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Quasilinear asymptotically periodic Schrödinger–Poisson system with subcritical growth

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Abstract

The aim of this paper is establishing the existence of a nontrivial solution for the following quasilinear Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + V(x)u - u\Delta(u^2) + K(x)\phi(x)u = g(x, u), & x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), & u > 0, \end{cases}$$

where V, K, g are continuous functions. To overcome the technical difficulties caused by the quasilinear term, we change the variable to guarantee the feasibility of applying the mountain pass theorem to solve the above problems. We use the mountain pass theorem and the concentration–compactness principle as basic tools to gain a nontrivial solution the system possesses under an asymptotic periodicity condition at infinity.

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Keywords: Quasilinear Schrödinger equation; Asymptotically periodic; Mountain pass theorem

1 Introduction and main results

In this paper, we consider the following quasilinear asymptotically periodic Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + V(x)u - u\Delta(u^2) + K(x)\phi(x)u = g(x, u), & x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), & u > 0, \end{cases} \quad (1.1)$$

where $V, K : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $g : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions.

The so-called quasilinear Schrödinger–Poisson system was introduced in [5, 25] and is a quantum mechanical model of extremely small devices in semiconductor nanostructures taking into account the quantum structure and longitudinal field oscillations during the beam propagation. As far as we know, this new significant physical system was not

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mentioned before [5, 25], even though it is worth exploring from a mathematical point of view.

System (1.1) without quasilinear potential $-u\Delta(u^2)$ is called the Schrödinger–Poisson system,

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi(x)u = g(x, u), & x \in \mathbb{R}^3, \\ -\Delta \phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

which has been widely studied, and many meaningful results were achieved for the sub-critical growth [7, 27, 44, 45, 47] or the critical exponent [46] under various assumptions on the potentials and nonlinearities. Furthermore, nontrivial solutions, radial solutions [10, 13, 29], ground state solutions [2, 4] and also semiclassical solutions [14, 23, 24] were obtained generally. Moreover, quasilinear Schrödinger equations with $-u\Delta(u^2)$:

$$-\Delta u + V(x)u - u\Delta(u^2) = g(x, u), \quad x \in \mathbb{R}^N,$$

have been accepted as models of several physical phenomena; we refer the reader to the introduction in [30] and references therein for a discussion on the subject. The existence of solutions of quasilinear equations was considered in [1, 6, 11, 12, 18, 43].

There are scarce existing results on system (1.1), which is a new combination of two different equations. So our discussion is worth considering and has an excellent prospect; this paper is also an innovation of the pioneering work. When $K(x) = 1$ in system (1.1), the authors in [8] took into account the systems by applying the perturbation method, and hence the existence of a sign-changing solution with precisely two nodal domains was derived. The authors in [15] studied the existence and asymptotic behavior of ground state in the whole space \mathbb{R}^3 for the quasilinear Schrödinger–Poisson system with asymptotically linear $f(t)$ with respect to t at infinity; see also [28] for more related results. Figueiredo and Siciliano [19, 20] paid close attention to two different systems with critical growth and obtained the existence and asymptotic behavior of solutions for the previous system in \mathbb{R}^3 and also achieved a similar result for the latter system in a bounded domain in \mathbb{R}^2 . The authors in [31] considered a system with radial potentials and discontinuous nonlinearity and obtained multiplicity results for radial solutions. By using Ekeland's variational principle and the mountain pass theorem the authors in [33] obtained the existence of the ground state solution for a generalized quasilinear Schrödinger–Poisson system in \mathbb{R}^3 . For related results on nonlinear problems based on variational methods, we refer to [21, 28, 32, 38–42], which are recent contributions to Kirchhoff-type problems. Tang [35] developed a direct method, a non-Nehari manifold method, to find a minimizing Cerami sequence for the energy functional outside the Nehari–Pankov manifold by using the diagonal method. Tang, Lin, and Yu [36] first introduced a local superquadratic condition instead of the classical global superquadratic condition. Chen and Tang [9] proposed a new approach to recover the compactness for the Cerami sequence.

Guided by the method of [34], our main goal is establishing the existence of a solution to problem (1.1) under an asymptotic periodicity condition at infinity.

Letting \mathcal{F} denote the class of functions $h \in C(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3)$ such that, for every $\varepsilon > 0$, the set $\{x \in \mathbb{R}^3 : |h(x)| \geq \varepsilon\}$ has finite Lebesgue measure, we suppose that V and K are perturbations of periodic functions at infinity. More specifically, we impose the following assumptions on V and K :

(V) there exist a constant $\alpha > 0$ and a function $V_0 \in C(\mathbb{R}^3, \mathbb{R})$, 1-periodic in x_i , $1 \leq i \leq 3$, such that $V_0 - V \in \mathcal{F}$ and

$$V_0(x) \geq V(x) \geq \alpha > 0, \quad \forall x \in \mathbb{R}^3.$$

(K) $K \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, and there exist a constant $k > 0$ and a function $K_0 \in C(\mathbb{R}^3, \mathbb{R}) \cap L^2(\mathbb{R}^3)$, 1-periodic in x_i , $1 \leq i \leq 3$, such that $K_0 - K \in \mathcal{F}$ and

$$K_0(x) \geq K(x) \geq k > 0, \quad \forall x \in \mathbb{R}^3.$$

For $g(x, s)$ and $G(x, s) = \int_0^s g(x, t) dt$, we need the following hypotheses:

(G₁) $g(x, s) = o(|s|)$ as $s \rightarrow 0^+$, uniformly in \mathbb{R}^3 ;

(G₂) there are constants $a_1, a_2 > 0$ and $8 \leq q_1 < 12$ such that

$$|g(x, s)| \leq a_1 + a_2 |s|^{q_1-1}, \quad \forall (x, s) \in \mathbb{R}^3 \times [0, +\infty);$$

(G₃) there exist a constant $q_2 > \frac{3(q_1-4)}{2}$ and a function $h_1 \in L^1(\mathbb{R}^3)$ such that

$$\frac{1}{4}g(x, s)s - G(x, s) \geq s^{q_2} - h_1(x), \quad \forall (x, s) \in \mathbb{R}^3 \times [0, +\infty),$$

where q_1 is given by hypothesis (G₂).

We observe that conditions (G₁) and (G₂) allow us to employ variational methods to study problem (1.1) and to verify that the associated functional has a local minimum at the origin. As a matter of fact, we study the functional associated with the modified problem. Condition (G₂) imposes a subcritical growth on g . However, under these hypotheses, this functional does not satisfy a compactness condition of Palais–Smale type since the domain is the whole space \mathbb{R}^3 . We also observe that from (G₂) and (G₃) we obtain $0 < q_2 \leq q_1$. Note that $22^* = \frac{4N}{N-2} = 12$ behaves like a critical exponent for problem (1.1). The asymptotic periodicity of g at infinity is given by the following condition:

(G₄) there exist a constant $4 \leq q_3 < 12$ and functions $h_2 \in \mathcal{F}$, and $g_0 \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R}^+)$, 1-periodic in x_i , $1 \leq i \leq 3$, such that

(i) $G(x, s) \geq G_0(x, s) = \int_0^s g_0(x, t) dt, \forall (x, s) \in \mathbb{R}^3 \times [0, +\infty),$

(ii) $|g(x, s) - g_0(x, s)| \leq h_2(x)s^{q_3-1}, \forall (x, s) \in \mathbb{R}^3 \times [0, +\infty),$

(iii) the function $s \rightarrow \frac{g_0(x, s)}{s^7}$ is nondecreasing in the variable $s > 0$.

Finally, we also suppose that G satisfies:

(G) $\frac{G(x, s)}{s^8} \rightarrow \infty$ as $s \rightarrow \infty$ uniformly in \mathbb{R}^3 .

The first main result is stated as follows.

Theorem 1.1 *Suppose that (V), (K), (G₁)–(G₄), and (G) are satisfied. Then system (1.1) admits a solution.*

We observe that in the particular case $V = V_0, K = K_0$, and $g = g_0$, Theorem 1.1 clearly gives us a solution for the periodic problem. Indeed, to show the existence of a solution for the periodic problem, condition (G₄) is not necessary. Consider the following problem:

$$\begin{cases} -\Delta u + V_0(x)u - u\Delta(u^2) + K_0(x)\phi(x)u = g_0(x, u), & x \in \mathbb{R}^3, \\ -\Delta\phi = K_0(x)u^2, & x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), & u > 0 \end{cases} \tag{1.2}$$

under the following hypotheses:

(V₀) $V_0 \in C(\mathbb{R}^3, \mathbb{R})$, 1-periodic in x_i , $1 \leq i \leq 3$, and there exists a constant $\alpha > 0$ such that

$$V_0(x) \geq \alpha > 0, \quad \forall x \in \mathbb{R}^3.$$

(K₀) $K_0 \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \cap C(\mathbb{R}^3)$, 1-periodic in x_i , $1 \leq i \leq 3$, and there exists a constant $k > 0$ such that

$$K_0(x) \geq k > 0, \quad \forall x \in \mathbb{R}^3.$$

Then with the function g_0 satisfying (G₁)–(G₃) and (G), we obtain the second main result.

