$$ {}_tD_b^{\alpha}u(t) := (-1)^n\frac{d^n}{dt^n}{}_tD_b^{\alpha-n}u(t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b (t-s)^{n-\alpha-1}u(s)\,ds, $$

for every $t \in [a,b]$, provided the right-hand sides are pointwise defined on $[a,b]$, where $n-1 \leq \alpha < n$ and $n \in \mathbb{N}$.

Here, $\Gamma(\alpha)$ is the gamma function, given by

$$ \Gamma(\alpha) := \int_0^{+\infty} z^{\alpha-1}e^{-z}\,dz. $$

The set $AC^n([a,b],\mathbb{R})$ corresponds to the space of functions $u : [a,b] \to \mathbb{R}$ such that $u \in C^{n-1}([a,b],\mathbb{R})$ and $u^{(n-1)} \in AC^n([a,b],\mathbb{R})$. Here, as usual, $C^{n-1}([a,b],\mathbb{R})$ denotes the set of mappings that are $(n-1)$ times continuously differentiable on $[a,b]$. In particular, $AC([a,b],\mathbb{R}) := AC^1([a,b],\mathbb{R})$.

Proposition 1 ([30]) *The following property of fractional integration holds*:

$$ \int_a^b \bigl[{}_aD_t^{-\alpha}u(t)\bigr]v(t)\,dt = \int_a^b \bigl[{}_tD_b^{-\alpha}v(t)\bigr]u(t)\,dt, \quad \alpha > 0, $$

provided that $u \in L^p([a,b],\mathbb{R})$, $v \in L^q([a,b],\mathbb{R})$ *and* $p \geq 1$, $q \geq 1$, $\frac{1}{p}+\frac{1}{q} \leq 1+\alpha$ *or* $p \neq 1$, $q \neq 1$, $\frac{1}{p}+\frac{1}{q} = 1+\alpha$.

Proposition 2 ([31]) *If* $u(a) = u(b) = 0$, $u \in L^\infty([a,b],\mathbb{R}^N)$, $v \in L^1([a,b],\mathbb{R})$, *or* $v(a) = v(b) = 0$, $v \in L^\infty([a,b],\mathbb{R}^N)$, $u \in L^1([a,b],\mathbb{R})$, *then*

$$ \int_a^b \bigl[{}_aD_t^{\alpha}u(t)\bigr]v(t)\,dt = \int_a^b \bigl[{}_tD_b^{\alpha}v(t)\bigr]u(t)\,dt, \quad 0 < \alpha \leq 1. $$

To establish a variational structure for the main problem, it is necessary to construct appropriate function spaces. Following [32], $C_0^\infty([0,T],\mathbb{R})$ denotes the set of all functions $g \in C^\infty([0,T],\mathbb{R})$ with $g(0) = g(T) = 0$.

Definition 2 ([24]) For $0 < \alpha_i \leq 1$, and for $1 \leq i \leq n$, the fractional derivative space $E_0^{\alpha i,p}$ is defined by

$$ E_0^{\alpha i,p} = \bigl\{u \in L^p\bigl([0,T],\mathbb{R}\bigr) : {}_0D_t^{\alpha_i}u_i \in L^p, u(0) = u(T) = 0\bigr\} $$

with the norm

$$\|u_i\|_{\alpha_i,p} = \left(\|u_i\|_{L^p}^p + \|{}_0D_t^{\alpha_i}u_i\|_{L^p}^p\right)^{\frac{1}{p}}, \quad \forall u_i \in E_0^{\alpha i,p}, \tag{2.1}$$

where

$$\|u_i\|_{L^p} = \left(\int_0^T |u_i|^2\,dt\right)^{\frac{1}{p}}$$

is the norm of $L^p([0,T],\mathbb{R})$ for $1 \le i \le n$, and

$$\|u_i\|_\infty = \max_{t\in[0,T]}\left|u_i(t)\right|. \tag{2.2}$$

From [24, Lemma 3.1], one finds that, for $0 < \alpha_i \le 1$ and $1 < p < +\infty$, the space $E_0^{\alpha_i,p}$ is a reflexive and separable Banach space.

