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General decay of solutions to a one-dimensional thermoelastic beam with variable coefficients

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Abstract
The vibrations of flexible structures in practice are described by nonlinear models of strings, beams, plates, and so on. This paper is concerned with longitudinal vibrations of a thermoelastic beam equation. Our main result is the general decay of the system. Using the multiplier method and some properties of the convex functions, we establish the general decay of energy.

MSC: 35L35; 35B35; 93D15

Keywords: flexible structure; longitudinal vibrations; decay; memory; convexity

1 Introduction

Generally speaking, the vibrations of flexible structures in practice are described by nonlinear models of strings, beams, plates, and so on. The linearized vibrations of flexible structures are usually governed by partial differential equations, in particular, by the second-order wave equation and the fourth-order Euler-Bernoulli beam equation [1]. Up till now, there are many results concerning the stability of wave and plate equations by adding some types of damping, for example, internal damping, boundary damping, thermal damping, and so on, most of which can be found in the literature. For general decay results on the wave equation, here we mention the works by Cao and Yao [2], Guesmia and Messaoudi [3], Said-Houari, Messaoudi, and Guesmia [4], Messaoudi [5–7], Messaoudi and Al-Gharabli [8, 9], Messaoudi and Soufyane [10], Mustafa and Messaoudi [11], Tatar [12], and Wu [13]. When the body vibrates, the balance of linear momentum reads

$$mu_{tt} - \sigma_x = f,$$

where σ is the stress. If the body is nonuniform, that is,

$$\sigma = \sigma(u_x, u_{xt}) = p(x)u_x + 2\delta(x)u_{xt},$$

then we can derive the equation of longitudinal vibrations of a flexible structure (see, e.g., Liu and Liu [14])

$$m(x)u_{tt} - \big(p(x)u_x + 2\delta(x)u_{xt}\big)_x = f,$$

where $u(x,t)$ represents the longitudinal displacement of a particle. The functions $m(x)$, $p(x)$, and $\delta(x)$ denote the mass per unit length of the structure, the coefficient of internal material damping, and a positive function related to the stress acting on the body, respectively. For this equation, Gorain [15] established the exponential stability of the solution.

For the thermal effect in flexible structures, that is,

$$\theta_t + q_x - \kappa u_{tx} = 0,$$

where $q(x,t)$ denotes the heat flux vector, and $\theta(x,t)$ is the temperature difference, we we can find the physical background in Carlson [16]. If we assume that the heat flux satisfies a different thermal law, we can obtain flexible structures with different thermal effects. For example, if the heat flux satisfies

$$q(t) + (1-\alpha)\theta_x(t) + \alpha \int_0^\infty g(s)\theta_x(t-s)\,ds = 0, \quad \alpha \in (0,1),$$

then we get the flexible structures with the Coleman-Gurtin law. The limit cases $\alpha = 0$ and $\alpha = 1$ correspond to the Fourier and Gurtin-Pipkin cases. Misra et al. [17] studied the thermoelastic flexible structure with Fourier law:

$$\begin{cases} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{xt})_x + \kappa\theta_x = f, \\ \theta_t - \theta_{xx} + \kappa u_{xt} = 0, \end{cases}$$

and proved the global well-posedness of the system. In addition, they established the exponential decay of energy. It is a remarkable fact that the assumption of Fourier's law causes an unrealistic property that a sudden disturbance at some point will be felt instantly everywhere else in the material. Green and Naghdi developed a damped model, called thermoelasticity of type III (see, e.g., [18–23]),

$$\begin{cases} u_{tt} - \alpha u_{xx} + +\beta\theta_x = f, \\ \theta_{tt} - \delta\theta_{xx} + \gamma u_{xtt} - \kappa\theta_{xtt} = 0. \end{cases}$$

For decay results of this system, we refer to Quintanilla and Racke [24] and Zhang and Zuazua [25] (see also [26]). For flexible structures with thermal effect, we also mention the work of Alves et al. [27], where the authors studied the thermoelastic flexible structure with second sound and proved the well-posedness and stability of the system.

