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Boundary value problems of fractional q-difference equations on the half-line

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Abstract

In this paper, we consider the boundary value problem of a class of nonlinear fractional q-difference equations involving the Riemann–Liouville fractional q-derivative on the half-line. By means of Schauder fixed point theorem and Leggett–Williams fixed point theorem, some results on existence and multiplicity of solutions are obtained.

MSC: 39A13; 34B18; 34A08

Keywords: Fractional q-difference equations; The half-line; Existence and multiplicity of solutions

1 Introduction

Quantum calculus, roughly speaking, is ordinary calculus without limits. There are several types of quantum calculus: h-calculus (also known as the calculus of finite differences), q-calculus, and Hahn's calculus. In this paper we are concerned with the q-calculus. The q-derivative and the q-integral were first defined by Jackson [1, 2] and had proven to have important applications in many subjects, like in hypergeometric series, complex analysis, particle physics, and quantum mechanics. For a general introduction to the q-calculus, we refer the reader to the book [3].

The origin of the fractional q-difference calculus can be traced back to the works in [4] by Al-salam and Agarwal. Perhaps due to the development of the fractional differential equations, an interest has been observed in studying boundary value problems of fractional q-difference equations, especially, about the existence of the solutions for the boundary value problems [5–10].

Boundary value problems on a half-line arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and models of gas pressure in a semi-infinite porous medium. Though much of the work on fractional calculus deals with finite domain, there is a considerable development on the topic involving unbounded domain [11–24].

In 2010, Zhao and Ge considered the following fractional boundary value problem [11]:

$$\begin{aligned} &D_{0+}^{\alpha}u(t)+f\big(t,u(t)\big)=0, \quad t\in(0,\infty), \alpha\in(1,2),\\ &u(0)=0, \quad \lim_{t\to\infty}D_{0+}^{\alpha-1}u(t)=\beta u(\xi), \end{aligned}$$

where $0 < \xi < \infty$, D^{a}_{0+} is the standard Riemann–Liouville fractional derivative. By means of fixed point theorems, sufficient conditions are obtained that guarantee the existence of solutions to the above boundary value problem.

In 2011, Zhao et al. studied the fractional multi-point boundary value problem [14]

$$D^{a}_{0+}u(t) + f\bigl(t, u(t)\bigr) = 0, \quad t \in (0, \infty),$$
$$u(0) = 0, \quad D^{a-1}_{q}u(+\infty) = \sum_{i=1}^{m} \alpha_i u(\xi_i),$$

where $1 < a \leq 2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < \infty$, D^{a}_{0+} is the standard Riemann–Liouville fractional derivative. By Leggett–Williams fixed point theorem, sufficient conditions that guarantee the existence of three positive solutions are obtained.

To the best of our knowledge, there is no paper considering fractional q-difference equations on the half-line. Theories and applications seem to be just being initiated. In this paper we will fill in the gap to investigate the existence of solutions for the following boundary value problem of nonlinear fractional q-difference equations on the half-line:

$$\bigl(D^{\alpha}_{q}u\bigr)(t) + f\bigl(t, u(t)\bigr) = 0, \quad 0 \leq t < \infty, \tag{1.1}$$

subject to the boundary conditions

$$u(0) = 0, \quad D^{\alpha-1}_{q}u(+\infty) = \sum_{i=1}^{m} a_i u(\xi_i), \tag{1.2}$$

where $0 < q < 1$, $1 < \alpha < 2$, $0 \leq \sum_{i=1}^{m} a_i \xi_i^{\alpha-1} < \Gamma_q(\alpha)$, $f : ([0, \infty), \mathbb{R}) \to [0, \infty)$ is a continuous function. Most results on the solution of boundary value problem of fractional q-difference equations that have been obtained concern the finite interval. In this paper, the range of variables is considered on the half-line. Since the Arzela–Ascoli theorem fails to work in the space C_∞, we use a modified compactness criterion to prove T is compact. We prove the existence and multiplicity results on positive solutions for boundary value problem (1.1)–(1.2) by utilizing Schauder fixed point theorem and Leggett–Williams fixed point theorem. Several existence results for solutions on the half-line are obtained. This work is motivated by papers [11, 14].

The paper is organized as follows. In Sect. 2, we introduce some definitions of q-fractional integral and differential operator together with some basic properties and lemmas to prove our main results. In Sect. 3, we investigate the existence of solutions for boundary value problem (1.1)–(1.2) by Schaefer fixed point theorem and Leggett–Williams fixed point theorem. In Sect. 4, we give an example to illustrate our main results.

2 Preliminaries

In the following section, we collect some definitions and lemmas about fractional q-integral and fractional q-derivative for the integrity of the theory, for which we refer to [25–27].

Definition 2.1 ([4]) Let $\alpha \geq 0$ and f be a function defined on $[0,b]$. The fractional q-integral of the Riemann–Liouville type is defined by $(I_q^0 f)(x) = f(x)$ and

$$\left(I_q^\alpha f\right)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t)\, d_q t, \quad \alpha > 0, x \in [0,b].$$

Definition 2.2 ([26]) The fractional q-derivative of the Riemann–Liouville type of order $\alpha \geq 0$ is defined by $(D_q^0 f)(x) = f(x)$ and

$$\left(D_q^\alpha f\right)(x) = \left(D_q^p I_q^{p-\alpha} f\right)(x), \quad \alpha > 0,$$

where p is the smallest integer greater than or equal to α.

As a particular case, it is easily seen that

$$D_q^\alpha x^{\mu-1} = \frac{\Gamma_q(\mu)}{\Gamma_q(\mu-\alpha)} x^{\mu-\alpha-1}. \tag{2.1}$$

Next, we list some properties about q-derivative and q-integral that are already known in the literature.

Lemma 2.1 ([4, 26]) *Let $\alpha, \beta \geq 0$ and f be a function defined on $[0,1]$. Then the following formulas hold*:
(i) $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$,
(ii) $(D_q^\alpha I_q^\alpha f) = f(x)$.

