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Analysis and explicit solvability of degenerate tensorial problems

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Abstract

We study a two-dimensional boundary value problem described by a tensorial equation in a bounded domain. Once its more general definition is given, we conclude that its analysis is linked to the resolution of an overdetermined hyperbolic problem and hence present some discussions and considerations. Secondly, for a simplified version of the original formulation, which leads to a degenerate problem on a rectangle, we prove the existence and uniqueness of a solution under proper assumptions on the data.

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1 Introduction, motivation, and structure of the paper

It is well known that the Dirichlet problem associated to a hyperbolic equation is often employed as an example of an ill posed problem in the theory of hyperbolic partial differential equations (see [1]). Nevertheless, a wide range of real problems arising in nature (gas dynamics, torsion theory of shells with alternating sign curvature, mechanical behaviors of bending structures, *etc.*) are in fact mathematically described through hyperbolic equations; thereafter, a deserving undertaking is developing a casuistry for which such problems are, indeed, well posed.

In this sense, the subject matter of this investigation is the existence and uniqueness of a solution to a tensorial boundary value problem whose analysis requires the study of a hyperbolic equation. Precisely, the corresponding formulation models the equilibrium of membrane structures used in civil engineering applications. It is worth underlining that these equilibrium equations are not new, but linked to those dealing with shell structures (see the fundamental monograph [2] and also [3])[a] and are given by

$$\begin{cases} \sigma_{xx,x} + \sigma_{xy,y} = 0 & \text{in } \Omega, \\ \sigma_{xy,x} + \sigma_{yy,y} = 0 & \text{in } \Omega, \\ z_{,xx}\sigma_{xx} + 2z_{,xy}\sigma_{xy} + z_{,yy}\sigma_{yy} = 0 & \text{in } \Omega, \\ \text{boundary conditions} & \text{on } \Gamma = \partial\Omega. \end{cases} \tag{1}$$

In system (1), $z = z(x, y)$ is a regular function defined in a bounded domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary $\Gamma = \partial\Omega$, and its graph represents the shape of the shell; similarly,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(x,y) = \begin{pmatrix} \sigma_{xx}(x,y) & \sigma_{xy}(x,y) \\ \sigma_{yx}(x,y) & \sigma_{yy}(x,y) \end{pmatrix}$$

is a symmetric second-order stress tensor, which determines the state (compression or tension) of the same shell.

The main difference between the equilibrium approach for shell structures and that for membrane ones are discussed, for instance, in [4–6] and references therein. Even though this paper goes beyond the technical and physical aspects addressed in the previous three references, it must be specified that, from the mathematical point of view, although on the one hand no assumption on the function z nor the tensor $\boldsymbol{\sigma}$ appearing in system (1) is required for the analysis of the equilibrium of shell structures, on the other hand considering membrane elements implies the following restrictions:

(H1) the graph of z represents an almost everywhere (a.e.) negative Gaussian curvature (alternating sign curvature) surface,[b] that is,

$$z_{,xx}z_{,yy} - z_{,xy}^2 < 0 \quad \text{a.e. in } \bar{\Omega};$$

(H2) the tensor $\boldsymbol{\sigma}$ is almost everywhere positive definite,[c] that is,

$$\sigma_{xx}\xi_1^2 + 2\sigma_{xy}\xi_1\xi_2 + \sigma_{yy}\xi_2^2 > 0 \quad \forall(\xi_1,\xi_2) \neq (0,0) \text{ and a.e. in } \bar{\Omega}.$$

In other words, *any* pair $(z,\boldsymbol{\sigma})$ satisfying system (1) models the equilibrium of a shell whose shape has not necessarily a constant sign curvature and has very general stress state (compression, tension, or both). Of course, for a given z with a.e. positive Gaussian curvature, an a.e. negative, a.e. positive, or alternating sign definite $\boldsymbol{\sigma}$ balancing z might be derived; nevertheless, *no one* of these cases would represent the equilibrium of a membrane structure. Indeed, a balanced pair $(z,\boldsymbol{\sigma})$ for a fixed z with a.e. negative Gaussian curvature and a.e. negative, a.e. positive, or alternating sign definite $\boldsymbol{\sigma}$ idealizes the equilibrium of a membrane structure in *only one case*.

Coming back to the framework of the equilibrium for membrane structures (which, as already commented, justifies this investigation), the discussion presented before naturally allows us to define two complementary approaches resulting from system (1):

($\mathcal{HP}$) a problem of hyperbolic type where the tensor $\boldsymbol{\sigma}$ is the unknown: given a function z with a.e. negative Gaussian curvature, find an a.e. positive definite tensor $\boldsymbol{\sigma}$ fulfilling (1);

($\mathcal{EP}$) a problem of elliptic type where the function z is the unknown: given an a.e. positive definite tensor $\boldsymbol{\sigma}$ satisfying the first two PDEs of (1), find a function z with a.e. negative Gaussian curvature fulfilling (1).

In this work, we mainly consider problem ($\mathcal{HP}$): furthermore, due to the high complementarity between ($\mathcal{HP}$) and ($\mathcal{EP}$), we might make mention to this latter approach, for which partial results are available in the literature.