Taking into account all considerations mentioned, in this paper, we consider the following longitudinal vibrations of a thermoelastic beam equation with past history:

$$m(x)u_{tt} - \big(p(x)u_x + 2\delta(x)u_{xt}\big)_x + \kappa\theta_{xt} = 0, \tag{1.1}$$

$$\theta_{tt} - \alpha\theta_{xx} - \int_0^\infty g(s)\theta_{xx}(t-s)\,ds + \kappa u_{xt} = 0, \tag{1.2}$$

where $x \in [0,L]$, and the constant κ is the coupling coefficient. The boundary conditions are given by

$$u(0,t) = u(L,t) = 0, \qquad \theta(0,t) = \theta(L,t) = 0, \quad t \geq 0, \tag{1.3}$$

and the initial conditions are

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad \theta(x,t)|_{t\le 0} = \theta_0(x,t), \qquad \theta_t(x,0) = \theta_1(x). \quad (1.4)$$

Our goal in this paper is to establish a general decay of solutions to problem (1.1)-(1.4) with exponential and polynomial decays as only particular cases. We use the multiplier method and some properties of convex functions to establish a general decay result.

To deal with the memory, motivated by Dafermos [28] and Giorgi et al. [29, 30], we define the new variable $\eta = \eta^t(x,s)$ by

$$\eta^t(x,s) = \theta(x,t) - \theta(x,t-s), \quad (t,s) \in [0,\infty) \times \mathbb{R}^+. \tag{1.5}$$

Then

$$\eta_t + \eta_s = \theta_t, \quad (x,t,s) \in \mathbb{R}^n \times \mathbb{R}^+ \times \mathbb{R}^+.$$

It follows from (1.5) that

$$\int_0^\infty g(s)\theta_{xx}(t-s)\,ds = \int_0^\infty g(s)\,ds\theta_{xx} - \int_0^\infty g(s)\eta^t_{xx}(s)\,ds.$$

Assuming for simplicity that $\alpha - \int_0^\infty g(s) = 1$, problem (1.1)-(1.4) is transformed into the new problem

$$m(x)u_{tt} - \bigl(p(x)u_x + 2\delta(x)u_{xt}\bigr)_x + \kappa\theta_{xt} = 0, \tag{1.6}$$

$$\theta_{tt} - \theta_{xx} + \int_0^\infty g(s)\eta_{xx}(s)\,ds + \kappa u_{xt} = 0, \tag{1.7}$$

$$\eta^t_t + \eta^t_s = \theta_t, \tag{1.8}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \qquad \eta^t(x,0) = 0, \quad x \in [0,L], \tag{1.9}$$

$$\theta(x,t)|_{t\le 0} = \theta_0(x,t), \qquad \theta_t(x,0) = \theta_1(x), \quad x \in [0,L], \tag{1.10}$$

$$\eta^0(x,s) = \eta_0(x,s), \quad (x,s) \in [0,L] \times \mathbb{R}^+, \tag{1.11}$$

$$u(0,t) = u(L,t) = 0, \qquad \theta(0,t) = \theta(L,t) = 0, \quad t \ge 0. \tag{1.12}$$

The plan of the paper is as follows. In Section 2, we give some assumptions and our main results. The proof of the general decay result is given in Section 3.

2 Assumptions and main results

By $L^q(0,L)$ ($1 \le q \le \infty$) and $H^1(0,L)$ we denote the standard Lebesgue integral and Sobolev spaces. The norm in a space B is denoted by $\|\cdot\|_B$. For simplicity, we use $\|u\|$ instead of $\|u\|_2$ when $q = 2$.

Now we give some assumptions used in this paper.

(A1) The functions $m(x), p(x), \delta(x) : [0,L] \to \mathbb{R}^+$ are functions of class $C^1(0,L)$, and there exist positive constants m_1, m_2, p_1, p_2, δ_1, and δ_2 such that

$$m_1 \le m(x) \le m_2, \qquad p_1 \le p(x) \le p_2, \quad \text{and} \quad \delta_1 \le \delta(x) \le \delta_2. \tag{2.1}$$

(A2) The function $g:\mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 a nonincreasing function satisfying

$$g(0) > 0, \qquad 1 - \int_0^\infty g(s)\,ds = l > 0, \tag{2.2}$$

and there exists a positive constant $a_0 < m_2$ such that, for any $u \in H_0^1(0,L)$,

$$a_0 \int_0^L u_x^2\,dx \le \int_0^L m(x)u_x^2\,dx - \int_0^L \left(\int_0^\infty g(s)\,ds\right) u_x^2\,dx. \tag{2.3}$$

In addition, there exists an increasing strictly convex function $G:\mathbb{R}^+ \to \mathbb{R}^+$ of class $C^1(\mathbb{R}^+) \cap C^2(\mathbb{R}^+)$ satisfying

$$G(0) = G'(0) = 0, \qquad \lim_{t\to\infty} G'(t) = \infty, \tag{2.4}$$

and

$$\int_0^\infty \frac{g(s)}{G^{-1}(-g'(s))}\,ds + \sup_{s\in\mathbb{R}^+} \frac{g(s)}{G^{-1}(-g'(s))} < \infty. \tag{2.5}$$