Theorem 1.2 *Suppose that (V₀), (K₀), (G₁)–(G₃), and (G) are satisfied. Then system (1.2) admits a solution.*

The remainder of this paper is organized as follows. In Sect. 2, we present two versions of the mountain pass theorem to be employed later. In Sect. 3, we introduce the variational framework associated with problem (1.1). In Sect. 4, we verify the geometric conditions of the mountain pass theorem and show the boundedness of the Cerami sequences associated with the minimax level. In Sect. 5, we prove the main results, and so the existence of a solution for problems (1.1) and (1.2) is established.

2 Preliminary

In this section, we present two versions of the mountain pass theorem due to Ambrosetti and Rabinowitz [3], which are essential tools in this paper.

Let E be a real Banach space, and let $I : E \rightarrow \mathbb{R}$ a functional of class C^1 . Let M be the set of critical points of I . For $b \in \mathbb{R}$, we define $I^b = \{u \in E : I \leq b\}$ and $M_b = \{u \in E : u \in M, I(u) = b\}$.

As we noted in the Introduction, the functional associated with problem (1.1) does not satisfy the Palais–Smale condition. To overcome this difficulty, we will use versions of the mountain pass theorem. To this end, we state the first version of this theorem (see [17, 37]).

We recall that $I \in C^1(E, \mathbb{R})$ satisfies the Cerami condition on level c , denoted by $(C)_c$, if any sequence $\{u_n\} \subset E$ satisfying

$$(i) \ I(u_n) \rightarrow c \quad \text{and} \quad (ii) \ \|I'(u_n)\|(1 + \|u_n\|) \rightarrow 0$$

as $n \rightarrow \infty$, possesses a convergent subsequence, where $\|\cdot\|_E$ denotes the norm on E . A function I satisfies the Cerami condition, denoted by (C) , if it satisfies $(C)_c$ for every $c \in \mathbb{R}$. We say that $\{u_n\} \subset E$ is a $(C)_c$ sequence if it satisfies (i)–(ii).

Theorem 2.1 *Let E be a real Banach space, and let $I \in C^1(E, \mathbb{R})$. Let S be a closed subset of E that disconnects E into distinct connected components E_1 and E_2 . Suppose further that $I(0) = 0$ and:*

- (I₁) $0 \in E_1$ and there is $\sigma > 0$ such that $\inf_{u \in S} I(u) \geq \sigma > 0$,
- (I₂) there is $e \in E_2$ such that $I(e) \leq 0$.

Then I possesses a $(C)_c$ sequence with $c \geq \sigma > 0$ given by

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)), \tag{2.1}$$

where

$$\Gamma = \{ \gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) \in I^0 \cap E_2 \}. \tag{2.2}$$

We will also need a local version of Theorem 2.1, which is proved in [26].

Theorem 2.2 *Let E be a real Banach space. Suppose that $I \in C^1(E, \mathbb{R})$ satisfies $I(0) = 0$, (I_1) , and (I_2) . If there exists $\gamma_0 \in \Gamma$, where Γ is defined by (2.2), such that*

$$c = \max_{t \in [0,1]} I(\gamma_0(t)) > 0, \tag{2.3}$$

then I possesses a nontrivial critical point $u \in M_c \cap \gamma_0([0, 1])$.

3 The variational framework

In this section, we introduce the variational framework associated with problem (1.1). Here, we consider the space $H^1(\mathbb{R}^3)$ endowed with one of the following norms:

$$\|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) dx \right)^{\frac{1}{2}},$$

$$\|u\|_0 = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V_0(x)u^2) dx \right)^{\frac{1}{2}}.$$

Note that, in view of condition (V), these norms are both equivalent to the standard norm on $H^1(\mathbb{R}^3)$. As far as we know, system (1.1) can be easily transformed into a single nonlinear Schrödinger equation with nonlocal term. Briefly, the Poisson equation is solved by using the Lax–Milgram theorem, so for all $u \in H^1(\mathbb{R}^3)$, there is a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta\phi = K(x)u^2$, which inserted into the first equation gives

$$-\Delta u + V(x)u - u\Delta(u^2) + K(x)\phi_u u = g(x, u).$$

In fact, for each $u \in H^1(\mathbb{R}^3)$, we define the operator T_u on $D^{1,2}(\mathbb{R}^3)$ by

$$T_u(v) = \int_{\mathbb{R}^3} K(x)u^2 v dx.$$

The Hölder inequality and the fact $K \in L^2(\mathbb{R}^3)$ yield that there is a constant $C > 0$ such that for every $v \in D^{1,2}(\mathbb{R}^3)$,

$$|T_u(v)| \leq C|K|_2 \|u\|^2 \|v\|_{D^{1,2}(\mathbb{R}^3)}.$$

Hence by the Riesz representation theorem there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3} \nabla\phi_u \nabla v dx = \int_{\mathbb{R}^3} K(x)u^2 v dx.$$

Thus ϕ_u is a weak solution of $-\Delta\phi_u = K(x)u^2$ and can be represented by

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y)}{|x-y|} u^2(y) dy.$$

Moreover, it is obvious that

$$\|\phi\|_{D^{1,2}(\mathbb{R}^3)} = \|T_u\|_{\mathcal{L}(D^{1,2},\mathbb{R})} \leq C|K|_2 \|u\|^2,$$

where $\mathcal{L}(D^{1,2}, \mathbb{R})$ denotes the linear space of linear function from $D^{1,2}$ to \mathbb{R} . Also, we have

$$\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2 dx \leq C|K|_2 \|u\|^2 \|\phi\|_{D^{1,2}(\mathbb{R}^3)} \leq C|K|_2^2 \|u\|^4. \tag{3.1}$$

Moreover,

$$\begin{aligned} |\phi_u(x)| &= \frac{1}{4\pi} \left| \int_{B_1(0)} \frac{K(y)}{|x-y|} u^2(y) dy + \int_{B_1^c(0)} \frac{K(y)}{|x-y|} u^2(y) dy \right| \\ &\leq \frac{1}{4\pi} \left(|K|_\infty \left(\int_{B_1(0)} \frac{1}{|x-y|^2} dy \right)^{\frac{1}{2}} |u|_4^2 + |K|_4 \left(\int_{B_1^c(0)} \frac{1}{|x-y|^4} dy \right)^{\frac{1}{4}} |u|_4^2 \right) \\ &< C(K) |u|_4^2. \end{aligned} \tag{3.2}$$

We observe that formally problem (1.1) is the Euler–Lagrange equation associated with the energy functional

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^3} (1 + 2u^2) |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx - \int_{\mathbb{R}^3} G(x, u) dx.$$

From the variational point of view, the first difficulty associated with problem (1.1) is finding an appropriate function space where the functional J is well defined. To avoid such a difficulty, we use the change of variable introduced in [30], that is, we consider $v = f^{-1}(u)$, where f is defined by

$$\begin{cases} f'(t) = \frac{1}{(1+2f^2(t))^{\frac{1}{2}}} & \text{on } [0, +\infty), \\ f(t) = -f(-t) & \text{on } [0, +\infty), \end{cases}$$

having the following properties proved in [11, 16].

Lemma 3.1 *The function f satisfies the following properties:*

- (1) f is uniquely defined, C^1 , and invertible;
- (2) $|f'(t)| \leq 1$ for all $t \in \mathbb{R}$;
- (3) $|f(t)| \leq t$ for all $t \in \mathbb{R}$;
- (4) $\frac{f(t)}{t} \rightarrow 1$ as $t \rightarrow 0$;
- (5) $\frac{f(t)}{\sqrt{t}} \rightarrow 2^{\frac{1}{4}}$ as $t \rightarrow +\infty$;
- (6) $\frac{f(t)}{2} \leq tf'(t) \leq f(t)$ for all $t \geq 0$;
- (7) $|f(t)| \leq 2^{\frac{1}{4}} |t|^{\frac{1}{2}}$ for all $t \in \mathbb{R}$;
- (8) $f^2(t) - f(t)f'(t)t \geq 0$ for all $t \in \mathbb{R}$;

(9) there exists a positive constant C such that

$$|f(t)| \geq \begin{cases} C|t|, & |t| \leq 1, \\ C|t|^{\frac{1}{2}}, & |t| \geq 1; \end{cases}$$

(10) $|f(t)f'(t)| \leq \frac{1}{\sqrt{2}}$ for all $t \in \mathbb{R}$.

As a consequence of Lemma 3.1, the following conclusion is proved in [17].

Corollary 3.1

- (1) The function $f(t)f'(t)t^{-1}$ is decreasing for all $t > 0$.
- (2) The function $f^3(t)f'(t)t^{-1}$ is increasing for all $t > 0$.