Lemma 2.2 ([4]) *Let $\alpha > 0$ and p be a positive integer. Then the following equality holds*:

$$\left(I_q^\alpha D_q^p f\right)(x) = \left(D_q^p I_q^\alpha f\right)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} \left(D_q^k f\right)(0).$$

Lemma 2.3 ([25]) *Let I and J be intervals containing zero such that $J \subseteq I$. Let f_n and f be functions defined in I, $n \in \mathbb{N}$ such that*

$$\lim_{n\to\infty} f_n(t) = f(t) \quad \textit{for all } t \in I, \textit{ and } f_n \textit{ tends uniformly to } f \textit{ on } J.$$

Then

$$\lim_{n\to\infty} \int_0^x f_n(t)\, d_q t = \int_0^x f(t)\, d_q t \quad \textit{for all } x \in I.$$

Lemma 2.4 ([28], Schauder fixed point theorem) *Let B be a Banach space with $C \subseteq B$ closed and convex. Assume that U is a relatively open subset of C with $0 \in U$ and $F : \overline{U} \to C$ is a continuous, compact map. Then either*
(1) *F has a fixed point in $\overline{U}$, or*
(2) *there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F u$.*

Let $E = (E, \|\cdot\|)$ be a Banach space, $P \subset E$ be a cone, κ be a nonnegative continuous concave functional on P, and $a, b, c > 0$ be constants. Define $P_c = \{x \in P : \|x\| < c\}$, $P(\kappa, a, b) = \{x \in P : a \leq \kappa(x), \|x\| < b\}$.

Lemma 2.5 (Leggett–Williams fixed point theorem) *Let $T:\overline{P_c}\to\overline{P_c}$ be a completely continuous operator, and let κ be a nonnegative continuous concave functional on K such that $\kappa(x)\le\|x\|$ for all $x\in\overline{P_c}$. Suppose that there exist $0<a<b<d\le c$ such that*

(A1) $\{x\in P(\kappa,b,d)|\kappa(x)>b\}\neq\emptyset$ *and* $\kappa(x)>b$ *for* $x\in P(\kappa,b,d)$;

(A2) $\|Tx\|<a$ *for* $\|x\|\le a$;

(A3) $\kappa(Tx)>b$ *for* $x\in P(\kappa,b,c)$ *with* $\|Tx\|>d$.

Then T has at least three fixed points x_1, x_2, and x_3 such that $\|x_1\|<a$, $b<\kappa(x_2)$, and $\|x_3\|>a$, with $\kappa(x_3)<b$.

Remark 2.1 If there holds $d=c$, then condition (A1) implies condition (A3) in Lemma 2.5.

The next result is important in the sequel.

Lemma 2.6 *Let $h\in C(\mathbb{R}^+)$ be a given function. Then the boundary value problem*

$$\bigl(D_q^\alpha u\bigr)(t)+h(t)=0,\quad 0\le t<\infty, \tag{2.2}$$

$$u(0)=0,\quad D_q^{\alpha-1}u(+\infty)=\sum_{i=1}^m a_i u(\xi_i), \tag{2.3}$$

has a unique solution

$$u(t)=\int_0^\infty G(t,qs)h(s)\,d_qs+\frac{t^{\alpha-1}}{\Gamma_q(\alpha)-\Delta}\sum_{i=1}^m a_i\int_0^\infty G(\xi_i,qs)h(s)\,d_qs,$$

where

$$\Delta=\sum_{i=1}^m a_i\xi_i^{\alpha-1},$$

and

$$G(t,qs)=\begin{cases}\frac{t^{\alpha-1}-(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}, & 0\le s\le t<\infty,\\ \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}, & 0\le t\le s<\infty,\end{cases}$$

is the Green function of boundary value problem (2.2)–(2.3).

Proof Let us begin with integrating on both sides of (2.2)

$$I_q^\alpha D_q^\alpha u(t)=-I_q^\alpha h(t).$$

In view of Definition 2.2 and Lemma 2.2, we deduce

$$u(t)=-\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}h(s)\,d_qs+c_1t^{\alpha-1}+c_2t^{\alpha-2}.$$

Applying the boundary condition $u(0)=0$, thus $c_2=0$, we have

$$u(t)=-\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}h(s)\,d_qs+c_1t^{\alpha-1}. \tag{2.4}$$

So,

$$u(\xi_i) = c_1\xi_i^{\alpha-1} - \frac{1}{\Gamma_q(\alpha)} \int_0^{\xi_i} (\xi_i - qs)^{(\alpha-1)} h(s)\, d_q s,$$

and

$$\sum_{i=1}^{m} a_i u(\xi_i) = c_1 \sum_{i=1}^{m} a_i \xi_i^{\alpha-1} - \frac{1}{\Gamma_q(\alpha)} \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - qs)^{(\alpha-1)} h(s)\, d_q s.$$

Differentiating on both sides of (2.4) and combining (2.1), we get

$$\begin{aligned} &D_q^{\alpha-1} u(t) \\ &\quad = -D_q^{\alpha-1} I_q^{\alpha} h(t) + D_q^{\alpha-1} c_1 t^{\alpha-1} = -D_q I_q^{1-(\alpha-1)} I_q^{\alpha} h(t) + c_1 \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\alpha+1)} t^{\alpha-(\alpha-1)-1} \\ &\quad = -D_q I_q^{2-\alpha} I_q^{\alpha} h(t) + c_1 \Gamma_q(\alpha) = -D_q I_q^{2} h(t) + c_1 \Gamma_q(\alpha) = -I_q h(t) + c_1 \Gamma_q(\alpha). \end{aligned}$$

Therefore,

$$D_q^{\alpha-1} u(\infty) = -\int_0^{+\infty} h(s)\, d_q s + c_1 \Gamma_q(\alpha).$$

Using the boundary condition $D_q^{\alpha-1} u(+\infty) = \sum_{i=1}^{\infty} a_i u(\xi_i)$, hence

$$-\int_0^{+\infty} h(s)\, d_q s + c_1 \Gamma_q(\alpha) = c_1 \sum_{i=1}^{m} a_i \xi_i^{\alpha-1} - \frac{1}{\Gamma_q(\alpha)} \sum_{i=1}^{m} a_i \int_0^{\xi_i} (\xi_i - qs)^{(\alpha-1)} h(s)\, d_q s.$$

It is easy to show that

$$\begin{aligned} c_1 = {}& \frac{1}{\Gamma_q(\alpha) - \sum_{i=1}^{m} a_i \xi_i} \int_0^{+\infty} h(s)\, d_q s \\ & - \frac{1}{\Gamma_q(\alpha) - \sum_{i=1}^{m} a_i \xi_i} \sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{(\xi_i - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s)\, d_q s. \end{aligned} \tag{2.5}$$