Lemma 1 ([24]) *Let* $0 < \alpha_i \le 1$, *for* $1 \le i \le n$ *and* $1 < p < +\infty$, *for all* $u_i \in E_0^{\alpha_i,p}$ *one has*

$$\|u_i\|_{L^p} \le \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)}\|{}_0D_t^{\alpha_i}u_i\|_{L^p}, \tag{2.3}$$

$$\|u_i\|_\infty \le \frac{T^{\alpha_i-\frac{1}{p}}}{\Gamma(\alpha_i)((\alpha_i-1)q+1)^{\frac{1}{q}}}\|{}_0D_t^{\alpha_i}u_i\|_{L^p}. \tag{2.4}$$

Hence, it is possible to consider $E_0^{\alpha_i,p}$ with respect to the norm

$$\|u_i\|_{\alpha_i,p} = \|{}_0D_t^{\alpha_i}u_i\|_{L^p} = \left(\int_0^T \left|{}_0D_t^{\alpha_i}u_i\right|^p dt\right)^{\frac{1}{p}}, \quad \forall u_i \in E_0^{\alpha_i,p}, \tag{2.5}$$

for $1 \le i \le n$, which is equivalent to (2.1).

Hereafter, let X be the Cartesian product of the n spaces $E_0^{\alpha_i,p}$, i.e., $X = E_0^{\alpha_1,p} \times E_0^{\alpha_2,p} \times \cdots \times E_0^{\alpha_n,p}$ equipped with the norm

$$\|u\| = \sum_{i=1}^n \|u_i\|_{\alpha_i,p}, \quad u = (u_1,u_2,\ldots,u_n),$$

where $\|u_i\|_{\alpha_i,p}$ is defined in (2.5). It is evident that X is compactly embedded in $C([0,T],\mathbb{R})^n$.

Definition 3 A weak solution of system (1.4) consists of any function $u = (u_1,u_2,\ldots,u_n) \in X$, such that, for all $v = (v_1,v_2,\ldots,v_n) \in X$, one finds that

$$\begin{aligned}
&\int_0^T \sum_{i=1}^n \left|{}_0D_t^{\alpha_i}u_i(t)\right|^{p-2}{}_0D_t^{\alpha_i}u_i(t){}_0D_t^{\alpha_i}v_i(t)\,dt \\
&\quad = \lambda\int_0^T \sum_{i=1}^n F_{u_i}\big(t,u_1(t),u_2(t),\ldots,u_n(t)\big)v_i(t)\,dt \\
&\qquad + \mu\int_0^T \sum_{i=1}^n G_{u_i}\big(t,u_1(t),u_2(t),\ldots,u_n(t)\big)v_i(t)\,dt + \int_0^T \sum_{i=1}^n h_i\big(u_i(t)\big)v_i(t)\,dt.
\end{aligned}$$

Remember the following result of [14, Theorem 1], with easy manipulations that are provided in the sequel.

Theorem 1 (Ricceri [14]) *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, bounded on bounded subsets of X, $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exists $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that

(a_1) $\frac{\sup_{\Phi(x)\leq r} \Psi(x)}{r} < \frac{\Phi(\overline{x})}{\Psi(\overline{x})}$;

(a_2) *for each $\lambda \in \Lambda_r; =]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)\leq r} \Psi(x)}[$, the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each compact interval $[a, b] \subseteq \Lambda_r$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1 functional $F : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\Phi'(x) - \lambda\Psi'(x) - \mu F'(x) = 0$$

has at least three solutions in X whose norms are less than ρ.

3 The main results

In the present section, the existence of multiple solutions for system (1.1) is discussed. For any $\varsigma > 0$, $K(\varsigma)$ denotes

$$\left\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \frac{1}{p}\sum_{i=1}^{n} |x_i|^p \leq \varsigma\right\}.$$

This set is one of the cornerstones of the given hypotheses for appropriate choices of ς. For $u = (u_1, u_2, \dots, u_n) \in X$ one has

$$\Upsilon(u) := \sum_{i=1}^{n} \Upsilon_i(u_i),$$

where

$$\Upsilon_i(x) := \int_0^T H_i\bigl(x(s)\bigr)\,ds \quad \text{and} \quad H_i(x) := \int_0^x h_i(z)\,dz, \quad 1 \leq i \leq n,$$

for every $t \in [0, T]$ and $x \in \mathbb{R}$. Moreover, let

$$c := \max_{1\leq i\leq n}\left\{\frac{T^{\alpha_i - \frac{1}{p}}}{\Gamma(\alpha_i)((\alpha_i - 1)q + 1)^{\frac{1}{q}}}\right\},$$

$$k := \min_{1\leq i\leq n}\left\{1 - \frac{L_i T^{\alpha_i p}}{(\Gamma(\alpha_i + 1))^p}\right\},$$

$$\tau := \max_{1\leq i\leq n}\left\{1 + \frac{L_i T^{p\alpha_i}}{(\Gamma(\alpha_i + 1))^p}\right\}.$$