So, after the change of variables from J , we obtain the following functional:

$$I(v) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x)f^2(v) dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_{f(v)}f^2(v) dx - \int_{\mathbb{R}^3} G(x,f(v)) dx, \tag{3.3}$$

which is well defined in $H^1(\mathbb{R}^3)$ and belongs to C^1 under the hypotheses (V) , (K) , (G_1) , and (G_2) .

Let $F(v) = \int_{\mathbb{R}^3} K(x)\phi_{f(v)}f^2(v) dx$. Then by Lemma 3.1(3) we have

$$F(v) \leq \int_{\mathbb{R}^3} K(x)\phi_v v^2 \leq C|K|_2^2 \|v\|^4$$

for $v \in H^1(\mathbb{R}^3)$. Then by the Lebesgue and Fubini theorems

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{F(v+tw) - F(v)}{t} \\ &= \lim_{t \rightarrow 0} \left(\int_{\mathbb{R}^3} \frac{K(x)}{t} \phi_{f(v+tw)} f^2(v+tw) dx - \int_{\mathbb{R}^3} \frac{K(x)}{t} \phi_{f(v)} f^2(v) dx \right) \\ &= \lim_{t \rightarrow 0} \left(\int_{\mathbb{R}^3} \frac{K(x)}{t} \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y)}{|x-y|} f^2(v+tw) dy f^2(v+tw) dx - \int_{\mathbb{R}^3} \frac{K(x)}{t} \phi_{f(v)} f^2(v) dx \right) \\ &= \lim_{t \rightarrow 0} \int_{\mathbb{R}^3} \frac{K(x)}{t} \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y)}{|x-y|} (f(v) + f'(v)tw + o(t))^2 dy (f(v) + f'(v)tw + o(t))^2 dx \\ &\quad - \lim_{t \rightarrow 0} \int_{\mathbb{R}^3} \frac{K(x)}{t} \phi_{f(v)} f^2(v) dx \\ &= \lim_{t \rightarrow 0} \left(\int_{\mathbb{R}^3} \frac{K(x)}{t} \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y)}{|x-y|} f^2(v) dy f^2(v) dx \right. \\ &\quad + \int_{\mathbb{R}^3} \frac{K(x)}{t} \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y)}{|x-y|} f^2(v) dy 2f(v)f'(v)tw dx \\ &\quad + \left. \int_{\mathbb{R}^3} \frac{K(x)}{t} \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(y)}{|x-y|} 2f(v)f'(v)tw dy f^2(v) dx \right) + o(t) \\ &\quad - \lim_{t \rightarrow 0} \int_{\mathbb{R}^3} \frac{K(x)}{t} \phi_{f(v)} f^2(v) dx \end{aligned}$$

$$\begin{aligned}
 &= 4 \lim_{t \rightarrow 0} \int_{\mathbb{R}^3} \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{K(x)K(y)}{|x-y|} f^2(v) f'(v) w \, dy \, dx \\
 &= 4 \int_{\mathbb{R}^3} K(x) \phi_{f(v)} f'(v) w \, dx.
 \end{aligned}$$

As a consequence, the critical points of I are weak solutions of the problem

$$-\Delta v + V(x) f(v) f'(v) + K(x) \phi_{f(v)} f'(v) = g(x, f(v)) f'(v), \quad v \in H^1(\mathbb{R}^3), \tag{3.4}$$

that is,

$$\begin{aligned}
 \langle I(v), w \rangle &= \int_{\mathbb{R}^3} (\nabla v \nabla w + V(x) f(v) f'(v) w) \, dx \\
 &\quad + \int_{\mathbb{R}^3} K(x) \phi_{f(v)} f'(v) w \, dx - \int_{\mathbb{R}^3} g(x, f(v)) f'(v) w \, dx \\
 &= 0
 \end{aligned}$$

for all $v, w \in H^1(\mathbb{R}^3)$. We observe (see [11, 16]) that if $v \in C^2(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$ is a critical point of the functional I , then the function $u = f(v)$ is a classical solution of problem (1.1).

Remark 3.1 We also observe that to obtain a nonnegative solution for (3.4), we set $g(x, s) = 0$ for all $x \in \mathbb{R}^3, s < 0$. Indeed, let v be a critical point of I . Taking $w = -v^-$, where $v^- = \max\{-v, 0\}$, we get

$$\int_{\mathbb{R}^3} (|\nabla v^-|^2 + V(x) f(v) f'(v) (-v^-)) \, dx + \int_{\mathbb{R}^3} K(x) \phi_{f(v)} f'(v) (-v^-) \, dx = 0.$$

Since $f(v) (-v^-) \geq 0$, we have

$$\int_{\mathbb{R}^3} |\nabla v^-|^2 \, dx = 0, \quad \int_{\mathbb{R}^3} \frac{V(x) f(v) (-v^-)}{\sqrt{1 + 2f^2(v)}} \, dx = 0,$$

and

$$\int_{\mathbb{R}^3} \frac{K(x) \phi_{f(v)} f'(v) (-v^-)}{\sqrt{1 + 2f^2(v)}} \, dx = 0.$$

Hence we may conclude that $v^- = 0$ almost everywhere in \mathbb{R}^3 and, therefore, $v = v^+ \geq 0$. As $u = f(v)$, we conclude that u is a nonnegative solution for problem (1.1).

Similarly, associated with the periodic problem, we have the functional $I_0 \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$ defined by

$$\begin{aligned}
 I_0(v) &= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} V_0(x) f^2(v) \, dx \\
 &\quad + \frac{1}{4} \int_{\mathbb{R}^3} K_0(x) \tilde{\phi}_{f(v)} f^2(v) \, dx - \int_{\mathbb{R}^3} G_0(x, f(v)) \, dx,
 \end{aligned} \tag{3.5}$$

where $G_0(x, s) = 0$ for all $x \in \mathbb{R}^3, s < 0$.

We also observe that for $\delta > 0$, because of (G_1) and (G_2) , there is a constant $C_\delta > 0$ such that

$$|g(x, s)| \leq \delta |s| + C_\delta |s|^{q_1-1}, \quad \forall (x, s) \in \mathbb{R}^3 \times \mathbb{R}, \tag{3.6}$$

$$|G(x, s)| \leq \frac{\delta}{2} |s|^2 + \frac{C_\delta}{q_1} |s|^{q_1}, \quad \forall (x, s) \in \mathbb{R}^3 \times \mathbb{R}. \tag{3.7}$$

4 Geometric properties

In this section, we present the variational results that will be used in the proof of main results. Firstly, we verify the geometric conditions of the mountain pass theorem. Then we present results concerning the behavior of the Cerami sequences of the associated functional. Finally, we establish two technical results necessary for the proof of Theorem 1.1. The following lemma shows that the (modified) functional associated with problem (1.1) satisfies the geometric properties of the mountain pass theorem.

Lemma 4.1 *Suppose that (V) , (G_1) , (G_2) , and (G) are satisfied. Then the functional I defined by (3.3) satisfies conditions (I_1) and (I_2) of Theorem 2.1.*

Proof For $\rho > 0$, define

$$S_\rho = \left\{ v \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} (|\nabla v|^2 + V(x)f^2(v)) \, dx = \rho^2 \right\}.$$

Since $\Psi : H^1(\mathbb{R}^3) \rightarrow \mathbb{R}$ given by

$$\Psi(v) = \int_{\mathbb{R}^3} (|\nabla v|^2 + V(x)f^2(v)) \, dx$$

is continuous, S_ρ is a closed subset that disconnects the space $H^1(\mathbb{R}^3) \rightarrow \mathbb{R}$. Taking $0 < \lambda < 1$ such that $\frac{q_1}{2} = \lambda + 6(1 - \lambda)$, by (V) , relation (3.7), Hölder’s inequality, Lemma 3.1(7), the Sobolev embedding theorem, and the result of [34] we have

$$\int_{\mathbb{R}^3} G(x, f(v)) \, dx \leq \frac{\delta}{2\alpha} \rho^2 + \frac{2^{3(1-\lambda)} C_0 C_\delta}{q_1 \alpha^\lambda} \rho^{2\lambda+6(1-\lambda)}$$

for every $v \in S_\rho$ and some constant $C_0 > 0$. Hence we have

$$I(v) \geq \left(\frac{1}{2} - \frac{\delta}{2\alpha} \right) \rho^2 - \frac{2^{3(1-\lambda)} C_0 C_\delta}{q_1 \alpha^\lambda} \rho^{2\lambda+6(1-\lambda)}$$

for every $v \in S_\rho$. Since $2\lambda + 6(1 - \lambda) > 2$, choosing $0 < \delta < \alpha$, we conclude that, for ρ sufficiently small,

$$c_0 := \inf_{S_\rho} I \geq \sigma > 0,$$

that is, condition (I_1) is satisfied.