The remaining structure of the paper is drawn as follows. In Section 2, we formulate the so-called General Problem associated with ($\mathcal{HP}$), which is a very broad (tensorial) boundary value problem modeling an optimal mechanical scenario appearing in membrane structures. As detailed in [4, 5, 7], the boundary of the domain is split into two parts; on a portion, mechanically corresponding to the boundary of the membrane tensioned by rigid elements (which admit any geometrical shape), a Dirichlet boundary condition is assumed, whereas on the remaining part, associated with the complementary boundary of the membrane tensioned by cables (which can be neither straight lines nor changing curvature curves), an unusual boundary relation is given. We discuss the main mathematical properties of this formulation, also in terms of other well-known results, and we conclude that this is an overdetermined, generally ill-posed, problem, for which the part of the domain with the singular boundary condition (free boundary) plays the role of a further unknown. In addition, Section 3 deals with the analytical resolution of the Reduced Problem, a simplified version of the General Problem, linked to a more restrictive physical situation, where the membrane is only tensioned by rigid elements: we examine a specific case in a rectangle, for which the resulting Dirichlet boundary problem admits an explicit unique solution. Specifically, once a polynomial for the function z is fixed in such a way that its graph identifies a surface with a.e. negative Gaussian curvature, by manipulating the tensorial expressions of the problem the main equation reads $cy^{2(n-1)}\sigma_{yy,xx} - \sigma_{yy,yy} = 0$ in $(0,a) \times (-b,0)$ with some $a,b,c > 0$ and n an integer greater than 1, exactly degenerating for $y = 0$. Connected to the last partial differential equation (PDE), the question of well posedness of boundary value problems for linear second-order PDEs of the form $\psi(y)u_{,xx} - u_{,yy} = 0$, where ψ is a sufficiently regular function with specific properties, has been studied in several works: contributions as [8–11] (and references therein) include discussions concerning the notorious special case of the mixed elliptic-hyperbolic Tricomi equation, obtained for $\psi(y) = y$, and provide a general comprehensive picture of the whole analysis. Also in line with these works, we cite paper [12], employed in this present investigation to prove the main result asserted in Theorem 3.1 of Section 3.2 and, in particular, to construct the claimed explicit solution $\boldsymbol{\sigma}$ to system (1). Finally, to mathematically point out the different physical behaviors between shells and membranes, in Section 3.3, we also solve the same Reduced Problem presented in Section 3.1 but in the case where no restriction on the sign of $\boldsymbol{\sigma}$ is required (Theorem 3.2); besides (in Section 3.4), we give a graphical representation of the derived solutions corresponding to the two mechanical situations. The closing Section 4 provides some final considerations.

2 The general problem

The following section includes some necessary tools used to our main purposes.

2.1 Definition of the domain and the boundary data

To formulate the General Problem associated with system (1), we need to properly define its domain and boundary data. The items below address these questions and are graphically represented in the left side of Figure 1.

Assumptions 2.1 We consider a function $z = z(x,y)$ with a.e. negative Gaussian curvature in $\bar{\Omega}$, in the sense of (H1), Ω being a bounded subset of $\mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$, obtained by the union of two portions, $\Gamma = \partial\Omega = \Gamma^r \cup \Gamma^c$, and having the following properties:[d]

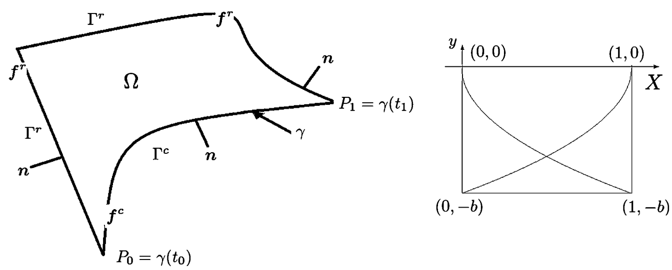

Figure 1 The General Problem. (Left) Representation of the domain and the boundary data. **(Right)** The Reduced Problem. Representation of the characteristic curves for the equation $(-y)^{2(n-1)}U_{,XX} - U_{,yy} = 0$ in the rectangle $(0,1) \times (-b,0)$.

(i) Γ^c is represented by a regular curve in $\mathbb{R}^2$ with no vanishing curvature whose parameterization is given by $\gamma(t) = (x(t), y(t))$, $t \in [t_0, t_1]$, and obtained by solving the ordinary differential equation

$$z_{,xx}(x')^2 + 2z_{,xy}x'y' + z_{,yy}(y')^2 = 0. \tag{2}$$

(ii) Γ^r is arbitrarily fixed but such that $\Gamma^r \cap \Gamma^c = \{P_0, P_1\}$, where $P_0 = \gamma(t_0)$ and $P_1 = \gamma(t_1)$.

(iii) $\mathbf{n}$ is the outward unit vector to Γ.

(iv) $\mathbf{f}^r = (f_1^r, f_2^r)$ and $\mathbf{f}^c = (f_1^c, f_2^c)$ are two regular vectorial fields, per unit length, defined on Γ^r and Γ^c, respectively; in addition, the continuity conditions $\mathbf{f}^r(P_0) = \mathbf{f}^c(P_0)$ and $\mathbf{f}^r(P_1) = \mathbf{f}^c(P_1)$ have to be satisfied.