Theorem 2 *Suppose that $k > 0$ and the conditions $(F1)$, $(F2)$, (G) and (H) are satisfied. Furthermore, assume that there exist a positive constant r and a function $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in X$ such that*

(i)

$$\frac{1}{p}\sum_{i=1}^{n} \|\omega_i\|_{\alpha_i}^{p} > \frac{r}{k};$$

(ii)

$$\frac{r\int_0^T F(t,\omega_1,\omega_2,\dots,\omega_n)\,dt}{\frac{1}{p}\sum_{i=1}^{n}\|\omega_i\|_{\alpha_i}^{p} - \Upsilon(\omega_1,\omega_2,\dots,\omega_n)} - \int_0^T \max_{(x_1,x_2,\dots,x_n)\in K(\frac{cr}{k})} F(t,x_1,x_2,\dots,x_n)\,dt > 0;$$

(iii)

$$\lim_{(|x_1|,|x_2|,\dots,|x_n|)\to(+\infty,+\infty,\dots,+\infty)} \sup \frac{\sup_{t\in[0,T]} F(t,x_1,x_2,\dots,x_n)}{\frac{1}{p}\sum_{i=1}^{n}|x_i|^p} \le 0.$$

Then, setting

$$\Lambda_r := \left]\frac{\frac{1}{p}\sum_{i=1}^{n}\|\omega_i\|_{\alpha_i}^{p} - \Upsilon(\omega_1,\omega_2,\dots,\omega_n)}{\int_0^T F(t,\omega_1,\omega_2,\dots,\omega_n)\,dt}, \frac{r}{\int_0^T \max_{(x_1,x_2,\dots,x_n)\in K(\frac{cr}{k})} F(t,x_1,x_2,\dots,x_n)\,dt}\right[,$$

for each compact interval $[a,b] \subseteq \Lambda_r$, there exists $\rho > 0$ with the following property: for every $\lambda \in [a,b]$ there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, system (1.4) admits at least three solutions in X whose norms are less than ρ.

Proof For each $u = (u_1, u_2, \dots, u_n) \in X$, define Φ, $\Psi : X \to \mathbb{R}$ as

$$\Phi(u) := \frac{1}{p}\sum_{i=1}^{n}\|u_i\|_{\alpha_i,p}^{p} - \Upsilon(u)$$

and

$$\Psi(u) := \int_0^T F\big(t, u_1(t), u_2(t), \dots, u_n(t)\big)\,dt.$$

Clearly, Φ and Ψ are continuously Gâteaux differentiable functionals whose Gâteaux derivatives at the point $u \in X$ are given by

$$\Phi'(u)(v) := \int_0^T \sum_{i=1}^{n} \big|{}_0D_t^{\alpha_i}u_i(t)\big|^{p-2}\,{}_0D_t^{\alpha_i}u_i(t)\,{}_0D_t^{\alpha_i}v_i(t)\,dt - \int_0^T \sum_{i=1}^{n} h_i\big(u_i(t)\big)v_i(t)\,dt,$$

$$\Psi'(u)(v) = \int_0^T \sum_{i=1}^{n} F_{u_i}\big(t, u_1(t), u_2(t), \dots, u_n(t)\big)v_i(t)\,dt,$$

for every $v = (v_1, v_2, \dots, v_n) \in X$. Hence, $\Phi - \lambda\Psi \in C^1(X, \mathbb{R})$. Moreover, $\Psi' : X \to X^*$ is a compact operator (see the proof of [21, Theorem 3.1]). Furthermore, similar to the proof of [22, Theorem 3.1], we can show that Φ is sequentially weakly lower semicontinuous. As concerns functional Φ, it is easy to show that Φ is bounded on each bounded subset of X and its derivative admits a continuous inverse on X^*. Moreover, we have $\Phi(0) = \Psi(0) = 0$.