To consider the new variable η, we define the weighted L^2-spaces

$$\mathcal{M} = L_g^2\big(\mathbb{R}^+, H_0^1(0,L)\big) = \left\{\eta : \mathbb{R}^+ \to H_0^1 : \int_0^\infty g(s)\big\|\eta_x(s)\big\|^2\,ds < \infty\right\},$$

which is a Hilbert space endowed with inner product and norm

$$(\eta,\zeta)_{\mathcal{M}} = \int_0^\infty g(s)\big(\eta_x(s), \zeta_x(s)\big)\,ds \quad \text{and} \quad \|\eta\|_{\mathcal{M}}^2 = \int_0^\infty g(s)\big\|\eta_x(s)\big\|^2\,ds.$$

Now we define the phase space

$$\mathcal{H} = H_0^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times \mathcal{M}$$

equipped with the norm

$$\big\|(u,v,\theta,\vartheta,\eta)\big\|_{\mathcal{H}}^2 = \|u_x\|^2 + \|v\|^2 + \|\theta_x\| + \|\vartheta\| + \|\eta\|_{\mathcal{M}}^2.$$

Using semigroup theory, we can easily prove the existence of solutions to problem (1.6)-(1.12); see, for example, Alves et al. [27].

Theorem 2.1 *Assume that* (2.1)-(2.2) *hold. Let* $U(t) = (u,u_t,\theta,\theta_t,\eta)$. *If the initial data* $U_0 \in \mathcal{H}$, *then problem* (1.6)-(1.12) *has a unique mild solution* $U(t) \in C([0,\infty),\mathcal{H})$ *with* $U(0) = U_0$.

Define the energy $E(t)$ of problem (1.6)-(1.12) by

$$E(t) = \frac{1}{2}\left[\int_0^L m(x)u_t^2\,dx + \int_0^L p(x)u_x^2\,dx + \int_0^L \theta_t^2\,dx + \int_0^L \theta_x^2\,dx + \|\eta\|_{\mathcal{M}}^2\right]. \tag{2.6}$$

The general decay result can be given in the following theorem.

Theorem 2.2 *Assume that* (2.1)-(2.5) *hold. Let* $(u_0, u_1, \theta_0(\cdot, 0), \theta_1, \eta_0) \in \mathcal{H}$ *be given. Suppose that there exists a constant* $C \geq 0$ *such that, for any* $s > 0$,

$$\|\eta_{0x}\| \leq C. \tag{2.7}$$

Then there exist positive constants k_1, k_2, ϵ_0 *such that, for any* $t \in \mathbb{R}^+$,

$$E(t) \leq k_1 H^{-1}(k_2 t), \tag{2.8}$$

where

$$H(s) = \int_s^1 \frac{1}{\tau G'(\epsilon_0 \tau)}\, d\tau.$$

3 General decay

In this section, to prove Theorem 2.2, we establish a general decay of solutions to problem (1.6)-(1.12). We need the following technical lemmas.

Lemma 3.1 *For the energy* $E(t)$ *defined in* (2.6), *there exists a constant* $c_1 > 0$ *such that, for any* $t > 0$,

$$E'(t) \leq -2 \int_0^L \delta(x) u_{xt}^2\, dx + \int_0^L g'(s) \big\| \eta_x(s) \big\|^2\, ds \leq 0. \tag{3.1}$$

Proof Multiplying (1.6) by u_t and (1.7) by θ_t and then integrating the result over $(0, L)$, we easily get the desired estimate (3.1). □

Lemma 3.2 *Under the assumptions of Theorem* 2.2, *for the functional* $\phi(t)$ *defined by*

$$\phi(t) = \int_0^L m(x) u(t) u_t(t)\, dx + \kappa \int_0^L \theta_x(t) u(t)\, dx,$$

there exists a positive constant c_1 *such that, for any* $t > 0$,

$$\phi'(t) \leq -\frac{1}{2} \int_0^L p(x) u_x^2\, dx + \int_0^L m(x) u_x^2\, dx + \frac{\kappa}{2} \int_0^L \theta_x^2\, dx + c_1 \int_0^L u_{xt}^2\, dx, \tag{3.2}$$

where $c_1 = \frac{\kappa}{2\lambda_1}$, *and* $\lambda_1 > 0$ *is the Poincaré constant.*

Proof Taking the derivative of $\phi(t)$ with respect to t and using (1.6), we get

$$\begin{aligned} \phi'(t) = {} & -2 \int_0^L \delta(x) u_x u_{xt}\, dx - \int_0^L p(x) u_x^2\, dx + \int_0^L m(x) u_t^2\, dx \\ & + \kappa \int_0^L \theta_x u_t\, dx. \end{aligned} \tag{3.3}$$