2.2 Mathematical formulation of the general problem

Let us now describe the details of the General Problem we are interested in.

General Problem 1 Under the hypothesis of Assumptions 2.1, find a symmetric and a.e. positive definite second-order tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x,y)$ in $\bar{\Omega}$ and a real function g defined in Γ^c such that

$$\begin{cases} \sigma_{xx,x} + \sigma_{xy,y} = 0 \quad \text{in } \Omega, & \text{(3a)} \\ \sigma_{xy,x} + \sigma_{yy,y} = 0 \quad \text{in } \Omega, & \text{(3b)} \\ z_{,xx}\sigma_{xx} + 2z_{,xy}\sigma_{xy} + z_{,yy}\sigma_{yy} = 0 \quad \text{in } \bar{\Omega}, & \text{(3c)} \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{f}^r \quad \text{on } \Gamma^r, & \text{(3d)} \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{f}^c \quad \text{on } \Gamma^c, & \text{(3e)} \end{cases}$$

and

$$\begin{cases} (gx')' = f_1^c \quad \text{on } \Gamma^c \ (t \in [t_0, t_1]), \\ (gy')' = f_2^c \quad \text{on } \Gamma^c \ (t \in [t_0, t_1]), \\ g(t_0) = g_0, \end{cases} \tag{4}$$

where g_0 is a given real number.

2.3 Analysis and discussion of the general problem

Since the vectorial field $\mathbf{f}^c$ has to satisfy both expressions (3e) and (4), it cannot be uniquely and arbitrarily assigned (see Counterexample 1). Essentially, this singularity is tied to the fact that the unknowns $\boldsymbol{\sigma}$ and g are coupled through $\mathbf{f}^c$ and that, even more, they are especially linked to the domain Γ^c; subsequently, the General Problem 1 represents an overdetermined system that commonly admits no solutions.

Moreover, so far we did not manage to derive a nontrivial analytical solution to the same problem; indeed, the question how to fix $(z, \boldsymbol{\sigma}, \gamma, \mathbf{f}^c, \mathbf{f}^r, g, g_0)$ such that all relations (2), (1), and (4) hold seems rather challenging. In line with this, the case where z is a linear function has no mathematical interest (and even less physical), since relation (3c) is automatically satisfied and, in addition, any $\gamma(t) = (x(t), y(t))$ is compatible with condition (2); hence the problem merely loses its intrinsic nature. The same mathematical and physical reasons make that functions behaving as $z(x,y) = \alpha^2x^2 - \beta^2y^2$ with $\alpha, \beta \in \mathbb{R}_0$ do not lead to stimulating issues, since (2) would infer straight lines parameterized as $\gamma(t) = (t, \mp(\alpha/\beta)t + \text{constant})$ for Γ^c, and essentially the General Problem would 'degenerate' to the forthcoming Reduced Problem (see p. 6).

Returning to overdetermined boundary value problems, there exists a large amount of literature dealing with the subject; in general, these problems are prescribed by a classical partial differential equation where both Dirichlet and Neumann boundary conditions are imposed on the boundary of the domain. Some meriting questions about the analysis are the proof of the existence of solutions, possibly uniqueness, and the study of their properties. The main characteristic of the overdetermined problems is that such an overdetermination makes the domain itself unknown (free boundary problems), or in general it cannot be arbitrarily assigned, resulting solvability only in precise domains; beyond the landmark result by Serrin [13], we refer also to [14–17] for contributions regarding both elliptic and hyperbolic equations.

Remark 1 As to the specific problem we are focusing on here, let us quote that the elliptic version of the General Problem 1, herein indicated with $(\mathcal{EP})$ and briefly defined in Section 1, represents the well-known form-finding problem of tensegrity structures. In line with this, there is a large literature concerning the analysis and the design of self-supporting membrane structures, and they are essentially based on appropriate discrete numerical methods (see, for instance, [18–20] and references therein).

On the other hand, as far as the continuous approach of problem $(\mathcal{EP})$ is concerned, this has been deeply discussed by one of the authors of this paper in recent investigations. We mention that the complete formulation of problem $(\mathcal{EP})$ corresponds to a boundary value problem, in the unknown z, described by an elliptic differential equation in Ω. The portion Γ^c of Γ is indeed constructed by means of $\boldsymbol{\sigma}$ (which in this case is fixed) and $\mathbf{f}^c$. Finally, the whole Γ is endowed with Dirichlet boundary conditions, but, in accordance to overdetermined problems, on Γ^c another relation involving $z_{,y}$ and replacing expression (2) has to be satisfied as well. The technical aspects for the construction of Γ^c and the definition of the complete boundary value problem are available in [21] and [7]; in particular, as for the General Problem, the questions of the existence and derivation of an explicit solution are still open. Conversely, in the last two aforementioned contributions an equivalent number of numerical procedures exactly tied to free boundary approaches are proposed and employed as resolution methods.