To verify condition (I_2) , it suffices to exhibit $w \in H^1(\mathbb{R}^3)$ such that

$$I(tw) \rightarrow -\infty \quad \text{as } t \rightarrow +\infty. \tag{4.1}$$

Indeed, consider $w \in H^1(\mathbb{R}^3)$. Hence by properties (3) of Lemma 3.1 we get, for every $t > 0$,

$$\begin{aligned} I(tw) &\leq \frac{t^2}{2} \int_{\mathbb{R}^3} (|\nabla w|^2 + V(x)w^2) \, dx + \frac{t^4}{4} \int_{\mathbb{R}^3} K(x)\phi_w w^2 \, dx - \int_{\mathbb{R}^3} G(x, f(tw)) \, dx \\ &\leq \frac{t^2}{2} \int_{\mathbb{R}^3} (|\nabla w|^2 + V(x)w^2) \, dx + \frac{t^4}{4} |K|_2 |\phi_w|_6 |w|_6^2 \\ &\quad - t^4 \int_{w \neq 0} \frac{G(x, f(tw)) f^8(tw)}{f^8(tw) (tw)^4} w^4 \, dx. \end{aligned}$$

Then by Lemma 3.1(5), assumption (G), and the Fatou lemma the last integral on the right-hand side tends to infinity with t . Hence $I(tw) \rightarrow -\infty$ as $t \rightarrow \infty$. \square

As a consequence of Theorem 2.1 and Lemma 3.1, we have the following corollary.

Corollary 4.1 *Suppose that (V), (G₁), (G₂), and (G) are satisfied. Then the functional I possesses a (C)_c sequence with c given by (2.1).*

Here we verify the boundedness of the (C)_c sequences associated with the functional I and a result concerning the behavior of such sequences.

Lemma 4.2 *Suppose that (V) and (G₁)–(G₃) are satisfied. Then every Cerami sequence {v_n} in H¹(ℝ³) associated with the functional I is bounded.*

Proof According to the proof of [34], if there exists a constant $M > 0$ such that

$$\int_{\mathbb{R}^3} (|\nabla v_n|^2 + V(x)f^2(v_n)) \, dx \leq M, \tag{4.2}$$

then {v_n} is bounded in H¹(ℝ³). So we only need to prove (4.2).

Now let {v_n} ∈ H¹(ℝ³) be an arbitrary Cerami sequence for I on level $c \in \mathbb{R}$, that is,

$$I(v_n) \rightarrow c, \quad \|I'(v_n)\| (1 + \|v_n\|) \rightarrow 0. \tag{4.3}$$

Taking $0 < \delta < \alpha$, by the first condition in (4.3), (3.7), and the condition (V) we have

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^3} (|\nabla v_n|^2 + V(x)f^2(v_n)) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_{v_n} f^2(v_n) \, dx \\ &= \int_{\mathbb{R}^3} G(x, f(v_n)) \, dx + c + o_n(1) \\ &\leq \frac{\delta}{2\alpha} \int_{\mathbb{R}^3} V(x)f^2(v_n) \, dx + \frac{C_\delta}{q_1} \int_{\mathbb{R}^3} |f(v_n)|^{q_1} \, dx + c + o_n(1), \end{aligned}$$

which yields

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx + \left(\frac{1}{2} - \frac{\delta}{2\alpha}\right) \int_{\mathbb{R}^3} V(x)f^2(v_n) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_{f(v_n)} f^2(v_n) \, dx \\ &\leq \frac{C_\delta}{q_1} \int_{\mathbb{R}^3} |f(v_n)|^{q_1} \, dx + c + o_n(1). \end{aligned} \tag{4.4}$$

By the second condition in (4.3) and Lemma 3.1(6) we have

$$\begin{aligned} \langle I'(v_n), v_n \rangle &\leq \int_{\mathbb{R}^3} (|\nabla v_n|^2 + V(x)f^2(v_n)) \, dx + \int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f^2(v_n) \, dx \\ &\quad - \int_{\mathbb{R}^3} \frac{1}{2}g(x, f(v_n))f(v_n) \, dx. \end{aligned}$$

Consequently, by (G_3) we have

$$\begin{aligned} I(v_n) - \frac{1}{2}\langle I'(v_n), v_n \rangle &\geq -\frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f^2(v_n) \, dx + \int_{\mathbb{R}^3} \left[\frac{1}{4}g(x, f(v_n))f(v_n) - G(x, f(v_n)) \right] \, dx \\ &\geq -\frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f^2(v_n) \, dx + \int_{\mathbb{R}^3} |f(v_n)|^{q_2} \, dx - \int_{\mathbb{R}^3} h_1(x) \, dx. \end{aligned}$$

Since $h_1 \in L^1(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} |f(v_n)|^{q_2} \, dx \leq C + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f^2(v_n) \, dx.$$

Because of Lemma 3.1(3), we have

$$\int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f^2(v_n) \, dx \leq \int_{\mathbb{R}^3} K(x)\phi_{v_n}v_n^2 \, dx \leq |K|_2|\phi_{v_n}|_6|v_n|_6^2.$$

Therefore

$$\int_{\mathbb{R}^3} |f(v_n)|^{q_2} \, dx \leq C_1. \tag{4.5}$$

As we noted in the Introduction, $q_2 \leq q_1$. If $q_2 = q_1$, then inequality (4.2) is given by (4.4), (4.5), and our choice of δ . Now consider $q_2 < q_1$. Let $0 < \lambda < 1$ be such that $q_1 = \lambda q_2 + 12(1 - \lambda)$. By Hölder's inequality, (4.4), (4.5), Lemma 3.1(7), and the Sobolev embedding theorem we have

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx + \left(\frac{1}{2} - \frac{\delta}{2\alpha} \right) \int_{\mathbb{R}^3} V(x)f^2(v_n) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f^2(v_n) \, dx \\ &\leq \frac{C_\delta}{q_1} \left(\int_{\mathbb{R}^3} |f(v_n)|^{q_2} \, dx \right)^\lambda \left(\int_{\mathbb{R}^3} |f(v_n)|^{12} \, dx \right)^{1-\lambda} + c + o_n(1) \\ &\leq \frac{C_\delta}{q_1} C_1^\lambda 2^{3(1-\lambda)} \left(\int_{\mathbb{R}^3} |v_n|^6 \, dx \right)^{1-\lambda} + c + o_n(1) \\ &\leq C \frac{C_\delta}{q_1} C_1^\lambda 2^{3(1-\lambda)} \left(\int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx \right)^{3(1-\lambda)} + c + o_n(1). \end{aligned}$$

Now note that by (G_3) that $3(1 - \lambda) = \frac{3(q_1 - q_2)}{12 - q_2} < 1$. Therefore estimate (4.2) is satisfied, and the lemma is proved. □

Lemma 4.3 *Suppose that (V), (G₁), and (G₂) are satisfied. Let {v_n} ⊂ H¹(ℝ³) be a (C)_c sequence with c given by (2.1), and let v_n → 0 weakly in H¹(ℝ³). Then there exist a sequence {y_n} ⊂ ℝ³ and r, η > 0 such that |y_n| → ∞ and*

$$\limsup_{n \rightarrow \infty} \int_{B_r(y_n)} |v_n|^2 dx \geq \eta > 0.$$

Proof Suppose on the contrary that

$$\limsup_{n \rightarrow \infty} \int_{B_r(y_n)} |v_n|^2 dx = 0, \quad \forall r > 0.$$

Then by Lion’s concentration compactness lemma (see [37, Lemma 1.21]) we have

$$u_n \rightarrow 0 \text{ in } L^p(\mathbb{R}^3) \quad \text{for } 2 < p < 6. \tag{4.6}$$

According to Lemma 3.1(3, 6), we have

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f'(v_n)v_n dx \\ & \leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f^2(v_n) dx \\ & \leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} K(x)\phi_{v_n}v_n^2 dx \leq |K|_\infty|\phi_{v_n}|_6 \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^3} v_n^{\frac{12}{5}} dx \right)^{\frac{5}{6}} = 0. \end{aligned} \tag{4.7}$$

Let 0 < β < 2 be sufficiently small such that 2 + β < q₁ < 12 – 2β. Now take 0 < λ < 1 such that q₁ = λ(2 + β) + (1 – λ)(12 – 2β). Applying Hölder’s inequality and Lemma 3.1(3, 7), we have

$$\begin{aligned} \int_{\mathbb{R}^3} |f(v_n)|^{q_1} dx & \leq \left(\int_{\mathbb{R}^3} |f(v_n)|^{2+\beta} dx \right)^\lambda \left(\int_{\mathbb{R}^3} |f(v_n)|^{2(6-\beta)} dx \right)^{1-\lambda} \\ & \leq 2^{\frac{(1-\lambda)(6-\beta)}{2}} \left(\int_{\mathbb{R}^3} |v_n|^{2+\beta} dx \right)^\lambda \left(\int_{\mathbb{R}^3} |v_n|^{6-\beta} dx \right)^{1-\lambda}. \end{aligned}$$