For convenience, we denote

$$\Delta = \sum_{i=1}^{m} a_i \xi_i^{\alpha-1}.$$

Utilizing (2.4) and (2.5), we get

$$\begin{aligned} u(t) = {}& \frac{t^{\alpha-1}}{\Gamma_q(\alpha) - \Delta} \int_0^{+\infty} h(s)\, d_q s - \frac{t^{\alpha-1}}{\Gamma_q(\alpha) - \Delta} \sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{(\xi_i - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s)\, d_q s \\ & - \frac{1}{\Gamma_q(\alpha)} \int_0^{t} (t - qs)^{(\alpha-1)} h(s)\, d_q s \\ = {}& \left(\frac{1}{\Gamma_q(\alpha)} + \frac{\Delta}{\Gamma_q(\alpha)(\Gamma_q(\alpha) - \Delta)} \right) t^{\alpha-1} \int_0^{+\infty} h(s)\, d_q s \end{aligned}$$

$$
\begin{aligned}
&- \frac{t^{\alpha-1}}{\Gamma_q(\alpha)-\Delta}\sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{(\xi_i - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s)\,d_qs \\
&- \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} h(s)\,d_qs \\
={}& \frac{1}{\Gamma_q(\alpha)} t^{\alpha-1}\int_0^{+\infty} h(s)\,d_qs - \frac{1}{\Gamma_q(\alpha)}\int_0^t (t-qs)^{(\alpha-1)} h(s)\,d_qs \\
&+ \frac{\Delta}{\Gamma_q(\alpha)(\Gamma_q(\alpha)-\Delta)} t^{\alpha-1}\int_0^{+\infty} h(s)\,d_qs \\
&- \frac{t^{\alpha-1}}{\Gamma_q(\alpha)-\Delta}\sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{(\xi_i - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)} h(s)\,d_qs \\
={}& \int_0^{\infty} G(t,qs)h(s)\,d_qs + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)-\Delta}\sum_{i=1}^{m} a_i \int_0^{\infty} G(\xi_i,qs)h(s)\,d_qs.
\end{aligned}
$$

Define

$$
G(t,qs) = \begin{cases} \frac{t^{\alpha-1}-(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}, & 0 \le s \le t < \infty, \\ \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}, & 0 \le t \le s < \infty. \end{cases}
$$

The proof is completed. □

The following properties of the Green function play important roles in this paper.

Lemma 2.7 *Function G defined in Lemma 2.6 satisfies the following properties*:

(P1) $G(t,qs) \ge 0$ *and* $G(t,qs) \le G(qs,qs)$ *for all* $0 \le s, t < \infty$.

(P2) *There exists a function* $\gamma \in C(\mathbb{R}^+)$ *such that*

$$
\min_{k \le t \le l} G(t,qs) \ge \gamma(qs) \sup_{0 \le t < \infty} G(t,qs) = \gamma(qs)G(qs,qs), \quad s \in (0,+\infty),
$$

where $0 < k < l < \infty$ *are constants.*

Proof It is clear that $G(t,qs) \ge 0$ for $s,t \in [0,+\infty)$. The monotonicity of $G(t,qs)$ implies

$$
\sup_{0 \le t < +\infty} G(t,qs) = G(qs,qs) = \frac{(qs)^{\alpha-1}}{\Gamma_q(\alpha)}, \quad s \in [0,+\infty). \tag{2.6}
$$

Now we consider the existence of γ. Setting $g_1(t,qs) = \frac{t^{\alpha-1}-(t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}$ and $g_2(t,qs) = \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}$, we have

$$
\begin{aligned}
\min_{k \le t \le l} G(t,qs) &= \begin{cases} g_1(l,qs), & s \in (0,k], \\ \min\{g_1(l,qs), g_2(k,qs)\}, & s \in [k,l], \\ g_2(k,qs), & s \in [l,\infty) \end{cases} \\
&= \begin{cases} \frac{l^{\alpha-1}-(l-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}, & s \in (0,r], \\ \frac{k^{\alpha-1}}{\Gamma_q(\alpha)}, & s \in [r,\infty), \end{cases}
\end{aligned} \tag{2.7}
$$

where $0 < r < \infty$ is the unique solution of the equation

$$l^{\alpha-1} - (l - qs)^{(\alpha-1)} = k^{\alpha-1}. \tag{2.8}$$

Hence

$$r(qs) = \begin{cases} \frac{l^{\alpha-1}-(l-qs)^{(\alpha-1)}}{(qs)^{\alpha-1}}, & 0 < s \leq r, \\ (\frac{k}{qs})^{\alpha-1}, & r \leq s < \infty. \end{cases} \tag{2.9}$$

This completes the proof of Lemma 2.7. □

3 Main results

We are now in a position to state and prove our main results in this paper.

Consider the space $C_\infty([0,\infty),\mathbb{R})$ defined by $C_\infty([0,\infty),\mathbb{R}) = \{u \in C([0,\infty),\mathbb{R}) : \lim_{t\to+\infty} \frac{u(t)}{1+t^{\alpha-1}} \text{ exists}\}$ with the norm

$$\|u\| = \sup_{t\in[0,\infty)} \frac{|u(t)|}{1 + t^{\alpha-1}}.$$

Lemma 3.1 ([29]) *C_∞ is a Banach space.*

For $u \in C_\infty$, we define the operator T by

$$Tu(t) = \int_0^\infty G(t,qs)f\bigl(s,u(s)\bigr)\,d_qs + \frac{t^{\alpha-1}}{\Gamma_q(\alpha) - \Delta}\sum_{i=1}^m a_i \int_0^\infty G(\xi_i,qs)f\bigl(s,u(s)\bigr)\,d_qs.$$

We list some conditions in this section for convenience:

(H1) Let $F(t,u) = f(t,(1 + t^{\alpha-1})u)$, $|F(t,u)| \leq \varphi(t)\omega(|u|)$ on $[0,\infty)$ with $\omega \in C([0,\infty),\mathbb{R})$ nondecreasing and $\varphi \in C([0,\infty))$, $\int_0^\infty \varphi(s)\,d_qs < +\infty$.

(H2) There exists a constant ρ such that, for any $t \in [0,\infty)$, $v_1, v_2 \in \mathbb{R}$,

$$\bigl|F\bigl(s,v_2(s)\bigr) - F\bigl(s,v_1(s)\bigr)\bigr| \leq \rho|v_2 - v_1|.$$

Since the Arzela–Ascoli theorem fails to work in the space C_∞, we need a modified compactness criterion to prove T is compact.