Counter example 1 (Ill posedness of the General Problem) Let us fix $z(x,y) = -A^2x^4 + 6B^2y^2$ (with $A, B \in \mathbb{R}_0$). By equation (2) we can choose as Γ^c the curve $\gamma(t) = (t, At^2/(2B))$, $t \in [t_0, t_1]$. In addition, $\sigma_{xx} = 1$, $\sigma_{xy} = \sigma_{yx} = 0$, and $\sigma_{yy} = A^2x^2/B^2$ is a symmetric and positive definite tensor a.e. in $\mathbb{R}^2$ that solves equations (3a), (3b), and (3c). As to the expression of $\mathbf{f}^c$, since $\mathbf{n} = (-At/B, 1)/\|\gamma'(t)\|$, relation (3e) infers $\mathbf{f}^c = (-At/B, A^2t^2/B^2)$ on Γ^c; thereafter from the first and last conditions of (4) we arrive at $g(t) = -At^2/(2B) + g_0 + At_0^2/(2B)$, which, in view of the second relation of (4), leads to the incongruence $-3A^2t^2/(2B^2) + g_0A/B + A^2t_0^2/(2B^2) = A^2t^2/B^2$ for all $t \in [t_0, t_1]$.

3 The reduced problem

3.1 Mathematical formulation of the reduced problem

Let us now introduce the Reduced Problem; essentially, its definition corresponds to setting $\Gamma^c = \emptyset$ in Assumptions 2.1. Therefore, Γ^c, $\mathbf{f}^c$, and g do not take part in the formulation, and, subsequently, we have $\Gamma = \Gamma^r = \partial\Omega$; moreover, for convenience, we avoid the superscript r for $\mathbf{f}^r$, and we directly consider $\mathbf{f}$ as a given vectorial field, per unit length, on $\Gamma = \partial\Omega$.

Reduced Problem 1 Under the hypothesis of Assumptions 2.1, let us set $\Gamma^c = \emptyset$. Find a symmetric and a.e. positive definite second-order tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(x,y)$ in $\bar{\Omega}$ such that

$$\begin{cases} \sigma_{xx,x} + \sigma_{xy,y} = 0 \quad \text{in } \Omega, & \text{(5a)} \\ \sigma_{xy,x} + \sigma_{yy,y} = 0 \quad \text{in } \Omega, & \text{(5b)} \\ z_{,xx}\sigma_{xx} + 2z_{,xy}\sigma_{xy} + z_{,yy}\sigma_{yy} = 0 \quad \text{in } \bar{\Omega}, & \text{(5c)} \\ \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{f} \quad \text{on } \Gamma. & \text{(5d)} \end{cases}$$

In the rest of this section, we show the existence and uniqueness of a solution to the Reduced Problem 1 defined in a rectangle.

3.2 A case of explicit resolution in a rectangle

For any $a, c_1 > 0$ and $n \in \mathbb{N}$ with $n > 1$, let us consider the rectangle $\Omega = (0,a) \times (-b, 0)$ with $b = n^{1/n}$ and the function $z(x,y) = c_1x^2 - c_2y^{2n}$, where $c_2 = a^2c_1/(n(2n-1))$, which satisfies $z_{,xx}z_{,yy} - z_{,xy}^2 < 0$ a.e. in $\bar{\Omega}$. Differentiating (5a) with respect to x and (5b) to y and subtracting the results from each other give

$$\sigma_{xx,xx} - \sigma_{yy,yy} = 0 \quad \text{in } \Omega. \tag{6}$$

On the other hand, in view of the expression of z, relation (5c) infers

$$\sigma_{xx} = \frac{c_2n(2n-1)}{c_1} y^{2(n-1)}\sigma_{yy} \quad \text{in } \bar{\Omega}. \tag{7}$$

Hence, (6) and (7) lead to

$$\frac{c_2n(2n-1)}{c_1} y^{2(n-1)}\sigma_{yy,xx} - \sigma_{yy,yy} = 0 \quad \text{in } \Omega, \tag{8}$$

which degenerates for $y = 0$. To endow this equation with the desired Dirichlet conditions for the unknown σ_{yy}, let us treat the vectorial field $\mathbf{f}$ on Γ, and let us write

$$\mathbf{f} = \begin{cases} (\gamma_1(y), \gamma_2(y)) & \text{on } x = a, \\ (-\gamma_1(y), \gamma_3(y)) & \text{on } x = 0, \\ (\theta_1(x), \theta_2(x)) & \text{on } y = 0, \\ (\theta_3(x), -\theta_2(x)) & \text{on } y = -b. \end{cases}$$

In the previous definition, $\gamma_i(y)$ and $\theta_i(x)$ (for $i = 1, 2, 3$) are continuous functions for $-b \leq y \leq 0$ and $0 \leq x \leq a$, respectively, which will be chosen later on; subsequently, taking $\mathbf{n} = (\mp 1, 0)$ respectively on $x = 0$ and $x = a$ and $\mathbf{n} = (0, \mp 1)$ respectively on $y = -b$ and $y = 0$, the boundary conditions (5d) read

$$\begin{cases} \sigma_{xx}(a, y) = \sigma_{xx}(0, y) = \gamma_1(y), \\ \sigma_{xy}(a, y) = \gamma_2(y) \quad \text{and} \quad \sigma_{xy}(0, y) = -\gamma_3(y), \\ \sigma_{xy}(x, 0) = \theta_1(x) \quad \text{and} \quad \sigma_{xy}(x, -b) = -\theta_3(x), \\ \sigma_{yy}(x, 0) = \sigma_{yy}(x, -b) = \theta_2(x). \end{cases} \tag{9}$$