It is shown that the required hypothesis $\Phi(\overline{x}) > r$ follows from (i) and the definition of Φ, by choosing $\overline{x} = \omega$. Indeed, since (1.5) holds for every $x_1, x_2 \in \mathbb{R}$ and $h_1(0) = h_2(0) = \cdots = h_n(0) = 0$, one has $|h_i(x)| \leq L_i|x|^{p-1}$, $1 \leq i \leq n$, for all $x \in \mathbb{R}$. Besides, it follows from (2.3) that

$$\begin{aligned}
\Phi(\omega) &\geq \frac{1}{p}\sum_{i=1}^{n}\|\omega_i\|_{\alpha_i,p}^{p} - \left|\int_0^T \sum_{i=1}^{n} H_i\big(\omega_i(t)\big)\,dt\right| \\
&\geq \frac{1}{p}\sum_{i=1}^{n}\|\omega_i\|_{\alpha_i,p}^{p} - \frac{1}{p}\sum_{i=1}^{n} L_i \int_0^T |\omega_i|^{p_i}\,dt \\
&\geq \sum_{i=1}^{n}\left(\frac{1}{p} - \frac{L_i T^{\alpha_i p}}{p(\Gamma(\alpha_i+1))^{p_i}}\right)\|\omega_i\|_{\alpha_i,p}^{p} \\
&\geq \frac{k}{p}\sum_{i=1}^{n}\|\omega_i\|_{\alpha_i,p}^{p} > r.
\end{aligned} \tag{3.1}$$

From (2.2) and (2.4), for every $u_i \in E_0^{\alpha_i,p}$ one has

$$\max_{t\in[0,T]}\big|u_i(t)\big|^p \leq c\|u_i\|_{\alpha_i,p}^{p}, \tag{3.2}$$

for each $u = (u_1, u_2, \ldots, u_n) \in X$. From (2.4), (3.1) and (3.2), for each $r > 0$ one obtains

$$\begin{aligned}
&\Phi^{-1}((-\infty;r]) \\
&\quad = \big\{u = (u_1, u_2, \ldots, u_n) \in X : \Phi(u) \leq r\big\} \\
&\quad \subseteq \left\{u = (u_1, u_2, \ldots, u_n) \in X : \frac{1}{p}\sum_{i=1}^{n}\|u_i\|_{\alpha_i,p}^{p} \leq \frac{r}{k}\right\} \\
&\quad \subseteq \left\{u = (u_1, u_2, \ldots, u_n) \in X : \frac{1}{p}\sum_{i=1}^{n}\frac{(\Gamma(\alpha_i))^p(((\alpha_i-1)q+1))^{\frac{p}{q}}}{T^{\alpha_i p-1}}\|u_i\|_{\infty}^{p} \leq \frac{r}{k}\right\} \\
&\quad \subseteq \left\{u = (u_1, u_2, \ldots, u_n) \in X : \frac{1}{p}\sum_{i=1}^{n}|u_i|^p \leq \frac{cr}{k}, \text{ for all } t \in [0,T]\right\}.
\end{aligned}$$

Then

$$\begin{aligned}
\sup_{u\in\Phi^{-1}((-\infty;r])}\Psi(u) &= \sup_{u\in\Phi^{-1}((-\infty;r])}\int_0^T F\big(t, u_1(t), u_2(t), \ldots, u_n(t)\big)\,dt \\
&\leq \int_0^T \max_{(x_1,x_2,\ldots,x_n)\in K(\frac{cr}{k})} F(t, x_1, x_2, \ldots, x_n)\,dt.
\end{aligned}$$

Therefore, from the condition (ii), one gets

$$\begin{aligned}\sup_{u\in\Phi^{-1}((-\infty;r])}\Psi(u) &\le \int_0^T \max_{(x_1,x_2,\dots,x_n)\in K(\frac{cr}{k})} F(t,x_1,x_2,\dots,x_n)\,dt\\ &< \frac{r\int_0^T F(t,\omega_1,\omega_2,\dots,\omega_n)\,dt}{\frac{1}{p}\sum_{i=1}^n \|\omega_i\|_{\alpha_i,p}^p - \Upsilon(\omega_1,\omega_2,\dots,\omega_n)}\\ &= \frac{r\int_0^T F(t,\omega_1,\omega_2,\dots,\omega_n)\,dt}{\frac{1}{p}\sum_{i=1}^n \|\omega_i\|_{\alpha_i,p}^p - \Upsilon(\omega_1,\omega_2,\dots,\omega_n)}\\ &= r\frac{\Psi(w)}{\Phi(w)},\end{aligned}$$