Noting that 2 < 2 + β, 6 – β < 6, from (3.6), Lemma 3.1(6), and (4.6) it follows that for all δ > 0,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |g(x, f(v_n))f'(v_n)v_n| dx \\ & \leq \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |g(x, f(v_n))f(v_n)| dx \\ & \leq \limsup_{n \rightarrow \infty} \left[\delta \int_{\mathbb{R}^3} |v_n|^2 dx + C_\delta 2^{\frac{(1-\lambda)(6-\beta)}{2}} \left(\int_{\mathbb{R}^3} |v_n|^{2+\beta} dx \right)^\lambda \left(\int_{\mathbb{R}^3} |v_n|^{6-\beta} dx \right)^{1-\lambda} \right] \\ & = \delta \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^3} |v_n|^2 dx. \end{aligned}$$

Hence, since $\delta > 0$ can be chosen arbitrarily small and the sequence $\{v_n\} \subset H^1(\mathbb{R}^3)$ is bounded, we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |g(x, f(v_n))f'(v_n)v_n| dx = 0. \tag{4.8}$$

Since $\langle I'(v_n), v_n \rangle \rightarrow 0$, from (4.7) and (4.8) we obtain

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} |\nabla v_n|^2 + V(x)f(v_n)f'(v_n)v_n dx = 0.$$

Using Lemma 3.1(6), we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} (|\nabla v_n|^2 + V(x)f^2(v_n)) dx = 0. \tag{4.9}$$

By an argument similar to that used to verify (4.7) and (4.8) we conclude that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f^2(v_n) dx = 0$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} G(x, f(v_n)) dx = 0.$$

These two equalities, together with (4.9), imply that $I(v_n) \rightarrow 0$, which contradicts $I(v_n) \rightarrow c > 0$. The lemma is proved. \square

Lemma 4.4 *Suppose that (V), (K), and (G₄) are satisfied. Let $\{v_n\} \subset H^1(\mathbb{R}^3)$ be a bounded sequence, and let $w_n(x) = w(x - y_n)$, where $w \in H^1(\mathbb{R}^3)$ and $(y_n) \subset \mathbb{R}^3$. If $|y_n| \rightarrow \infty$, then, as $n \rightarrow \infty$, we have*

- (1) $\int_{\mathbb{R}^3} [V_0(x) - V(x)]f(v_n)f'(v_n)w_n dx \rightarrow 0$,
- (2) $\int_{\mathbb{R}^3} [g_0(x, f(v_n)) - g(x, f(v_n))]f'(v_n)w_n dx \rightarrow 0$,
- (3) $\int_{\mathbb{R}^3} K_0(x)\tilde{\phi}_{f(v_n)}f(v_n)f'(v_n)w_n dx - K(x)\phi_{f(v_n)}f(v_n)f'(v_n)w_n dx \rightarrow 0$.

Proof The proof of (1) and (2) can found in [34], so we only need to prove (3). Let $\zeta(x) = K_0(x) - K(x)$. Then by Lemma 3.1(2, 3) we have

$$\begin{aligned} & \int_{\mathbb{R}^3} |K_0(x)\tilde{\phi}_{f(v_n)}f(v_n)f'(v_n)w_n - K(x)\phi_{f(v_n)}f(v_n)f'(v_n)w_n| dx \\ & \leq \int_{\mathbb{R}^3} |K_0(x)\tilde{\phi}_{v_n} - K(x)\phi_{v_n}||v_n||w_n| dx \\ & = \frac{1}{4\pi} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K(y)}{|x-y|} |v_n||w_n| dy (K_0(x) - K(x))v_n^2 dx \right. \\ & \quad + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{K(y)}{|x-y|} v_n^2 dy (K_0(x) - K(x))|v_n||w_n| dx \\ & \quad \left. + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{(K_0(y) - K(y))}{|x-y|} v_n^2 dy (K_0(x) - K(x))|v_n||w_n| dx \right) \\ & = I_1 + I_2 + I_3. \end{aligned}$$

We consider, for example, I_2 . Then the other two can be obtained in the same way.

$$\begin{aligned}
 I_2 &= \int_{B_R(0)} (K_0(x) - K(x))\phi_{v_n}|v_n||w_n| \, dx + \int_{\mathbb{R}^3 \setminus (B_R(0))} (K_0(x) - K(x))\phi_{v_n}|v_n||w_n| \, dx \\
 &= I_{21} + I_{22}.
 \end{aligned}$$

Given $\delta > 0$, since $w \in H^1(\mathbb{R}^3) \subset L^p(\mathbb{R}^3)$ with $2 \leq p \leq 6$, we can find $0 < \varepsilon < \delta$ such that, for every measurable set $\Omega \subset \mathbb{R}^3$ satisfying $|\Omega| < \varepsilon$,

$$\int_{\Omega} |w|^2 \, dx < \delta, \quad \int_{\Omega} |w|^6 \, dx < \delta. \tag{4.10}$$

We fix $\varepsilon > 0$ and set $D_\varepsilon(R) = \{x \in \mathbb{R}^3 : |K_0(x) - K(x)| \geq \varepsilon, |x| \geq R\}$. By condition (K) we can find $R > 0$ such that $|D_\varepsilon(R)| < \varepsilon$. Then, according to (3.2), applying Lemma 3.1(2, 3), Hölder’s inequality, condition (K), $h \in \mathcal{F}$, and (4.10), we get

$$\begin{aligned}
 |I_{22}| &= |\phi_{v_n}|_\infty \left(\int_{D_\varepsilon(R)} (K_0(x) - K(x))|v_n||w_n| \, dx \right. \\
 &\quad \left. + \int_{\mathbb{R}^3 \setminus (B_R(0) \cup D_\varepsilon(R))} (K_0(x) - K(x))|v_n||w_n| \, dx \right) \\
 &\leq |\phi_{v_n}|_\infty \left(|h|_\infty \int_{D_\varepsilon(R)} |v_n||w_n| \, dx + \varepsilon \int_{\mathbb{R}^3 \setminus (B_R(0) \cup D_\varepsilon(R))} |v_n||w_n| \, dx \right) \\
 &\leq |\phi_{v_n}|_\infty \left(|h|_\infty |v_n|_2 \left(\int_{D_\varepsilon(R)} |w_n|^2 \, dx \right)^{\frac{1}{2}} + \varepsilon |v_n|_2 |w_n|_2 \right) \\
 &\leq C(\delta^{\frac{1}{2}} + \delta).
 \end{aligned}$$

Applying Hölder’s inequality, condition (K), Lemma 3.1(2, 3), and the fact that $\{v_n\} \subset H^1(\mathbb{R}^3)$ is bounded, we can find \tilde{C} such that

$$\begin{aligned}
 |I_{21}| &\leq |\phi_{v_n}|_\infty \int_{B_R(0)} |K_0(x) - K(x)||v_n||w_n| \, dx \\
 &\leq |\phi_{v_n}|_\infty |h|_\infty |v_n|_2 \left(\int_{B_R(0)} |w_n|^2 \, dx \right)^{\frac{1}{2}} \\
 &\leq \tilde{C} \left(\int_{B_R(-y_n)} |w(x)|^2 \, dx \right)^{\frac{1}{2}}.
 \end{aligned}$$

Hence, since $w \in H^1(\mathbb{R}^3)$ and $|y_n| \rightarrow \infty$, there is $n_0 \in \mathbb{N}$ such that

$$\int_{B_R(0)} (K_0(x) - K(x))\phi_{v_n}|v_n||w_n| \, dx \leq \tilde{C}\delta, \quad \forall n \geq n_0.$$

Since $\delta > 0$ can be chosen arbitrarily small, the last three inequalities imply that

$$\int_{\mathbb{R}^3} K_0(x)\tilde{\phi}_{f(v_n)}f(v_n)f'(v_n)w_n \, dx - K(x)\phi_{f(v_n)}f(v_n)f'(v_n)w_n \, dx \rightarrow 0$$

strongly in $L^1(\mathbb{R}^3)$ as $n \rightarrow \infty$. □

Lemma 4.5 (see [34]) *Suppose that $2 \leq q \leq 12$ and $h \in \mathcal{F}$. Let $\{v_n\} \subset H^1(\mathbb{R}^3)$ be a sequence such that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^3)$. Then*

$$\int_{\mathbb{R}^3} h(x)|f(v_n)|^q dx \rightarrow \int_{\mathbb{R}^3} h(x)|f(v)|^q dx \tag{4.11}$$

as $n \rightarrow \infty$.

5 Proofs of the main results

In this section, we prove Theorems 1.1 and 1.2 by verifying that the functionals I and I_0 , defined by (3.3) and (3.5), respectively, have nontrivial critical points.