Lemma 3.2 ([29]) *Let $V = \{u \in C_\infty : \|u\| < l\}$ $(l > 0)$, $V_1 = \{\frac{u(t)}{1+t^{\alpha-1}} : u \in V\}$. If V_1 is equicontinuous on any compact intervals of $[0,+\infty)$ and equiconvergent at infinity, then V is relatively compact on C_∞.*

Remark 3.1 V_1 is called equiconvergent at infinity if and only if, for any given $\varepsilon > 0$, there exists $N = N(\varepsilon) > 0$ such that

$$\left|\frac{u(t_1)}{1 + t_1^{\alpha-1}} - \frac{u(t_2)}{1 + t_2^{\alpha-1}}\right| < \varepsilon \quad \text{for all } u \in V_1, t_1, t_2 \geq N.$$

Define the cone $P \subset C_\infty$ by

$$P = \bigl\{u \in C_\infty : u(t) \geq 0, t \in [0,+\infty)\bigr\}.$$

Lemma 3.3 *If* (H1)–(H2) *hold, then* $T:P\to P$ *is completely continuous.*

Proof We divide the proof into three steps.

Step 1: We show that $T:P\to P$ is continuous.

In view of the continuity and nonnegativity of G and f, we have $Tu(t)\geq 0$ for $t\in[0,\infty)$.

For any $u\in\Omega$, by (H1) we obtain

$$\int_0^\infty f\bigl(s,u(s)\bigr)\,d_qs\leq\omega(L)\int_0^\infty\varphi(s)\,d_qs<\infty,$$

and

$$\lim_{t\to+\infty}\frac{(Tu)(t)}{1+t^{\alpha-1}}=\int_0^\infty\frac{1}{\Gamma_q(\alpha)}f\bigl(s,u(s)\bigr)\,d_qs+\frac{1}{\Gamma_q(\alpha)-\Delta}\sum_{i=1}^m a_i\int_0^\infty G(\xi_i,qs)f\bigl(s,u(s)\bigr)\,d_qs.$$

So, $\lim_{t\to+\infty}\frac{(Tu)(t)}{1+t^{\alpha-1}}$ exists. Thus, $T(P)\subset P$.

Let $v(t)=\frac{u(t)}{1+t^{\alpha-1}}$. Then by (H1), we have $f(s,u(s))=F(s,\frac{u_n(s)}{1+s^{\alpha-1}})=F(s,v(s))$. Taking $u_n\to u$ as $n\to+\infty$ in C_∞, by (H2) we deduce

$$\begin{aligned}\bigl|f\bigl(t,u_n(t)\bigr)-f\bigl(t,u(t)\bigr)\bigr|&=\left|F\left(t,\frac{u_n(t)}{1+s^{\alpha-1}}\right)-F\left(t,\frac{u(t)}{1+t^{\alpha-1}}\right)\right|=\bigl|F\bigl(t,v_n(t)\bigr)-F\bigl(t,v(t)\bigr)\bigr|\\&\leq\rho\bigl|v_n(t)-v(t)\bigr|\to 0\quad\text{uniformly on }[0,\infty).\end{aligned}$$

So $f(t,u_n(t))\to f(t,u(t))$ uniformly on $[0,\infty)$. By Lemma 2.3 we have

$$\lim_{n\to\infty}\int_0^\infty f\bigl(s,u_n(s)\bigr)\,d_qs=\int_0^\infty f\bigl(s,u(s)\bigr)\,d_qs.\tag{3.1}$$

Then, combining (3.1), we have

$$\begin{aligned}&\left|\frac{Tu_n(t)}{1+t^{\alpha-1}}-\frac{Tu(t)}{1+t^{\alpha-1}}\right|\\&\quad=\left|\int_0^\infty\frac{G(t,qs)}{1+t^{\alpha-1}}\bigl(f\bigl(s,u_n(s)\bigr)-f\bigl(s,u(s)\bigr)\bigr)\,d_qs\right.\\&\qquad\left.+\frac{t^{\alpha-1}}{\Gamma_q(\alpha)-\Delta}\sum_{i=1}^m a_i\int_0^\infty\frac{G(\xi_i,qs)}{1+t^{\alpha-1}}\bigl(f\bigl(s,u_n(s)\bigr)-f\bigl(s,u(s)\bigr)\bigr)\,d_qs\right|\\&\quad\leq\left|\int_0^\infty\frac{G(t,qs)}{1+t^{\alpha-1}}\bigl(f\bigl(s,u_n(s)\bigr)-f\bigl(s,u(s)\bigr)\bigr)\,d_qs\right|\\&\qquad+\left|\frac{t^{\alpha-1}}{\Gamma_q(\alpha)-\Delta}\sum_{i=1}^m a_i\int_0^\infty\frac{G(\xi_i,qs)}{1+t^{\alpha-1}}\bigl(f\bigl(s,u_n(s)\bigr)-f\bigl(s,u(s)\bigr)\bigr)\,d_qs\right|\\&\quad\leq\left|\frac{1}{\Gamma_q(\alpha)}\int_0^\infty\bigl(f\bigl(s,u_n(s)\bigr)-f\bigl(s,u(s)\bigr)\bigr)\,d_qs\right|\\&\qquad+\left|\frac{\sum_{i=1}^m a_i\xi_i^{\alpha-1}}{\Gamma_q(\alpha)(\Gamma_q(\alpha)-\Delta)}\int_0^\infty\bigl(f\bigl(s,u_n(s)\bigr)-f\bigl(s,u(s)\bigr)\bigr)\,d_qs\right|.\end{aligned}$$

Hence

$$\|Tu_n - Tu\| = \sup_{t\in[0,\infty)}\left|\frac{Tu_n(t) - Tu(t)}{1 + t^{\alpha-1}}\right| \to 0, \quad \text{as } n \to \infty.$$

So, T is continuous.

Step 2: We show that T is uniformly bounded.

Let Ω be any bounded subset of P, *i.e.*, there exists $L > 0$ such that $\|u\| \leq L$ for each $u \in \Omega$. It suffices to show that $T(u)$ is bounded in P. In fact, for $u \in \Omega$, we have

$$\begin{aligned}
\|Tu\| &= \sup_{t\in[0,\infty)}\left(\int_0^\infty \frac{G(t,qs)}{1+t^{\alpha-1}} f\big(t,u(s)\big)\,d_qs \right.\\
&\quad \left. + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)-\Delta}\sum_{i=1}^m a_i \int_0^\infty \frac{G(\xi_i,qs)}{1+t^{\alpha-1}} f\big(t,u(s)\big)\,d_qs\right)\\
&\leq \int_0^\infty \frac{1}{\Gamma_q(\alpha)}\big|f\big(t,u(s)\big)\big|\,d_qs + \frac{\sum_{i=1}^m a_i\xi_i^{\alpha-1}}{\Gamma_q(\alpha)-\Delta}\int_0^\infty \frac{1}{\Gamma_q(\alpha)}\big|f\big(t,u(s)\big)\big|\,d_qs\\
&= \left(\frac{1}{\Gamma_q(\alpha)} + \frac{\sum_{i=1}^m a_i\xi_i^{\alpha-1}}{(\Gamma_q(\alpha)-\Delta)\Gamma_q(\alpha)}\right)\int_0^\infty \left|f\left(s, \frac{u(s)(1+s^{\alpha-1})}{1+s^{\alpha-1}}\right)\right|\,d_qs\\
&= \left(\frac{1}{\Gamma_q(\alpha)} + \frac{\Delta}{(\Gamma_q(\alpha)-\Delta)\Gamma_q(\alpha)}\right)\int_0^\infty \left|F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right)\right|\,d_qs\\
&\leq \frac{\omega(L)}{\Gamma_q(\alpha)-\Delta}\int_0^\infty \varphi(s)\,d_qs < \infty.
\end{aligned}$$

Hence, $T(\Omega)$ is uniformly bounded.