from which assumption (a_1) of Theorem 1 follows. Fix $0<\epsilon<\frac{1}{pTc\lambda}$; from (iii) there is a constant τ_ϵ such that

$$F(t,x_1,x_2,\dots,x_n) \le \epsilon\sum_{i=1}^n |x_i|^p + \tau_{\epsilon_i} \tag{3.3}$$

for every $t\in[0,T]$ and for every $(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$. Taking (2.4) into account, from (3.3), it follows that, for each $u\in X$,

$$\begin{aligned}\Phi(u)-\lambda\Psi(u) &= \frac{1}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i,p}^p - \lambda\int_0^T F(t,u_1,u_2,\dots,u_n)\,dt\\ &\ge \frac{1}{p}\sum_{i=1}^n \|u_i\|_{\alpha_i,p}^p - T\lambda c\epsilon\sum_{i=1}^n \|u_i\|_{\alpha_i,p}^p - \lambda\tau_\epsilon\\ &\ge \left(\frac{1}{p} - T\lambda c\epsilon\right)\sum_{i=1}^n \|u_i\|_{\alpha_i,p}^p - \lambda\tau_\epsilon,\end{aligned}$$

and thus

$$\lim_{\|u\|\to+\infty}\bigl(\Phi(u)-\lambda\Psi(u)\bigr) = +\infty,$$

which means the functional $\Phi(u)-\lambda\Psi(u)$ is coercive for every parameter λ, in particular, for every $\lambda\in\Lambda\subset]\frac{\Phi(\omega)}{\Psi(\omega)},\frac{r}{\sup_{\Phi(u)\le r}\Psi(u)}[$. Then also condition (a_2) holds.

In addition, since $G:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ is a measurable function with respect to $t\in[0,T]$ for every $(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$ belonging to C^1 with respect to $(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$ for a.e. $t\in[0,T]$ satisfying condition (**G**), the functional

$$F(u) = \int_0^T G\bigl(t,u_1(t),u_2(t),\dots,u_n(t)\bigr)\,dt$$

is well defined and continuously Gâteaux differentiable on X with a compact derivative, and

$$F'(u) = \int_0^T \sum_{i=1}^n G_{u_i}\bigl(t,u_1(t),u_2(t),\dots,u_n(t)\bigr)v_i(t)\,dt$$

for all $(u_1, u_2, \dots, u_n)$, $(v_1, v_2, \dots, v_n) \in X$. Thus, all the hypotheses of Theorem 1 are satisfied. Also note that the solutions of the equation

$$\Phi'(x) - \lambda\Psi'(x) - \mu F'(x) = 0$$

are exactly the solutions of (1.4) (see [21]). So, the conclusion follows from Theorem 1. □

Example 1 Consider the following fractional boundary value problem:

$$\begin{cases} {}_tD_T^{0,75}\phi_3({}_0D_t^{0,75}u_1(t)) = \lambda F_{u_1}(t,u_1(t),u_2(t)) + \mu G_{u_1}(t,u_1(t),u_2(t)) + h_1(u_1(t)) \\ \quad \text{a.e. } t \in [0,T], \\ {}_tD_T^{0,6}\phi_3({}_0D_t^{0,6}u_2(t)) = \lambda F_{u_2}(t,u_1(t),u_2(t)) + \mu G_{u_1}(t,u_1(t),u_2(t)) + h_2(u_2(t)) \\ \quad \text{a.e. } t \in [0,T], \\ u_1(0) = u_2(0) = u_1(1) = u_2(1) = 0, \end{cases} \tag{3.4}$$

where $\alpha_1 = 0.75$, $\alpha_2 = 0.6$, $p = 3$, $T = 1$, $h_1(u_1) = (\sin(\frac{u_1}{2}))^2$, $h_2(u_2) = (\arctan(\frac{u_2}{3}))^2$ and $G : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is an arbitrary function which is measurable with to respect to $t \in [0,1]$ for every $(x_1,x_2) \in \mathbb{R}^2$ and is C^1 with respect to $(x_1,x_2) \in \mathbb{R}^2$ for a.e. $t \in [0,1]$, satisfying