Additionally, if $\gamma_1 = \gamma_1(y)$ is such that the function

$$\gamma(y) = \frac{\gamma_1(y) c_1}{c_2 n(2n-1) y^{2(n-1)}}$$

is itself continuous in $-b \leq y \leq 0$, then in light of (7), (8), and (9) we arrive at the boundary value problem

$$\begin{cases} \frac{c_2 n(2n-1)}{c_1} y^{2(n-1)} \sigma_{yy,xx} - \sigma_{yy,yy} = 0 \quad \text{in } \Omega, \\ \sigma_{yy}(x, 0) = \sigma_{yy}(x, -b) = \theta_2(x), \\ \sigma_{yy}(0, y) = \sigma_{yy}(a, y) = \gamma(y), \end{cases} \tag{10}$$

for which we fix the following proper assumptions:

$$\gamma(0) = \gamma(-b) = \theta_2(0) = \theta_2(a) = K \in \mathbb{R} \quad \text{and} \quad \gamma_2(0) = H \in \mathbb{R}. \tag{11}$$

Hence, let us translate the unknown σ_{yy} through

$$u(x, y) = \sigma_{yy}(x, y) - \left[\gamma(y) + \theta_2(x) - K\right], \tag{12}$$

and successively let us rescale x by the homogeneous dilation mapping $(0, 1)$ onto $(0, a)$ by $x(X) = aX$; the new variable $U(X, y) = u(aX, y)$ and data $\Theta(X) = \theta_2(ax)$ are so obtained. These two transformations, in conjunction with the relation $c_2 = a^2 c_1/(n(2n-1))$, reduce (10) to

$$\begin{cases} y^{2(n-1)} U_{,XX} - U_{,yy} + \Theta''(X) y^{2(n-1)} - \gamma''(y) = 0 \quad \text{in } (0, 1) \times (-b, 0), \\ U(X, 0) = U(X, -b) = 0, \\ U(0, y) = U(1, y) = 0. \end{cases} \tag{13}$$

Now, let us impose $\Theta''(X)y^{2(n-1)} - \gamma''(y) = 0$, that is, $\Theta''(X) = \gamma''(y)/y^{2(n-1)} = \lambda$ for some $\lambda \in \mathbb{R}$. This provides, thanks to the first continuity conditions (11), which for $-b \le y \le 0$ and $0 \le x \le 1$ are $\gamma(0) = \gamma(-b) = \Theta(0) = \Theta(1) = K$,

$$\Theta(X) = \frac{\lambda}{2}X^2 - \frac{\lambda}{2}X + K \quad \text{and} \quad \gamma(y) = \frac{\lambda}{2n(2n-1)}y^{2n} + \frac{\lambda b^{2n-1}}{2n(2n-1)}y + K; \tag{14}$$

as a consequence, problem (13) is equivalent to

$$\begin{cases} (-y)^{2(n-1)}U_{,XX} - U_{,yy} = 0 & \text{in } (0,1) \times (-b,0), \\ U(X,0) = U(X,-b) = 0, \\ U(0,y) = U(1,y) = 0. \end{cases} \tag{15}$$

According to the theory of second-order linear PDE's, and since $b = n^{1/n}$, the characteristic curves associated with the equation $(-y)^{2(n-1)}U_{,XX} - U_{,yy} = 0$ and exactly passing through the vertexes of the rectangle $(0,1) \times (-b,0)$ are (see the right side of Figure 1)

$$X = \frac{1}{n}(-y)^n \quad \text{and} \quad 1 - X = \frac{1}{n}(-y)^n.$$

Thereafter we can rely on the main statement given in [12] and apply its result to the boundary value problem (15); hence we conclude that it admits a unique solution in $(0,1) \times (-b,0)$ that is continuously differentiable everywhere in its closure, possibly except along the mentioned characteristics. As to our specific case, in view of the homogeneous boundary conditions, $U(X,y) \equiv 0$ (and hence also $u(x,y) \equiv 0$) is the unique function with such properties solving problem (15).

Coming back to the tensorial unknown $\boldsymbol{\sigma}$ in $\bar{\Omega}$, expression (14) produces through the relations $X = x/a$, (12), and (7)

$$\begin{cases} \sigma_{yy}(x,y) = \frac{\lambda}{2a^2}x^2 - \frac{\lambda}{2a}x + \frac{\lambda}{2n(2n-1)}y^{2n} + \frac{\lambda b^{2n-1}}{2n(2n-1)}y + K & \text{in } \bar{\Omega}, \\ \sigma_{xx}(x,y) = a^2 y^{2(n-1)}\sigma_{yy}(x,y) & \text{in } \bar{\Omega}. \end{cases} \tag{16}$$

As to $\sigma_{xy} = \sigma_{yx}$, from (5b) we deduce

$$\sigma_{xy}(x,y) = -\int \sigma_{yy,y}\,dx = -\frac{2n\lambda y^{2n-1} + \lambda b^{2n-1}}{2n(2n-1)}x + h(y),$$

so that imposing (5a), we get

$$h'(y) = \frac{a}{2}\lambda y^{2(n-1)} \quad \Leftrightarrow \quad h(y) = \frac{a\lambda y^{2n-1}}{2(2n-1)} + h_0, \quad h_0 \in \mathbb{R}.$$