Now, we show that $T(\Omega)$ is equicontinuous on any compact interval.

First, for any given $Q > 0$, $t_1, t_2 \in [0, Q]$, and $u \in \Omega$, without loss of generality, we assume that $t_2 > t_1$, we deduce

$$\begin{aligned}
&\left|\frac{Tu(t_2)}{1+t_2^{\alpha-1}} - \frac{Tu(t_1)}{1+t_1^{\alpha-1}}\right|\\
&\quad\leq \left|\int_0^\infty \frac{G(t_2,qs)}{1+t_2^{\alpha-1}} f\big(s,u(s)\big)\,d_qs - \int_0^\infty \frac{G(t_1,qs)}{1+t_1^{\alpha-1}} f\big(s,u(s)\big)\,d_qs\right|\\
&\qquad + \left|\left(\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right)\sum_{i=1}^m a_i\int_0^\infty \frac{G(\xi_i,qs)}{\Gamma_q(\alpha)-\Delta} f\big(s,u(s)\big)\,d_qs\right|\\
&\quad\leq \left|\int_0^\infty \frac{G(t_2,qs)}{1+t_2^{\alpha-1}} f\big(s,u(s)\big)\,d_qs - \int_0^\infty \frac{G(t_1,qs)}{1+t_2^{\alpha-1}} f\big(s,u(s)\big)\,d_qs\right|\\
&\qquad + \left|\int_0^\infty \frac{G(t_1,qs)}{1+t_2^{\alpha-1}} f\big(s,u(s)\big)\,d_qs - \int_0^\infty \frac{G(t_1,qs)}{1+t_1^{\alpha-1}} f\big(s,u(s)\big)\,d_qs\right|\\
&\qquad + \left|\left(\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right)\sum_{i=1}^m a_i\int_0^\infty \frac{G(\xi_i,qs)}{\Gamma_q(\alpha)-\Delta} f\big(s,u(s)\big)\,d_qs\right|.
\end{aligned}$$

On the other hand, for all $u \in \Omega$, $t_1 \to t_2$, we have that

$$\begin{aligned}
&\left|\int_0^\infty \frac{G(t_2,qs)}{1+t_2^{\alpha-1}} f\bigl(s,u(s)\bigr)\,d_qs - \int_0^\infty \frac{G(t_1,qs)}{1+t_2^{\alpha-1}} f\bigl(s,u(s)\bigr)\,d_qs\right| \\
&\quad\le \int_0^{t_1} \left|\frac{(t_2^{\alpha-1}-t_1^{\alpha-1}) - ((t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)})}{\Gamma_q(\alpha)(1+t_2^{\alpha-1})}\right| f\bigl(s,u(s)\bigr)\,d_qs \\
&\qquad + \int_{t_1}^{t_2} \left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}-(t_2-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)(1+t_2^{\alpha-1})}\right| f\bigl(s,u(s)\bigr)\,d_qs \\
&\qquad + \int_{t_2}^{\infty} \left|\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Gamma_q(\alpha)(1+t_2^{\alpha-1})}\right| f\bigl(s,u(s)\bigr)\,d_qs \\
&\quad\le \int_0^{t_1} \frac{(t_2^{\alpha-1}-t_1^{\alpha-1}) + ((t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)})}{\Gamma_q(\alpha)(1+t_2^{\alpha-1})} f\bigl(s,u(s)\bigr)\,d_qs \\
&\qquad + \frac{3Q^{\alpha-1}}{\Gamma_q(\alpha)(1+t_2^{\alpha-1})}\int_{t_1}^{t_2} f\bigl(s,u(s)\bigr)\,d_qs + \frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Gamma_q(\alpha)(1+t_2^{\alpha-1})}\int_{t_2}^{\infty} f\bigl(s,u(s)\bigr)\,d_qs \\
&\quad\le \omega(L)\int_0^{t_1} \frac{(t_2^{\alpha-1}-t_1^{\alpha-1}) + ((t_2-qs)^{(\alpha-1)} - (t_1-qs)^{(\alpha-1)})}{\Gamma_q(\alpha)(1+t_2^{\alpha-1})} \varphi(s)\,d_qs \\
&\qquad + \omega(L)\frac{3Q^{\alpha-1}}{\Gamma_q(\alpha)(1+t_2^{\alpha-1})}\int_{t_1}^{t_2} \varphi(s)\,d_qs + \omega(L)\frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Gamma_q(\alpha)(1+t_2^{\alpha-1})}\int_{t_2}^{\infty} \varphi(s)\,d_qs \\
&\quad\to 0.
\end{aligned} \tag{3.2}$$

Similar to (3.2), for all $u \in \Omega$, $t_1 \to t_2$, we have

$$\begin{aligned}
&\left|\int_0^\infty \frac{G(t_1,qs)}{1+t_2^{\alpha-1}} f\bigl(s,u(s)\bigr)\,d_qs - \int_0^\infty \frac{G(t_1,qs)}{1+t_1^{\alpha-1}} f\bigl(s,u(s)\bigr)\,d_qs\right| \\
&\quad\le \int_0^\infty \frac{G_1(t,qs)|(1+t_1^{\alpha-1})-(1+t_2^{\alpha-1})|}{(1+t_1^{\alpha-1})(1+t_2^{\alpha-1})} f\bigl(s,u(s)\bigr)\,d_qs \\
&\quad\le \frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|}{(1+t_2^{\alpha-1})}\int_0^\infty \frac{G(t_1,qs)}{1+t_1^{\alpha-1}} f\bigl(s,u(s)\bigr)\,d_qs \\
&\quad\le \frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|}{(1+t_2^{\alpha-1})}\int_0^\infty \frac{1}{\Gamma_q(\alpha)}\omega(L)\varphi(s)\,d_qs \le \omega(L)\frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|}{\Gamma_q(\alpha)(1+t_2^{\alpha-1})}\int_0^\infty \varphi(s)\,d_qs \to 0,
\end{aligned}$$