$$\sup_{|(x_1,x_2)|\le M} \left|G_{u_i}(t,x_1,x_2)\right| \in L^1([0,T]),$$

for every $M > 0$ and $i = 1,2$. Moreover, for all $(t,x_1,x_2) \in [0,1] \times \mathbb{R}^2$, put $F(t,x_1,x_2) = (1 + t^2)H(x_1,x_2)$, where

$$H(x_1,x_2) = \begin{cases} (x_1^3 + x_2^3)^2, & x_1^3 + x_2^3 \le 1, \\ 2\sqrt{x_1^3 + x_2^3} - (x_1^3 + x_2^3), & x_1^3 + x_2^3 > 1. \end{cases}$$

Obviously, $F(t,0,0) = 0$ for all $t \in [0,1]$, and a direct calculation shows that

$$c \approx 1.0727, \qquad k \approx 0.3559.$$

By choosing, for instance,

$$\omega_1(t) = \Gamma(1,25)t(1-t), \qquad \omega_2(t) = \Gamma(1,4)t(1-t),$$

and $r = \frac{1}{10^3}$ all assumptions of Theorem 2 are satisfied. In fact, $\omega_i(0) = \omega_i(1) = 0$, $i = 1,2$, and

$${}_0D_t^{0,75}\omega_1(t) = t^{0,25} - \frac{2\Gamma(1,25)}{\Gamma(2,25)}t^{1,25}, \qquad {}_0D_t^{0,6}\omega_2(t) = t^{0,4} - \frac{2\Gamma(1,4)}{\Gamma(2,4)}t^{1,4}.$$

Then one has

$$\|\omega_1\|_{0.75}^3 \approx 0.0498, \qquad \|\omega_2\|_{0.6}^3 \approx 0.0233,$$

which implies that the condition (i) holds, and

$$\frac{\int_0^1 \max_{(x_1,x_2)\in\pi(\frac{cr}{k})} F(t,x_1,x_2)\,dt}{r} = \frac{12c^2 r}{k^2} \approx 0.1090$$
$$< \frac{\int_0^1 F(t,\omega_1,\omega_2)\,dt}{\frac{1}{3}\sum_{i=1}^2 \|\omega_i\|_{\alpha_i}^3 - \Upsilon(\omega_1,\omega_2)} \approx 0.5548$$

and

$$\lim_{(|x_1|,|x_2|)\to(+\infty,+\infty)} \sup \frac{\sup_{t\in[0,1]} F(t,u_1,u_2)}{\frac{1}{3}\sum_{i=1}^2 |u_i|^3} = 0.$$

Thus, conditions (ii) and (iii) are satisfied. Then, in view of Theorem 2 for each $\lambda \in]1.8025, 9.1743[$, system (3.4) has at least three weak solutions in $X = E_0^{0,75,3} \times E_0^{0,6,3}$.

Next, it is desirable to give a verifiable consequence of Theorem 2 for a fixed text function ω. For a given constant $\gamma \in (0,\frac{1}{2})$ and for all $1 \leq i \leq n$, set

$$C_i(\alpha_i,\gamma) = \frac{1}{p(\gamma T)^p}\left\{\int_0^{\gamma T} t^{p(1-\alpha_i)}\,dt + \int_{\gamma T}^{(1-\gamma)T} \left(t^{1-\alpha_i} - (t-\gamma T)^{1-\alpha_i}\right)^p dt \right.$$
$$\left. + \int_{(1-\gamma)T}^{T} \left(t^{1-\alpha_i} - (t-\gamma T)^{1-\alpha_i}\right) - \left(1 - \left((1-\gamma)T\right)^{1-\alpha_i}\right)^p \right\},$$
$$\triangle = \min_{1\leq i\leq n}\left\{\sum_{i=1}^n C_i(\alpha_i,\gamma)\right\},$$
$$\triangle' = \max\left\{\sum_{i=1}^n C_i(\alpha_i,\gamma)\right\}.$$

Corollary 1 *Let assumption* (iii) *in Theorem* 2 *hold. Assume that there exist positive constants d and η such that $\frac{d}{\triangle ckn} \geq \eta^p$, and also*

(j) $F(t,x_1,x_2,\ldots,x_n) \geq 0$, *for each* $(t,x_1,x_2,\ldots,x_n) \in [0,T]\times[0;+\infty)\times\cdots\times[0;+\infty)$;

(jj) $\frac{\int_0^T \max_{(x_1,x_2,\ldots,x_n)\in K(d)} F(t,x_1,x_2,\ldots x_n)\,dt}{kd} < \frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\Gamma(2-\alpha_1)\eta,\ldots,\Gamma(2-\alpha_n)\eta)\,dt}{nc\tau\triangle'\eta^p}$.