Now, taking into account the second position in (11) and the boundary conditions (9), the last two expressions yield

$$\begin{aligned} \sigma_{xy}(x,y) &= \sigma_{yx}(x,y) \\ &= -\frac{2n\lambda y^{2n-1} + \lambda b^{2n-1}}{2n(2n-1)}x + \frac{a\lambda y^{2n-1}}{2(2n-1)} + H + \frac{a\lambda b^{2n-1}}{2n(2n-1)} \quad \text{in } \bar{\Omega}. \end{aligned} \tag{17}$$

Lately, to guarantee the positive definiteness of $\boldsymbol{\sigma}$ a.e. in $\bar{\Omega}$ in the sense of (H2), we have to impose, inter alia, that $\sigma_{xx}\sigma_{yy} - (\sigma_{xy})^2 > 0$ a.e. in $\bar{\Omega}$. From (16) and (17) we obtain that $\sigma_{xx}(x,0) = 0$ for $x \in [0,a]$, whereas $\sigma_{xy}(x,0) = -\lambda b^{2n-1}x/(2n(2n-1)) + H + a\lambda b^{2n-1}/(2n(2n-1))$ for $x \in [0,a]$; therefore, without specific assumptions and relations on a, b, n, λ, and H, generally, $\sigma_{xx}(x,0)\sigma_{yy}(x,0) - (\sigma_{xy}(x,0))^2 < 0$ in $[0,a]$. Under these conditions, since for continuity arguments, there would exist a constant $\varepsilon > 0$ such that $\sigma_{xx}\sigma_{yy} - (\sigma_{xy})^2 < 0$ in $(0,a) \times (-\varepsilon, 0)$, in (17) we have to impose $H = \lambda = 0$ obtaining $\sigma_{xy}(x,y) = \sigma_{yx}(x,y) \equiv 0$ in $\bar{\Omega}$. In addition, to avoid the nil solution $\boldsymbol{\sigma} \equiv \mathbf{0}$, we choose a strictly positive value for K, and from (16) we explicitly write $\sigma_{yy}(x,y) = K$ and $\sigma_{xx}(x,y) = Ka^2y^{2(n-1)}$ in $\bar{\Omega}$ and also obtain the following formulas for the functions $\gamma_i(y)$ and $\theta_i(x)$ defining the boundary conditions (9):

$$\gamma_1(y) = a^2Ky^{2(n-1)}, \qquad \theta_2(x) = K, \qquad \theta_1(x) = \theta_3(x) = \gamma_2(y) = \gamma_3(y) = 0.$$

So we have proved our main result.

Theorem 3.1 *Let $a, c_1 > 0$ and $n \in \mathbb{N}$ with $n > 1$. Moreover, for $b = n^{1/n}$ and $c_2 = a^2c_1/(n(2n-1))$, the rectangle $\Omega = (0,a) \times (-b,0)$ and the function $z(x,y) = c_1x^2 - c_2y^{2n}$ are given. Then, for any fixed $K > 0$ and vectorial field (per unit length) on $\Gamma = \partial\Omega$*

$$\mathbf{f} = \begin{cases} (a^2Ky^{2(n-1)}, 0) & \text{on } x = a, \\ (-a^2Ky^{2(n-1)}, 0) & \text{on } x = 0, \\ (0, K) & \text{on } y = 0, \\ (0, -K) & \text{on } y = -b, \end{cases}$$

the symmetric and a.e. positive definite tensor

$$\boldsymbol{\sigma}(x,y) = \begin{pmatrix} a^2Ky^{2(n-1)} & 0 \\ 0 & K \end{pmatrix} \quad \text{in } \bar{\Omega} \tag{18}$$

is the unique classical solution of the Reduced Problem 1.

3.3 The case of no restriction on the sign definiteness of σ

By retracing the proof of Theorem 3.1 we observe that, behind other technical reasons, the final expression of the solution $\boldsymbol{\sigma}$ derived in (18) is deeply tied to the requirement of the a.e. positivity definiteness of such a tensor; conversely, as announced in the introductory comments of Section 1, if this restriction is omitted, then for the same function $z(x,y) = c_1x^2 - c_2y^{2n}$, the unique solution in $\bar{\Omega}$ exhibits a more general representation, precisely given by (16) and (17). Subsequently, we have this further result, which we state without further comments.