Proof of Theorem 1.1 Firstly, by Corollary 4.1 we can find a Cerami sequence $\{v_n\}$ on level c , that is, such that

$$I(v_n) \rightarrow c \geq \sigma > 0 \quad \text{and} \quad \|I'(v_n)\|(1 + \|v_n\|) \rightarrow 0 \tag{5.1}$$

as $n \rightarrow \infty$ with c given by Theorem 2.1. Applying Lemma 4.3, we may assume, without loss of generality, that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^3)$.

Step 1. Note that, for every $z \in C_0^\infty(\mathbb{R}^3)$,

$$\begin{aligned} & \langle I'(v_n), z \rangle - \langle I'(v), z \rangle \\ &= \int_{\mathbb{R}^3} \nabla(v_n - v) \nabla z dx + \int_{\mathbb{R}^3} V(x)[f(v_n)f'(v_n) - f(v)f'(v)]z dx \\ & \quad + \int_{\mathbb{R}^3} K(x)[\phi_{f(v_n)}f(v_n)f'(v_n) - \phi_{f(v)}f(v)f'(v)]z dx \\ & \quad - \int_{\mathbb{R}^3} [g(x, f(v_n))f'(v_n) - g(x, f(v))f'(v)]z dx. \end{aligned} \tag{5.2}$$

Since $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^3)$, we have that $v_n \rightarrow v$ in $L^s_{loc}(\mathbb{R}^3)$ with $1 \leq s < 6$. Then, up to a subsequence,

$$\begin{aligned} & v_n(x) \rightarrow v(x) \quad \text{a.e. on } \Omega = \text{supp } z \text{ as } n \rightarrow \infty, \\ & |v_n(x)| \leq |\omega_s(x)| \quad \text{for every } n \in \mathbb{N} \text{ and a.e. on } \Omega, \text{ with } \omega_s \in L^s. \end{aligned}$$

Consequently, as $n \rightarrow \infty$,

$$\begin{aligned} & f(v_n)f'(v_n) \rightarrow f(v)f'(v), \\ & \phi_{f(v_n)}f(v_n)f'(v_n) \rightarrow \phi_{f(v)}f(v)f'(v), \\ & g(x, f(v_n))f'(v_n) - g(x, f(v))f'(v), \end{aligned}$$

a.e. on Ω . Furthermore, from conditions (V) and (K), Lemma 3.1(2, 3), and (3.2) we have

$$|V(x)f(v_n)f'(v_n)z| \leq |V(x)v_nz| \leq |V|_\infty|\omega_2||z|$$

and

$$|K(x)\phi_{f(v_n)}f(v_n)f'(v_n)z| \leq |K(x)\phi_{v_n}v_nz| \leq |K|_\infty|\phi_{v_n}|_\infty|\omega_2||z|.$$

Moreover, by (3.6) and Lemma 3.1(2, 3, 6, 7) we have, for $|v_n(x)| \leq 1$,

$$|g(x, f(v_n))f'(v_n)z| \leq \delta|v_n||z| + C_\delta|f(v_n)|^{q_1-1}|z| \leq \delta|\omega_2||z| + C_\delta|\omega_2||z|.$$

On the other hand, if $|v_n(x)| > 1$, then

$$\begin{aligned} |g(x, f(v_n))f'(v_n)z| &\leq \delta|v_n||z| + C_\delta|f(v_n)|^{q_1-1}|f'(v_n)||z| \\ &\leq \delta|\omega_2||z| + C_\delta|f(v_n)|^{q_1-1}\frac{|f(v_n)|}{|v_n|}|z| \\ &\leq \delta|\omega_2||z| + 2^3C_\delta|\omega_5|^5|z|. \end{aligned}$$

The two estimates imply that there exists a function $\psi \in L^1(\Omega)$ such that

$$|g(x, f(v_n))f'(v_n)z| \leq \psi \quad \text{in } \mathbb{R}^3. \tag{5.3}$$

Using (5.2), (5.3), the Lebesgue dominated convergence theorem, and the weak convergence $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$, for every $z \in C_0^\infty(\mathbb{R}^3)$, we obtain

$$\langle I'(v_n), z \rangle \rightarrow \langle I'(v), z \rangle.$$

Since $C_0^\infty(\mathbb{R}^3)$ is dense in $H^1(\mathbb{R}^3)$ and $I'(v_n) \rightarrow 0$, we conclude that $I'(v) = 0$. In other words, we have that v is a critical point of I .

Step 2. Now we need to prove that $v \neq 0$. Assume that $v = 0$.

By Lemma 4.3 there exist a sequence $(y_n) \subset \mathbb{R}^3$ and r and $\eta > 0$ such that $|y_n| \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\limsup_{n \rightarrow \infty} \int_{B_r(y_n)} |v_n|^2 dx \geq \eta > 0, \quad \forall n \in \mathbb{N}. \tag{5.4}$$

Without loss of generality, we may assume that $(y_n) \subset \mathbb{Z}^3$. Then defining $u_n(x) = v_n(x + y_n)$, $n \in \mathbb{N}$, we have $\|u_n\|_0 = \|v_n\|_0$ for all $n \in \mathbb{N}$. Thus, taking a subsequence if necessary, there exists $u \in H^1(\mathbb{R}^3)$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{weakly in } H^1(\mathbb{R}^3), \\ u_n &\rightarrow u \quad \text{strongly in } L^2_{\text{loc}}(\mathbb{R}^3), \\ u_n(x) &\rightarrow u(x) \quad \text{a.e. on } \mathbb{R}^3. \end{aligned}$$

From (5.4) we get that $u \neq 0$.

We claim that u is a critical point of I_0 . Indeed, we first observe that

$$\langle I'_0(u_n), z \rangle \rightarrow \langle I'_0(u), z \rangle, \quad \forall z \in C_0^\infty(\mathbb{R}^3), \tag{5.5}$$

as $n \rightarrow \infty$. Effectively, writing

$$\begin{aligned} & \langle I'_0(u_n), z \rangle - \langle I'_0(u), z \rangle \\ &= \int_{\mathbb{R}^3} \nabla(u_n - u) \nabla z \, dx + \int_{\mathbb{R}^3} V_0(x) [f(u_n) f'(u_n) - f(u) f'(u)] z \, dx \\ & \quad + \int_{\mathbb{R}^3} K_0(x) [\tilde{\phi}_{f(u_n)} f(u_n) f'(u_n) - \tilde{\phi}_{f(u)} f(u) f'(u)] z \, dx \\ & \quad - \int_{\mathbb{R}^3} [g_0(x, f(u_n)) - g_0(x, f(u))] f'(u_n) z \, dx + o(1). \end{aligned} \tag{5.6}$$

From the arguments used before we deduce that

$$\begin{aligned} & \int_{\mathbb{R}^3} \nabla(u_n - u) \nabla z \, dx \rightarrow 0, \\ & \int_{\mathbb{R}^3} V_0(x) [f(u_n) f'(u_n) - f(u) f'(u)] z \, dx \rightarrow 0, \end{aligned}$$

and

$$\int_{\mathbb{R}^3} K_0(x) [\tilde{\phi}_{f(u_n)} f(u_n) f'(u_n) - \tilde{\phi}_{f(u)} f(u) f'(u)] z \, dx \rightarrow 0,$$

as $n \rightarrow \infty$. Hence, to prove (5.5), it remains to analyze the last integral in (5.6). Note that

$$\begin{aligned} [g_0(x, f(u_n)) - g_0(x, f(u))] f'(u_n) z \, dx &= [g_0(x, f(u_n)) - g(x, f(u_n))] f'(u_n) z \, dx \\ & \quad + [g(x, f(u_n)) - g_0(x, f(u))] f'(u_n) z \, dx. \end{aligned} \tag{5.7}$$

Now, by condition (G_4) (ii) and the arguments used in the proof of (5.3) we get $\tilde{\psi} \in L^1(\Omega)$, $\Omega = \text{supp } z$, such that

$$|[g(x, f(u_n)) - g_0(x, f(u))] f'(u_n) z| \leq \tilde{\psi}. \tag{5.8}$$

Hence, applying the Lebesgue dominated convergence theorem once more, we obtain

$$[g(x, f(u_n)) - g_0(x, f(u_n))] f'(u_n) z \rightarrow [g(x, f(u)) - g_0(x, f(u))] f'(u) z \tag{5.9}$$

in $L^1(\Omega)$. The claim (5.3) and the Lebesgue dominated convergence theorem also yield

$$g(x, f(u_n)) f'(u_n) z \rightarrow g(x, f(u)) f'(u) z$$

in $L^1(\Omega)$. Furthermore, from (5.3), (5.8), and the Lebesgue dominated convergence theorem we again have

$$g_0(x, f(u)) f'(u_n) z \rightarrow g_0(x, f(u)) f'(u) z$$

in $L^1(\Omega)$. Consequently,

$$[g(x, f(u_n)) - g_0(x, f(u))] f'(u_n) z \rightarrow [g(x, f(u)) - g_0(x, f(u))] f'(u) z \tag{5.10}$$

in $L^1(\Omega)$. Relations (5.7), (5.9), and (5.10) establish the verification of (5.5). On the other hand, considering $z_n(x) = z(x - y_n)$ for all $n \in \mathbb{N}$, in view of the periodicities of V_0, K_0 , and g_0 , we get

$$\langle I'_0(u_n), z \rangle = \langle I'_0(v_n), z_n \rangle, \quad \forall n \in \mathbb{N}. \tag{5.11}$$

Moreover, applying Lemma 4.4, we have

$$|\langle I'_0(v_n), z_n \rangle - \langle I'(v_n), z_n \rangle| \rightarrow 0, \quad \forall n \in \mathbb{N}, \tag{5.12}$$

as $n \rightarrow \infty$. Since $\|z_n\|_0 = \|z\|_0$ for all $n \in \mathbb{N}$, by (5.11) and (5.12) we conclude that

$$\langle I'_0(v_n), z_n \rangle \rightarrow 0$$

as $n \rightarrow \infty$. This limit, (5.11), and (5.5) show that u is a critical point of I_0 , as claimed. \square

Step 3.