Then, setting

$$\Lambda_1 := \left(\frac{n\tau\triangle'\eta^p}{\int_{\gamma T}^{(1-\gamma)T} F(t,\Gamma(2-\alpha_1)\eta,\ldots,\Gamma(2-\alpha_n)\eta)\,dt},\right.$$
$$\left.\frac{kd}{\int_0^T \max_{(x_1,x_2,\ldots,x_n)\in K(d)} F(t,x_1,x_2,\ldots x_n)\,dt}\right)$$

for each compact interval $[a,b] \subseteq \Lambda_1$, there exists $\rho > 0$ with the following property: for every $\lambda \in]a,b[$, there exists δ such that, for each $\mu \in [0,\delta]$, system (1.4) admits at least three solutions in X whose norms are less than ρ.

Proof For $\gamma \in (0, \frac{1}{2})$ choose $\omega(t) = (\omega_1(t), \dots, \omega_n(t))$ for every $t \in [0, T]$ with

$$\omega_i(t) = \begin{cases} \frac{\Gamma(2-\alpha_i)\eta}{\gamma T} t, & t \in [0; \gamma T), \\ \Gamma(2-\alpha_i)\eta, & t \in [\gamma T; (1-\gamma)T], \\ \frac{\Gamma(2-\alpha_i)\eta}{\gamma T}(t-T), & t \in ((1-\gamma)T; T], \end{cases}$$

for $1 \leq i \leq n$, Clearly $\omega_i(0) = \omega_i(T) = 0$ and $\omega_i \in L^2([0,T], \mathbb{R})$ for $1 \leq i \leq n$,A direct calculation shows that

$${}_0D_t^{\alpha_i}\omega_i(t) = \begin{cases} \frac{\eta}{\gamma T} t^{1-\alpha_i}, & t \in [0; \gamma T), \\ \frac{\eta}{\gamma T}(t^{1-\alpha_i} - (t-\gamma T)^{1-\alpha_i}), & t \in [\gamma T; (1-\gamma)T], \\ \frac{\eta}{\gamma T}(t^{1-\alpha_i} - (t-\gamma T)^{1-\alpha_i} - (t-(1-\gamma)T)^{1-\alpha_i}), & t \in ((1-\gamma)T; T], \end{cases}$$

for $1 \leq i \leq n$. Furthermore,

$$\begin{aligned} \int_0^T \left|{}_0D_t^{\alpha_i}\omega_i(t)\right|^p dt &= \left(\frac{\eta}{\gamma T}\right)^p \left\{ \int_0^{\gamma T} t^{(1-\alpha_i)p}\, dt + \int_{hT}^{(1-\gamma)T} \left(t^{1-\alpha_i} - (t-\gamma T)^{1-\alpha_i}\right)^p dt \right. \\ &\quad \left. + \int_{(1-h)T}^{T} \left(t^{1-\alpha_i} - (t-\gamma T)^{1-\alpha_i} - \left(t-(1-\gamma)T\right)^{1-\alpha_i}\right)^p dt \right\} \\ &= p\eta^p C_i(\alpha_i, h), \end{aligned}$$

for $1 \leq i \leq n$. Thus, $\omega \in X$, and

$$\|\omega_i\|_{\alpha_i,p}^p = p\eta^p C_i(\alpha_i, h),$$

with $1 \leq i \leq n$. This and (3.1) imply that

$$\begin{aligned} \Phi(\omega) = \Phi(\omega_1, \dots, \omega_n) &= \frac{1}{p}\sum_{i=1}^n \|\omega_i\|_{\alpha_i,p}^p - \Upsilon(\omega_i) \\ &\geq \frac{k}{p}\sum_{i=1}^n \|\omega_i\|_{\alpha_i,p}^p \\ &\geq k\eta^p \sum_{i=1}^n C_i(\alpha_i, h) \\ &\geq nk\triangle\eta^p. \end{aligned} \tag{3.5}$$