Theorem 3.2 *Let $a, c_1 > 0$ and $n \in \mathbb{N}$ with $n > 1$. Moreover, for $b = n^{1/n}$ and $c_2 = a^2c_1/(n(2n-1))$, the rectangle $\Omega = (0,a) \times (-b,0)$ and the function $z(x,y) = c_1x^2 - c_2y^{2n}$*

are given. Then for any fixed $H, K, \lambda \in \mathbb{R}$ *and vectorial field* (*per unit length*) *on* $\Gamma = \partial\Omega$

$$\mathbf{f} = \begin{cases} (\gamma_1(y), \gamma_2(y)) & \text{on } x = a, \\ (-\gamma_1(y), \gamma_3(y)) & \text{on } x = 0, \\ (\theta_1(x), \theta_2(x)) & \text{on } y = 0, \\ (\theta_3(x), -\theta_2(x)) & \text{on } y = -b, \end{cases}$$

where

$$\begin{cases} \gamma_1(y) = a^2 y^{2(n-1)}\left(\frac{\lambda}{2n(2n-1)}y^{2n} + \frac{\lambda b^{2n-1}}{2n(2n-1)}y + K\right), \\ \gamma_2(y) = H + a\lambda\frac{ny^{2n-1} - b^{2n-1}(n-1)}{2n(2n-1)}, \\ \gamma_3(y) = -H - a\lambda\frac{b^{2n-1} + y^{2n-1}}{2(2n-1)}, \\ \theta_1(x) = \frac{\lambda b^{2n-1}}{2n(2n-1)}(an - x) + H, \\ \theta_2(x) = \frac{\lambda}{2a^2}x^2 - \frac{\lambda}{2a}x + K, \\ \theta_3(x) = -\frac{\lambda b^{2n-1}}{2n}x - H, \end{cases}$$

the symmetric tensor

$$\begin{cases} \sigma_{yy}(x,y) = \frac{\lambda}{2a^2}x^2 - \frac{\lambda}{2a}x + \frac{\lambda}{2n(2n-1)}y^{2n} + \frac{\lambda b^{2n-1}}{2n(2n-1)}y + K & \text{in } \bar{\Omega}, \\ \sigma_{xx}(x,y) = a^2 y^{2(n-1)}\sigma_{yy}(x,y) & \text{in } \bar{\Omega}, \\ \sigma_{xy}(x,y) = \sigma_{yx}(x,y) = \frac{\lambda y^{2n-1}}{2n-1}\left(\frac{a}{2} - x\right) + \frac{\lambda b^{2n-1}}{2n(2n-1)}(an - x) + H & \text{in } \bar{\Omega} \end{cases} \tag{19}$$

is the unique classical solution of the Reduced Problem 1.

3.4 Two specific examples: representation of the solution

To give an explicit example to each one of the results claimed in Theorems 3.1 and 3.2, we analyze Figures 2 and 3. They graphically show the behavior of the tensor $\boldsymbol{\sigma}$, which solves the Reduced Problem 1, once in the hypothesis of such theorems the same surface $z = c_1x^2 - c_2y^{2n}$ and the rectangle $\Omega = (0, a) \times (-b, 0)$ are fixed by means of the values $n = 3$, $c_1 = 4$, and $a = 5$ (the surface and the domain are shown at the top-left corners of Figures 2 and 3).

More precisely, for $K = 5$ in expression (18), Figure 2 represents the case of the equilibrium between the stress and shape of a membrane structure. We can realize that the component σ_{xx} is positive a.e. in Ω and increases as $y \to -b$ and constant values of x (see the below part of the top-right corner of Figure 2); in the limit, it exactly corresponds to a zone on the membrane with major tension, along the x-direction, with respect to others (Figure 2, above part, top-right). As to σ_{yy}, it is constant and positive in Ω, so that the corresponding tension along the y-direction is uniformly distributed on the surface (see the lower-left corner of Figure 2); finally, the lower-right corner of Figure 2 highlights the nil contribution of $\sigma_{yx} = 0$ in Ω, that is, the absence of shear stress on the membrane.

Conversely, if in (19) we set $\lambda = 4$, $K = 0.5$, and $H = 2$, the features of the solution $\boldsymbol{\sigma}$ are summarized in Figure 3, which models the balance between the stress and shape for a shell structure. Relaxing the assumption on the sign definiteness of $\boldsymbol{\sigma}$, we obtain not only positive expressions for the components σ_{xx} and σ_{yy} on the whole Ω, but also regions of

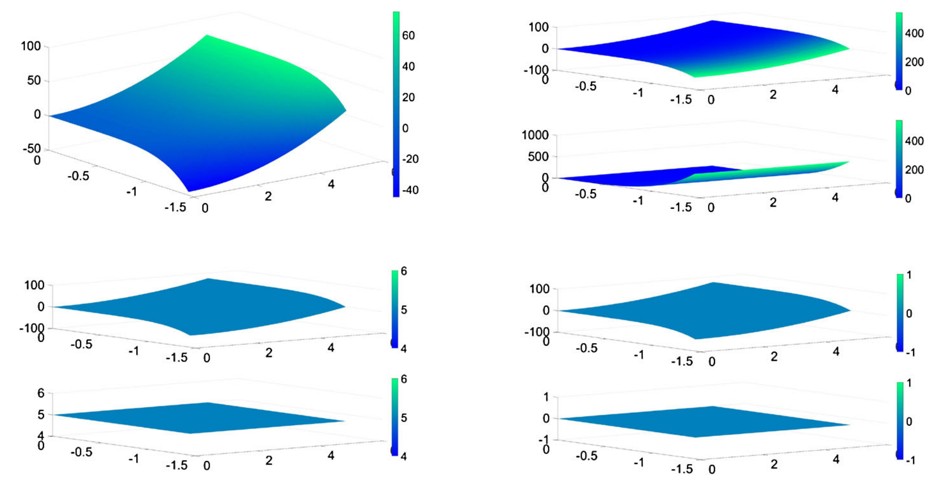

Figure 2 ***Membrane structure*: the case of positive definiteness and graphical representation of the solution σ. (Top-Left)** Graph of the function z in Ω. **(Top-Right)** Above: distribution of the component σ_{xx} on the surface z. Below: representation of the component σ_{xx} in the rectangle Ω. **(Lower-Left)** Above: distribution of the component σ_{yy} on the surface z. Below: representation of the component σ_{yy} in the rectangle Ω. **(Lower-Right)** Above: distribution of the component σ_{xy} on the surface z. Below: representation of the component σ_{xy} in the rectangle Ω.