and

$$\begin{aligned}
&\left|\left(\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right)\sum_{i=1}^m a_i\int_0^\infty \frac{G(\xi_i,qs)}{\Gamma_q(\alpha)-\Delta} f\bigl(s,u(s)\bigr)\,d_qs\right| \\
&\quad\le \frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}\int_0^\infty \frac{\sum_{i=1}^m a_i\xi_i^{\alpha-1}}{\Gamma_q(\alpha-\Delta)\Gamma_q(\alpha)} f\bigl(s,u(s)\bigr)\,d_qs \\
&\quad\le \frac{|t_2^{\alpha-1}-t_1^{\alpha-1}|}{(1+t_2^{\alpha-1})(1+t_1^{\alpha-1})}\frac{\omega(L)\Delta}{\Gamma_q(\alpha-\Delta)\Gamma_q(\alpha)}\int_0^\infty \varphi(s)\,d_qs \to 0.
\end{aligned}$$

Hence $T(\Omega)$ is equicontinuous on any compact intervals of $[0,\infty)$.

Step 3: We show that T is equiconvergent at ∞. For any $u \in \Omega$, by (H1)

$$\int_0^\infty f\bigl(s,u(s)\bigr)\,d_qs \le \omega(L)\int_0^\infty \varphi(s)\,d_qs < \infty. \tag{3.3}$$

On the other hand, since $\lim_{t\to\infty}\frac{t^{\alpha-1}}{1+t^{\alpha-1}}=1$, there exists $T_1>0$ such that, for any $t_2>t_1>T_1$,

$$\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\le\left|1-\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}\right|+\left|1-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|<\varepsilon. \tag{3.4}$$

Similarly, there exists a constant $M>0$ such that $\lim_{t\to\infty}\frac{(t-M)^{(\alpha-1)}}{1+t^{\alpha-1}}=1$ and there exists T_2 such that, for any $t_2>t_1>T_2$ and $0\le s\le M$,

$$\begin{aligned}\left|\frac{(t_2-s)^{(\alpha-1)}}{1+t_2^{\alpha-1}}-\frac{(t_1-s)^{(\alpha-1)}}{1+t_1^{\alpha-1}}\right|&\le\left|1-\frac{(t_2-s)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|+\left|1-\frac{(t_1-s)^{(\alpha-1)}}{1+t_1^{\alpha-1}}\right|\\&\le\left|1-\frac{(t_2-M)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right|+\left|1-\frac{(t_1-M)^{(\alpha-1)}}{1+t_1^{\alpha-1}}\right|<\varepsilon.\end{aligned} \tag{3.5}$$

Choose $N>\max\{T_1,T_2\}$, then for any $u\in\Omega$, $t_2>t_1>N$ and $t_1\to t_2$, by (3.4)–(3.5), we have

$$\begin{aligned}&\left|\int_0^\infty\frac{G(t_2,qs)}{1+t_2^{\alpha-1}}f(s,u(s))\,d_qs-\int_0^\infty\frac{G(t_1,qs)}{1+t_1^{\alpha-1}}f(s,u(s))\,d_qs\right|\\&\quad\le\frac{1}{\Gamma_q(\alpha)}\int_0^{t_1}\left|\frac{t_2^{\alpha-1}-(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}-(t_1-qs)^{(\alpha-1)}}{1+t_1^{\alpha-1}}\right|f(s,u(s))\,d_qs\\&\qquad+\frac{1}{\Gamma_q(\alpha)}\int_{t_1}^{t_2}\left|\frac{t_2^{\alpha-1}-(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|f(s,u(s))\,d_qs\\&\qquad+\frac{1}{\Gamma_q(\alpha)}\int_{t_2}^{\infty}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|f(s,u(s))\,d_qs\\&\quad\le\frac{1}{\Gamma_q(\alpha)}\int_0^{t_1}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}-\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}+\frac{(t_1-qs)^{(\alpha-1)}}{1+t_1^{\alpha-1}}\right|f(s,u(s))\,d_qs\\&\qquad+\frac{1}{\Gamma_q(\alpha)}\int_{t_1}^{t_2}\left(\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}+\frac{(t_2-qs)^{(\alpha-1)}}{1+t_2^{\alpha-1}}\right)f(s,u(s))\,d_qs\\&\qquad+\frac{1}{\Gamma_q(\alpha)}\int_{t_2}^{\infty}\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|f(s,u(s))\,d_qs\\&\quad\le\frac{1}{\Gamma_q(\alpha)}\left(\int_0^{t_1}2\varepsilon f(s,u(s))\,d_qs+\int_{t_1}^{t_2}(\varepsilon+1)f(s,u(s))\,d_qs+\int_{t_2}^{\infty}\varepsilon f(s,u(s))\,d_qs\right)\\&\quad\le\frac{\varepsilon\omega(L)}{\Gamma_q(\alpha)}\left(\int_0^{t_1}\varphi(s)\,d_qs+\int_0^\infty\varphi(s)\,d_qs\right)+\omega(L)\int_{t_1}^{t_2}\varphi(s)\,d_qs\\&\quad\to 0\quad\text{uniformly as } t_2,t_1>N,\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&\left|\left(\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right)\sum_{i=1}^m a_i\int_0^\infty\frac{G(\xi_i,qs)}{\Gamma_q(\alpha)-\Delta}f(s,u(s))\,d_qs\right|\\&\quad\le\left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}}-\frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right|\sum_{i=1}^m a_i\int_0^\infty\frac{\xi_i^{\alpha-1}}{(\Gamma_q(\alpha)-\Delta)\Gamma_q(\alpha)}f(s,u(s))\,d_qs\end{aligned}$$

$$\leq \left|\frac{t_2^{\alpha-1}}{1+t_2^{\alpha-1}} - \frac{t_1^{\alpha-1}}{1+t_1^{\alpha-1}}\right| \frac{\Delta\omega(L)}{(\Gamma_q(\alpha)-\Delta)\Gamma_q(\alpha)} \int_0^\infty \varphi(s)\,d_qs$$
$$\to 0 \quad \text{uniformly as } t_2, t_1 > N. \tag{3.7}$$

It can be easily seen from (3.6) and (3.7) that, for any $\varepsilon > 0$, there exists a sufficiently large $N > 0$ such that, for any $u \in \Omega$,

$$\left|\frac{(Tu)(t_1)}{1+t_1^{\alpha-1}} - \frac{(Tu)(t_2)}{1+t_2^{\alpha-1}}\right| < \varepsilon \quad \text{for all } t_1, t_2 \geq N.$$

This implies that $T: P \to P$ is equiconvergent at ∞.