Similarly to (3.1) and (3.5) one has

$$\Phi(\omega) \leq n\tau\triangle'\eta^p.$$

Let $r = \frac{kd}{c}$. From $\frac{d}{\triangle ckn} < \eta^p$, it is found as a result that

$$\frac{1}{p}\sum_{i=1}^n \|\omega_i\|_{\alpha_i,p}^p \geq \Phi(\omega) \geq nk\triangle\eta^p \geq nk\triangle \times \frac{d}{\triangle ckn} = \frac{r}{k},$$

which is assumption (i) of Theorem 2.

On the other hand, by using assumption (j), one can infer

$$\begin{aligned}\Psi(\omega) &:= \int_0^T F\big(t,\omega_1(t),\omega_2(t),\dots,\omega_n(t)\big)\,dt \\ &\geq \int_{\gamma T}^{(1-\gamma)T} F\big(t,\Gamma(2-\alpha_1)\eta,\Gamma(2-\alpha_2)\eta,\dots,\Gamma(2-\alpha_n)\eta\big)\,dt.\end{aligned}$$

Moreover, by condition (jj) one gets

$$\begin{aligned}&\frac{\int_0^T \max_{(x_1,x_2,\dots,x_n)\in K(\frac{cr}{k})} F(t,x_1,x_2,\dots x_n)\,dt}{r} \\ &= \frac{c\int_0^T \max_{(x_1,x_2,\dots,x_n)\in K(d)} F(t,x_1,x_2,\dots x_n)\,dt}{kd} \\ &< \frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\Gamma(2-\alpha_1)\eta,\dots,\Gamma(2-\alpha_n)\eta)\,dt}{n\tau\triangle'\eta^p} \\ &\leq \frac{\int_{\gamma T}^{(1-\gamma)T} F(t,\Gamma(2-\alpha_1)\eta,\dots,\Gamma(2-\alpha_n)\eta)\,dt}{\Phi(\omega)} \\ &\leq \frac{p\int_0^T F(t,\omega_1,\omega_2,\dots,\omega_n)\,dt}{\sum_{i=1}^n \|\omega_i\|_{\alpha_i,p}^p - p\Upsilon(\omega_1,\omega_2,\dots,\omega_n)},\end{aligned}$$

which implies that (ii) is satisfied. Thus, all the assumptions of Theorem 2 are satisfied and the proof is complete. □

Corollary 2 *Let $F:\mathbb{R}^n\to\mathbb{R}^n$ be a C^1-function and $F(0,\dots,0)=0$. Assume that there exist positive constants d and η such that $\frac{d}{\triangle ckn}<\eta^p$, and also*

(H) $F(x_1,\dots,x_n)\geq 0$, *for each* $(x_1,\dots,x_n)\in[0;+\infty)\times\cdots\times[0;+\infty)$;

(HH) $\frac{\max_{(x_1,x_2,\dots,x_n)\in K(d)} F(x_1,x_2,\dots x_n)}{kd} < \frac{(1-2\gamma)F(\Gamma(2-\alpha_1)\eta,\dots,\Gamma(2-\alpha_n)\eta)}{nc\tau\triangle'\eta^p}$;

(HHH) $\lim_{(|x_1|,|x_2|,\dots,|x_n|)\to(+\infty,+\infty,\dots,+\infty)}\sup\frac{F(x_1,x_2,\dots,x_n)}{\frac{1}{p}\sum_{i=1}^n|x_i|^p}\leq 0$.

Then, setting

$$\begin{aligned}\Lambda_2 := \bigg(&\frac{n\tau\triangle'\eta^p}{T(1-2\gamma)F(\Gamma(2-\alpha_1)\eta,\dots,\Gamma(2-\alpha_n)\eta)}, \\ &\frac{kd}{cT\max_{(x_1,x_2,\dots,x_n)\in K(d)} F(x_1,x_2,\dots x_n)}\bigg),\end{aligned}$$

for each compact interval $[a,b]\subseteq\Lambda_2$, there exists $\rho>0$ with the following property: for every $\lambda\in]a,b[$, there exists $\delta>0$ such that, for each $\mu\in[0,\delta]$, system (1.4) admits at least three solutions in X whose norms are less than ρ.

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