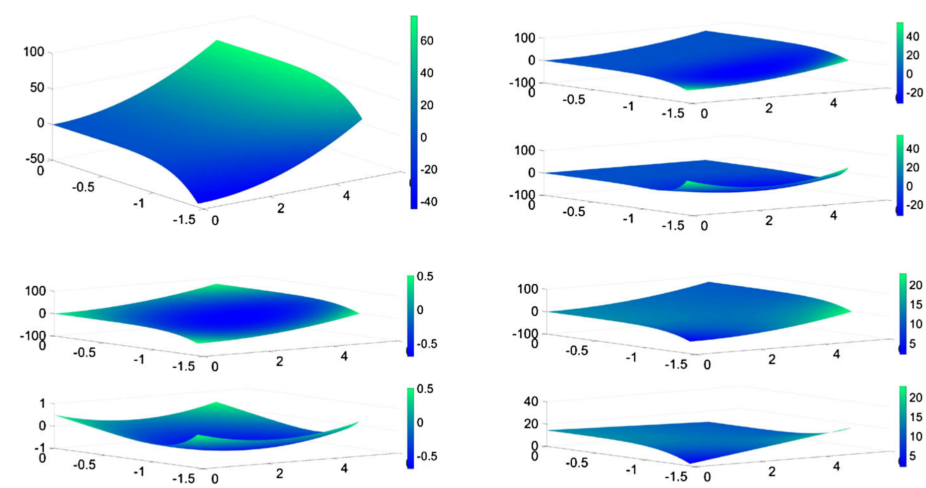

Figure 3 ***Shell structure*: the case of no restriction on the sign definiteness and graphical representation of the solution σ. (Top-Left)** Graph of the function z in Ω. **(Top-Right)** Above: distribution of the component σ_{xx} on the surface z. Below: representation of the component σ_{xx} in the rectangle Ω. **(Lower-Left)** Above: distribution of the component σ_{yy} on the surface z. Below: representation of the component σ_{yy} in the rectangle Ω. **(Lower-Right)** Above: distribution of the component σ_{xy} on the surface z. Below: representation of the component σ_{xy} in the rectangle Ω.

the rectangle where they are negative (see the below part of the top-right and lower-left corner of Figure 3, respectively); this aspect identifies zones of the shell where tensions or compressions are present along both the x- and y-directions (same corners of Figure 3, but the above part).

By the above we stress again that the general solution for the tensor $\boldsymbol{\sigma}$ given by relation (18) represents a very particular and simplified case of solution (19). Such a leap has not to appear surprising since, indeed, it is intimately linked to the different natures of the problems: in particular, when a membrane is considered, a strong limitation on the state of its stress tensor that exactly balances its shape is naturally expected and absolutely consistent with the mechanical problem.

4 Conclusions

This paper is devoted to a two-dimensional boundary value system described by a tensorial equation in a bounded domain. Its more general definition leads to the resolution of an overdetermined hyperbolic problem, whose analysis is complex and represents a challenging open question in the field. Indeed, for a simplified version, whose formulation is given by a degenerate problem on a rectangle, the existence and uniqueness of a solution under proper assumptions on the data can be proven. Behind its pure mathematical interest, this research is motivated by natural applications to real mechanic problems, linked to the equilibrium of membrane and shell structures. In this sense, the derived solutions achieved throughout the paper are totally consistent with the expected results.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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Endnotes

[a] In this paper the partial derivative of a function f with respect to a certain variable w is indicated with $f_{,w}$; similar symbols concerning higher-order derivatives (double or mixed) are introduced in a natural way.

[b] With some abuse of language, we also use sentences as *z has an a.e. negative Gaussian curvature* or *z is a function with a.e. negative Gaussian curvature* or similar; in any case, no misunderstanding will be possible from the context.

[c] Obviously, if $\sigma_{xx}\xi_1^2 + 2\sigma_{xy}\xi_1\xi_2 + \sigma_{yy}\xi_2^2 < 0\ \forall(\xi_1, \xi_2) \neq (0,0)$ and a.e. in $\bar{\Omega}$, we say that the tensor $\boldsymbol{\sigma}$ is almost everywhere negative definite. We say that the tensor is indefinite (or alternating sign definite) a.e. in $\bar{\Omega}$ if it is a.e. negative definite for some values of $\bar{\Omega}$ and a.e. positive definite for the others.

[d] Let us remark that, as indicated in the paragraphs of Section 1 dealing with the *General Problem*, the superscripts r and c stand for *rigid* and *cable*.

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