Claim $I_0(u) \leq c$.

To show this fact, we apply the definition of $\{u_n\}$ to get

$$\begin{aligned} I(v_n) - \frac{1}{2} \langle I'(v_n), v_n \rangle &= \frac{1}{2} \int_{\mathbb{R}^3} V_0(x) [f^2(u_n) - f(u_n)f'(u_n)u_n] dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^3} (V(x) - V_0(x)) [f^2(v_n) - f(v_n)f'(v_n)v_n] dx \\ &\quad + \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f^2(v_n) dx - \frac{1}{2} \int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f(v_n)f'(v_n)v_n dx \\ &\quad + \int_{\mathbb{R}^3} \frac{1}{2} g(x, f(v_n))f'(v_n)v_n - G(x, f(v_n)) dx. \end{aligned} \tag{5.13}$$

Now, taking a subsequence if necessary, by property (8) of Lemma 3.1 and Fatou’s lemma we obtain

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^3} V_0(x) [f^2(u_n) - f(u_n)f'(u_n)u_n] dx \\ \geq \frac{1}{2} \int_{\mathbb{R}^3} V_0(x) [f^2(u) - f(u)f'(u)u] dx. \end{aligned} \tag{5.14}$$

We observe that as $v_n \rightharpoonup 0$ weakly in $H^1(\mathbb{R}^3)$, by Lemma 4.4 with $h = V - V_0$ and Lemma 3.1(6) we get

$$\liminf_{n \rightarrow \infty} \frac{1}{2} \int_{\mathbb{R}^3} (V(x) - V_0(x)) [f^2(v_n) - f(v_n)f'(v_n)v_n] dx = 0. \tag{5.15}$$

We also claim that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \left[\frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f^2(v_n) dx - \frac{1}{2} \int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f(v_n)f'(v_n)v_n dx \right] \\ \geq \frac{1}{4} \int_{\mathbb{R}^3} K_0(x)\tilde{\phi}_{f(u)}f^2(u) dx - \frac{1}{2} \int_{\mathbb{R}^3} K_0(x)\tilde{\phi}_{f(u)}f(u)f'(u)u dx \end{aligned} \tag{5.16}$$

and

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} \left(\frac{1}{2} g(x, f(v_n)) f'(v_n) v_n - G(x, f(v_n)) \right) dx \\ & \geq \int_{\mathbb{R}^3} \left(\frac{1}{2} g_0(x, f(u)) f'(u) u - G_0(x, f(u)) \right) dx. \end{aligned} \tag{5.17}$$

Assuming that the claims are true, we use (5.1), (5.13)–(5.17), and the fact that u is a critical point of I_0 to get

$$\begin{aligned} c & \geq \frac{1}{2} \int_{\mathbb{R}^3} V_0(x) [f^2(u) - f(u) f'(u) u] dx + \frac{1}{4} \int_{\mathbb{R}^3} K_0(x) \tilde{\phi}_{f(u)} f^2(u) dx \\ & \quad - \frac{1}{2} \int_{\mathbb{R}^3} K_0(x) \tilde{\phi}_{f(u)} f(u) f'(u) u dx \\ & \quad + \int_{\mathbb{R}^3} \left(\frac{1}{2} g_0(x, f(u)) f'(u) u - G_0(x, f(u)) \right) dx \\ & = I_0(u) - \frac{1}{2} \langle I'_0(u), u \rangle = I_0(u), \end{aligned} \tag{5.18}$$

that is, $I_0(u) \leq c$.

Step 4. We will verify that $\max_{t \geq 0} I_0(tu) = I_0(u)$. Define the function $\eta(t) := I_0(tu)$ for $t \geq 0$. Since u is a critical point of I_0 , it follows that $u > 0$ (see the argument further). Hence by Fubini’s theorem we may write

$$\begin{aligned} \eta'(t) & = t \left\{ \int_{\mathbb{R}^3} |\nabla u|^2 dx - \int_{\mathbb{R}^3} \left[\frac{g_0(x, f(t|u)) f'(t|u)}{t|u|} \right. \right. \\ & \quad \left. \left. - \frac{V_0(x) f(t|u) f'(t|u)}{t|u|} - \frac{K_0(x) \tilde{\phi}_{f(t|u)} f(t|u) f'(t|u)}{t|u|} \right] u^2 dx \right\}. \end{aligned}$$

Note that, for fixed $x \in \mathbb{R}^3$, the function $\xi : (0, +\infty \rightarrow \mathbb{R})$ defined by

$$\xi(s) = \frac{g_0(x, f(s)) f'(s)}{s} - \frac{V_0(x) f(s) f'(s)}{s} - \frac{K_0(x) \tilde{\phi}_{f(s)} f(s) f'(s)}{s}$$

is increasing. Indeed, this is a direct consequence of (G_4) (iii) and Corollary 3.1(2) applied to

$$\xi(s) = \frac{g_0(x, f(s)) f^3(s) f'(s)}{f^7(s) s} f^4(s) + V_0(x) \left(-\frac{f(s) f'(s)}{s} \right) + K_0(x) \tilde{\phi}_{f(s)} \left(-\frac{f(s) f'(s)}{s} \right).$$

Now we observe that $\eta'(1) = 0$ since u is a critical point of I_0 . Moreover, we have that $\eta'(t) > 0$ for $0 < t < 1$ and $\eta'(t) < 0$ for $t > 1$. Therefore

$$I_0(u) = \eta(1) = \max_{t \geq 0} \eta(t) = \max_{t \geq 0} I_0(tu).$$

Consequently, by (5.17), (G_4) (i), and the definition of c we have

$$c \leq \max_{t \geq 0} I(tu) \leq \max_{t \geq 0} I_0(tu) = I_0(u) \leq c.$$

This implies that there exists $\gamma \in \Gamma$ such that (2.3) holds. In view of Theorem 2.2, we possess a critical point v on level c . From $c \geq \sigma > 0 = I(0)$ we have that v is a nonzero critical point of I . This concludes the proof of Theorem 1.1, except for claims (5.16) and (5.17).

Step 5. We will now show that $v > 0$ in \mathbb{R}^3 . Since $v \geq 0$ in \mathbb{R}^3 is a weak solution of the equation

$$-\Delta v = w, \quad x \in \mathbb{R}^3,$$

where

$$w =: f'(v(x)) [g(x, f(v(x))) - V(x)f(v(x)) - K(x)\phi_{f(v(x))}f(v(x))],$$

by conditions (V), (K), (G₁), and (G₂) we have

$$|w| \leq f'(v) [\delta |f(v)| + C_\delta |f(v)|^{q_1-1}] \leq C\delta + \tilde{C}_\delta |f(v)|^{\frac{q_1-1}{2}},$$

thanks to Lemma 3.1(2), (7), and (10). Let $p_0 = \frac{12}{q_1-1} > 1$. Since $v \in L^6(\mathbb{R}^3)$, it follows that $w \in L^{p_0}(B_R)$ with arbitrary $R > 0$. By elliptic regularity theory, $v \in W^{2,p_0}(B_R)$. Using a standard bootstrap argument, we may conclude that $v \in W^{2,p_0}(B_R)$ for every $p_0 \geq 2$. Hence $v \in C_{loc}^{1,\alpha}$ for some $\alpha \in (0, 1)$. Now suppose by contradiction that there is $x_0 \in \mathbb{R}^3$ such that $v(x_0) = 0$. Equation (3.4) can be rewritten as

$$-\Delta v + c(x)v = V(x)f'(v)(v - f(v)) + K(x)\phi_{f(v)}f'(v)(v - f(v)) + g(x, f(v))f'(v) \geq 0,$$

where $c(x) = V(x)f'(v) + K(x)\phi_{f(v)}f'(v) > 0$ for $x \in \mathbb{R}^3$ is a continuous function. We also note that from Lemma 3.1(3) we have $v - f(v) \geq 0$. Hence, applying the strong maximum principle for weak solutions (see [22]), we obtain that $v \equiv 0$, which contradicts the fact that $v \not\equiv 0$.