By using Lemma 3.2, we obtain that $T: P \to P$ is completely continuous. This completes the proof. □

Theorem 3.1 *Assume that* (H1)–(H2) *hold. There exists a constant* $\nu > 0$ *such that* ω *and* φ *satisfy the following condition*:

$$\frac{\nu(\Gamma_q(\alpha)-\Delta)}{\omega(\nu)\int_0^\infty \varphi(s)\,d_qs} > 1. \tag{3.8}$$

Then boundary value problem (1.1)–(1.2) *has an unbounded solution* $u(t)$ *such that*

$$0 \leq \frac{u(t)}{1+t^{\alpha-1}} \leq \nu \quad \textit{for } t \in [0,\infty).$$

Proof Let

$$U = \{u \in P : \|u\| \leq \nu\}.$$

We claim that $u \neq \lambda Tu$ for $u \in \partial U$ and $\lambda \in (0,1)$. The claim is immediate, since if there exists $u \in \partial U$ with $u = \lambda Tu$, then for $\lambda \in (0,1)$, we have

$$\begin{aligned}
\|u\| &= \sup_{t\in[0,\infty)} \left|\frac{\lambda Tu(t)}{1+t^{\alpha-1}}\right| \leq \sup_{t\in[0,\infty)} \left|\frac{Tu(t)}{1+t^{\alpha-1}}\right| \\
&= \sup_{t\in[0,\infty)} \Biggl(\int_0^\infty \frac{G(t,qs)}{1+t^{\alpha-1}} f\bigl(s,u(s)\bigr)\,d_qs \\
&\quad + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)-\Delta} \sum_{i=1}^m a_i \int_0^\infty \frac{G(\xi_i,qs)}{1+t^{\alpha-1}} f\bigl(s,u(s)\bigr)\,d_qs\Biggr) \\
&\leq \int_0^\infty \frac{1}{\Gamma_q(\alpha)} \bigl|f\bigl(s,u(s)\bigr)\bigr|\,d_qs + \frac{\sum_{i=1}^m a_i\xi^{\alpha-1}}{\Gamma_q(\alpha)-\Delta} \int_0^\infty \frac{1}{\Gamma_q(\alpha)} \bigl|f\bigl(s,u(s)\bigr)\bigr|\,d_qs \\
&= \left(\frac{1}{\Gamma_q(\alpha)} + \frac{\Delta}{(\Gamma_q(\alpha)-\Delta)\Gamma_q(\alpha)}\right) \int_0^\infty \left|f\left(s, \frac{u(s)(1+s^{\alpha-1})}{1+s^{\alpha-1}}\right)\right|\,d_qs \\
&= \left(\frac{1}{\Gamma_q(\alpha)} + \frac{\Delta}{(\Gamma_q(\alpha)-\Delta)\Gamma_q(\alpha)}\right) \int_0^\infty \left|F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right)\right|\,d_qs \\
&\leq \frac{\omega(\nu)}{\Gamma_q(\alpha)-\Delta} \int_0^\infty \varphi(s)\,d_qs.
\end{aligned}$$

So, for $u \in \partial U$, we obtain

$$\nu \leq \frac{\omega(\nu)}{\Gamma_q(\alpha) - \Delta} \int_0^\infty \varphi(s)\, d_q s,$$

that is,

$$\frac{\nu(\Gamma_q(\alpha) - \Delta)}{\omega(\nu) \int_0^\infty \varphi(s)\, d_q s} \leq 1,$$

which contradicts with (3.8). By Lemma 2.4, boundary value problem (1.1)–(1.2) has an unbounded solution $u = u(t)$ such that

$$0 \leq \frac{u(t)}{1 + t^{\alpha-1}} \leq \nu \quad \text{for } t \in [0, \infty).$$

The proof is completed. □

We define a nonnegative continuous concave functional $\kappa(u) = \min_{t\in[k,l]} \frac{|u(t)|}{1+t^{\alpha-1}}$ for $u \in P$, k, and l is defined by (2.8). Denote

$$N_1 = \frac{\Gamma_q(\alpha) - \Delta}{\int_0^\infty \varphi(s)\, d_q s}, \qquad N_2 = \frac{(1 + l^{\alpha-1})(\Gamma_q(\alpha) - \Delta)}{(\Gamma_q(\alpha) - \Delta + \sum_{i=1}^m a_i k^{\alpha-1}) \int_k^l \gamma(qs) G(qs, qs)\, d_q s}.$$

Theorem 3.2 *Assume that there exist constants a, b, c with $0 < a < b < c$ such that $f(t,u)$ satisfies* (H1)–(H2) *and the following conditions*:

(H3) $\omega(u) < N_1 a$ *for all* $u \in [0, a]$; $\omega(u) \leq N_1 c$ *for all* $u \in [0, c]$.

(H4) $F(t, \frac{u}{1+t^{\alpha-1}}) > N_2 b$ *for all* $(t, \frac{u}{1+t^{\alpha-1}}) \in [k, l] \times [b, c]$.

Then the boundary value problem (1.1)–(1.2) *has at least three solutions u_1, u_2, u_3 such that $\|u_1\| < a$, $b < \kappa(u_2(t))$ and $\|u_3\| > a$ with $\kappa(u_3(t)) < b$ for $t \in [0, \infty)$.*

Proof Firstly, we show that $T : \overline{P_c} \to \overline{P_c}$ is a completely continuous operator.

In fact, for $u \in \overline{P_c}$, then $\|u\| \leq c$, by (H1) and (H3), we get

$$\begin{aligned}
\|Tu\| &= \sup_{t\in[0,\infty)} \frac{Tu(t)}{1 + t^{\alpha-1}} \\
&\leq \int_0^\infty \frac{G(t, qs)}{1 + t^{\alpha-1}} \left| f\left(s, \frac{u(s)(1 + s^{\alpha-1})}{1 + s^{\alpha-1}}\right)\right| d_q s \\
&\quad + \frac{t^{\alpha-1}}{\Gamma_q(\alpha) - \Delta} \sum_{i=1}^m a_i \int_0^\infty \frac{G(\xi_i, qs)}{1 + t^{\alpha-1}} \left| f\left(s, \frac{u(s)(1 + s^{\alpha-1})}{1 + s^{\alpha-1}}\right)\right| d_q s \\
&\leq \left(\frac{1}{\Gamma_q(\alpha)} + \frac{\Delta}{\Gamma_q(\alpha)(\Gamma_q(\alpha) - \Delta)}\right) \int_0^\infty F\left(s, \frac{u(s)}{1 + s^{\alpha-1}}\right) d_q s \\
&\leq \frac{1}{\Gamma_q(\alpha) - \Delta} \int_0^\infty \varphi(s) \omega\left(\frac{|u|}{1 + s^{\alpha-1}}\right) d_q s \\
&\leq \frac{1}{\Gamma_q(\alpha) - \Delta} \int_0^\infty \varphi(s) \omega(|u|)\, d_q s \leq \frac{N_1 c}{\Gamma_q(\alpha) - \Delta} \int_0^\infty \varphi(s)\, d_q s \leq c.
\end{aligned}$$

Hence, $\|Tu\| \leq c$, that is, $T : \overline{P_c} \to \overline{P_c}$. In view of Lemma 3.3, $T : \overline{P_c} \to \overline{P_c}$ is completely continuous.