Step 6. Finally, we conclude the proof of Theorem 1.1 by showing that (5.16) and (5.17) hold. Similarly to the proof of Lemma 4.4(3), we have

$$\int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f(v_n)f'(v_n)v_n \, dx \rightarrow \int_{\mathbb{R}^3} K_0(x)\tilde{\phi}_{f(v_n)}f(v_n)f'(v_n)v_n \, dx$$

and

$$\int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f^2(v_n) \, dx \rightarrow \int_{\mathbb{R}^3} K_0(x)\tilde{\phi}_{f(v_n)}f^2(v_n) \, dx$$

strongly in $L^1(\mathbb{R}^3)$ as $n \rightarrow \infty$. Consequently, by the periodicity of K_0 ,

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{4} \int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f^2(v_n) \, dx - \frac{1}{2} \int_{\mathbb{R}^3} K(x)\phi_{f(v_n)}f(v_n)f'(v_n)v_n \, dx \\ &= \liminf_{n \rightarrow \infty} \frac{1}{4} \int_{\mathbb{R}^3} K_0(x)\tilde{\phi}_{f(u_n)}f^2(u_n) \, dx - \frac{1}{2} \int_{\mathbb{R}^3} K_0(x)\tilde{\phi}_{f(u_n)}f(u_n)f'(u_n)u_n \, dx. \end{aligned} \tag{5.19}$$

Moreover, by Lemma 3.1(6), for $s \geq 0$, we have

$$\begin{aligned} & \frac{1}{2}K_0(x)\tilde{\phi}_{f(s)}f^2(s) - K_0(x)\tilde{\phi}_{f(s)}f(s)f'(s)s \\ &= \frac{1}{2}[K_0(x)\tilde{\phi}_{f(s)}f^2(s) - K(x)\phi_{f(s)}f^2(s)] + \frac{1}{2}K(x)\phi_{f(s)}f^2(s) - K(x)\phi_{f(s)}f(s)f'(s)s \\ & \quad + K(x)\phi_{f(s)}f(s)f'(s)s - K_0(x)\tilde{\phi}_{f(s)}f(s)f'(s)s \\ & \geq -\frac{1}{2}K(x)\phi_{f(s)}f^2(s) + [K(x)\phi_{f(s)}f(s)f'(s)s - K_0(x)\tilde{\phi}_{f(s)}f(s)f'(s)s] \\ & \geq -\frac{1}{2}K(x)\phi_s s^2 + [K(x)\phi_{f(s)}f^2(s) - K_0(x)\tilde{\phi}_{f(s)}f^2(s)] \\ & \geq -\frac{1}{2}K(x)\phi_s s^2 - [K_0(x)\tilde{\phi}_s s^2 - K(x)\phi_s s^2]. \end{aligned}$$

Claim $\int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2 dx \rightarrow \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx$.

Proof By the result in [7], if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then we have $u_n^2 \rightharpoonup u^2$ in $L^3(\mathbb{R}^3)$ and $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$, so that $\phi_{u_n} \rightharpoonup \phi_u$ in $L^6(\mathbb{R}^3)$. Therefore

$$\begin{aligned} \int_{\mathbb{R}^3} K(x)\phi_{u_n}u_n^2 dx - \int_{\mathbb{R}^3} K(x)\phi_u u^2 dx &= \int_{\mathbb{R}^3} K(x)\phi_{u_n}(u_n^2 - u^2) dx \\ & \quad + \int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_u)u^2 dx. \end{aligned}$$

Since

$$\int_{\mathbb{R}^3} (K(x)\phi_{u_n})^{\frac{3}{2}} dx \leq |K|_{\frac{3}{2}} |\phi_{u_n}|_{\frac{6}{5}}^{\frac{3}{2}} < +\infty,$$

so that $K(x)\phi_{u_n} \in L^{\frac{3}{2}}(\mathbb{R}^3)$, we have $\int_{\mathbb{R}^3} K(x)\phi_{u_n}(u_n^2 - u^2) dx \rightarrow 0$. Similarly,

$$\int_{\mathbb{R}^3} (K(x)u^2)^{\frac{6}{5}} dx \leq |K|_{\frac{6}{5}} |u|_{\frac{6}{5}}^{\frac{12}{5}} < +\infty,$$

so that $K(x)u^2 \in L^{\frac{6}{5}}(\mathbb{R}^3)$, and thus $\int_{\mathbb{R}^3} K(x)(\phi_{u_n} - \phi_u)u^2 dx \rightarrow 0$. □

We obtain $\int_{\mathbb{R}^3} K_0(x)\tilde{\phi}_{u_n}u_n^2 dx \rightarrow \int_{\mathbb{R}^3} K_0(x)\tilde{\phi}_u u^2 dx$ in the same way. Thus by Fatou's lemma we have

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} \left[\frac{1}{4}K_0(x)\tilde{\phi}_{f(u_n)}f^2(u_n) - \frac{1}{2}K_0(x)\tilde{\phi}_{f(u_n)}f(u_n)f'(u_n)u_n \right. \\ & \quad \left. + \frac{1}{2}K(x)\phi_{u_n}u_n^2 + K_0(x)\tilde{\phi}_{u_n}u_n^2 - K(x)\phi_{u_n}u_n^2 \right] dx \\ & \geq \int_{\mathbb{R}^3} \left[\frac{1}{4}K_0(x)\tilde{\phi}_{f(u)}f^2(u) - \frac{1}{2}K_0(x)\tilde{\phi}_{f(u)}f(u)f'(u)u \right. \\ & \quad \left. + \frac{1}{2}K(x)\phi_u u^2 + K_0(x)\tilde{\phi}_u u^2 - K(x)\phi_u u^2 \right] dx. \end{aligned} \tag{5.20}$$

Relations (5.19) and (5.20) conclude the verification of (5.16). The proof of (5.17) can be found in [34]. The proof of Theorem 1.1 is completed. □

Proof of Theorem 1.2 We argue as in the initial steps of the proof of Theorem 1.1. Since g_0 satisfies (G_1) , (G_2) , and (G) , applying Corollary 4.1, we can find a sequence $\{v_n\} \subset H^1(\mathbb{R}^3)$ such that

$$I_0(v_n) \rightarrow c \geq \sigma > 0, \quad \|I'_0(v_n)\| (1 + \|v_n\|) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \tag{5.21}$$

where c is given by Theorem 2.1. Because of Lemma 4.2, we may suppose, without loss of generality, that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^3)$. From this and (3.6) we have that v is a critical point of I_0 , that is, $I'_0(v) = 0$. To finish the proof of the theorem, it suffices to assume that $v = 0$.

By Lemma 4.3 there exist a sequence $(y_n) \subset \mathbb{R}^3$ and $r, \eta > 0$ such that $|y_n| \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\limsup_{n \rightarrow \infty} \int_{B_r(y_n)} |v_n|^2 dx \geq \eta > 0, \quad \forall n \in \mathbb{N}. \tag{5.22}$$

Without loss of generality we may assume that $(y_n) \subset \mathbb{Z}$. Defining $u_n(x) = v_n(x + y_n)$, $n \in \mathbb{N}$, we have that $\|u_n\|_0 = \|v_n\|_0$ for all $n \in \mathbb{N}$. Consequently, taking a subsequence if necessary, there exists $u \in H^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$, $u_n \rightarrow u$ strongly in $L^2_{loc}(\mathbb{R}^3)$, and $u_n(x) \rightarrow u(x)$ almost everywhere on \mathbb{R}^3 . We claim that u is a critical point of I_0 . Indeed, given $w \in H^1(\mathbb{R}^3)$, from (V) , (K) , (G_1) , and (G_2) we get

$$\langle I'_0(u_n), w \rangle \rightarrow \langle I'_0(u), w \rangle \quad \text{as } n \rightarrow \infty. \tag{5.23}$$

On the other hand, considering $w_n(x) = w(x - y_n)$ for all $n \in \mathbb{N}$, in view of the periodicities of V_0, K_0 , and g_0 , we get

$$\langle I'_0(u_n), w \rangle \rightarrow \langle I'_0(v_n), w_n \rangle, \quad n \in \mathbb{N}.$$

Consequently, from (5.21) and the fact that $\|w_n\|_0 = \|w\|_0$ for all $n \in \mathbb{N}$ we conclude that

$$\langle I'_0(u_n), w \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

This limit, together with (5.23), shows that u is a critical point of I_0 . The claim is proved. Furthermore, (5.22) implies that $u \neq 0$. Theorem 1.2 is proved. □

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Availability of data and materials

We declare that materials described in the manuscript are freely available to any scientist wishing to use them for noncommercial purposes, without breaching participant confidentiality.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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