In an analogous way, condition (H3) implies that condition (A2) of Lemma 2.5 is satisfied. Let $u(t) = \frac{(b+c)(1+t^{\alpha-1})}{2}$, $0 \leq t < \infty$. Then it is easy to see $\|u\| = \frac{(b+c)}{2}$, $u(\frac{(b+c)(1+t^{\alpha-1})}{2}) = \frac{(b+c)}{2} \in P(\kappa, b, c)$, $\kappa(\frac{(b+c)(1+t^{\alpha-1})}{2}) = \frac{(b+c)}{2} > b$, so $\{u \in P(\kappa, b, c) | \kappa(u) > b\} \neq \emptyset$. If $u \in P(\kappa, b, c)$, then $b \leq \frac{u(t)}{1+t^{\alpha-1}} \leq c$ for $t \in [k, l]$. By (H4) and Lemma 2.7, we get

$$\begin{aligned}
\kappa(u) &= \min_{t\in[k,l]} \frac{|u(t)|}{1+t^{\alpha-1}} \\
&= \min_{t\in[k,l]} \left| \int_0^\infty \frac{G(t,qs)}{1+t^{\alpha-1}} f\left(s, \frac{u(s)(1+s^{\alpha-1})}{1+s^{\alpha-1}}\right) d_q s \right. \\
&\quad \left. + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)-\Delta} \sum_{i=1}^m a_i \int_0^\infty \frac{G(\xi_i,qs)}{1+t^{\alpha-1}} f\left(s, \frac{u(s)(1+s^{\alpha-1})}{1+s^{\alpha-1}}\right) d_q s \right| \\
&\geq \int_0^\infty \frac{G(t,qs)}{1+l^{\alpha-1}} f\left(s, \frac{u(s)(1+s^{\alpha-1})}{1+s^{\alpha-1}}\right) d_q s \\
&\quad + \frac{k^{\alpha-1}}{\Gamma_q(\alpha)-\Delta} \sum_{i=1}^m a_i \int_0^\infty \frac{G(\xi_i,qs)}{1+l^{\alpha-1}} f\left(s, \frac{u(s)(1+s^{\alpha-1})}{1+s^{\alpha-1}}\right) d_q s \\
&\geq \int_0^\infty \frac{\gamma(qs)G(qs,qs)}{1+l^{\alpha-1}} f\left(s, \frac{u(s)(1+s^{\alpha-1})}{1+s^{\alpha-1}}\right) d_q s \\
&\quad + \frac{k^{\alpha-1}}{\Gamma_q(\alpha)-\Delta} \sum_{i=1}^m a_i \int_0^\infty \frac{\gamma(qs)G(qs,qs)}{1+l^{\alpha-1}} f\left(s, \frac{u(s)(1+s^{\alpha-1})}{1+s^{\alpha-1}}\right) d_q s \\
&= \left(\frac{1}{1+l^{\alpha-1}} + \frac{\sum_{i=1}^m a_i k^{\alpha-1}}{\Gamma_q(\alpha)-\Delta}\right) \int_k^l \gamma(qs)G(qs,qs) F\left(s, \frac{u(s)}{1+s^{\alpha-1}}\right) d_q s \\
&> \left(\frac{1}{1+l^{\alpha-1}} + \frac{\sum_{i=1}^m a_i k^{\alpha-1}}{\Gamma_q(\alpha)-\Delta}\right) \int_k^l \gamma(qs)G(qs,qs) N_2 b \, d_q s = b.
\end{aligned}$$

So condition (A1) of Lemma 2.5 is satisfied.

By Lemma 2.5 and Remark 2.1, there exist three solutions u_1, u_2, u_3 such that $\|u_1\| < a$, $b < \kappa(u_2(t))$, and $\|u_3\| > a$ with $\kappa(u_3(t)) < b$, which completes the proof. □

4 Example

In this section, we will give an example to expound our main results.

Example 4.1 Consider the following boundary value problem:

$$\left(D_q^{\frac{3}{2}} u\right)(t) + f\left(t, u(t)\right) = 0, \quad t \geq 0, \tag{4.1}$$

$$u(0) = 0, \quad D_q^{\frac{1}{2}} u(+\infty) = \sum_{i=1}^m a_i u(\xi_i), \tag{4.2}$$

here $\alpha = \frac{3}{2}$, $0 \leq \Delta = \sum_{i=1}^m a_i \xi_i^{\alpha-1} < \Gamma_q(\frac{3}{2})$, $f(t,u) = \sqrt{|\frac{u}{1+t^{\frac{1}{2}}}|} e^{-t}$, $F(t,u) = \sqrt{|u|} e^{-t}$.

Choose $\omega(u) = \sqrt{u}$, $\varphi(t) = e^{-t}$, $k > (\frac{1}{\Gamma_q(\frac{3}{2})-\Delta})^2$, we have:

(i) $0 \leq \sum_{i=1}^m a_i \xi_i^{\alpha-1} < \Gamma_q(\alpha)$;

(ii) $f : [0, +\infty) \times \mathbb{R} \to [0, +\infty)$ is continuous;

(iii) $|F(t,u)| = \varphi(t)\omega(|u|)$ on $[0,\infty) \times \mathbb{R}$ with $\omega \in C([0,\infty),\mathbb{R})$ nondecreasing and $\int_0^\infty \varphi(s)\,d_qs < +\infty$;

(iv) $\frac{k(\Gamma_q(\alpha)-\Delta)}{\omega(k)\int_0^\infty \varphi(s)\,d_qs} = \sqrt{k}(\Gamma_q(\frac{3}{2}) - \Delta) > 1$.

Thus, from Theorem 3.1, problem (4.1)–(4.2) has a positive solution u such that

$$0 \le \frac{u(t)}{1+\sqrt{t}} \le k \quad \text{for } t \in [0,\infty).$$

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