Young's inequality, Poincaré's inequality, and (2.1) give us

$$-2 \int_0^L \delta(x) u_x u_{xt}\, dx \leq \frac{1}{2} \int_0^L p(x) u_x^2\, dx + \frac{4\delta_2^2}{p_1} \int_0^L u_{xt}^2\, dx$$

and

$$\kappa \int_0^L \theta_x u_t \, dx \le \frac{\kappa}{2} \int_0^L \theta_x^2 \, dx + \frac{\kappa}{2\lambda_1} \int_0^L u_{xt}^2 \, dx,$$

which, together with (3.3), give us (3.2). □

Lemma 3.3 *Under the assumptions of Theorem* 2.2, *the functional* $\psi(t)$ *defined by*

$$\psi(t) = \int_0^L \theta(t)\theta_t(t) \, dx + \kappa \int_0^L u_x(t)\theta(t) \, dx$$

satisfies

$$\psi'(t) \le -\frac{1}{2} \int_0^L \theta_x^2 \, dx + \left(\frac{\kappa}{2} + 1\right) \int_0^L \theta_t^2 \, dx + \frac{\kappa}{2\lambda_1} \int_0^L u_{xt}^2 \, dx + \frac{1-l}{2} \|\eta\|_{\mathcal{M}}^2 \tag{3.4}$$

for all $t > 0$.

Proof It follows from (1.7) that

$$\psi'(t) = -\int_0^L \theta_x^2 \, dx + \int_0^L \theta_t^2 \, dx - \int_0^L \theta_x(t) \int_0^\infty g(s)\eta_x(s) \, ds + \kappa \int_0^L u_x \theta_t \, dx. \tag{3.5}$$

Using Young's and Poincaré's inequalities, we infer that

$$-\int_0^L \theta_x(t) \int_0^\infty g(s)\eta_x(s) \, ds \, dx \le \frac{1}{2} \int_0^L \theta_x^2 \, dx + \frac{1-l}{2} \|\eta\|^2$$

and

$$\kappa \int_0^L u_x \theta_t \, dx \le \frac{\kappa}{2} \int_0^L \theta_t^2 \, dx + \frac{\kappa}{2\lambda_1} \int_0^L u_{xt}^2 \, dx,$$

which, along with (3.5), implies (3.4). □

Lemma 3.4 *Under the assumptions of Theorem* 2.2, *for the functional* $\chi(t)$ *defined by*

$$\chi(t) = -\int_0^L \theta_t(t) \int_0^\infty g(s)\eta(s) \, ds \, dx,$$

there exists a positive constant c_2 *such that, for any* $t > 0$,

$$\begin{aligned}\chi'(t) \le{} & -\frac{1-l}{2} \int_0^L \theta_t^2 \, dx + \frac{\kappa^2}{2\lambda_1} \int_0^L u_{xt}^2 \, dx + 2(1-l)\|\eta\|_{\mathcal{M}}^2 \\ & - c_2 \int_0^\infty g'(s) \big\| \eta_x(s) \big\|^2 \, ds. \end{aligned} \tag{3.6}$$

Proof Direct differentiation using (1.7) implies

$$\begin{aligned}\chi'(t) = & \underbrace{\int_0^L \theta_x\left(\int_0^\infty g(s)\eta_x(s)\,ds\right)dx}_{:=I_1} + \underbrace{\int_0^L\left(\int_0^\infty g(s)\eta_x(s)\,ds\right)^2 dx}_{:=I_2} \\ & \underbrace{-\kappa\int_0^L u_t\int_0^\infty g(s)\eta_x(s)\,ds\,dx}_{:=I_3} \underbrace{-\int_0^L \theta_t\int_0^\infty g(s)\eta_t(s)\,ds\,dx.}_{:=I_4}\end{aligned} \tag{3.7}$$

It follows from Hölder's and Young's inequalities that

$$I_1 \le \frac{1}{2}\int_0^L \theta_x^2\,dx + \frac{1-l}{2}\|\eta\|_{\mathcal{M}}^2, \qquad I_2 \le (1-l)\|\eta\|_{\mathcal{M}}^2,$$

and

$$I_3 \le \frac{\kappa^2}{2\lambda_1}\int_0^L u_{xt}^2\,dx + \frac{1-l}{2}\|\eta\|_{\mathcal{M}}^2.$$

Noting (1.8) and using Young's inequality, we have

$$\begin{aligned}I_4 &= -(1-l)\int_0^L \theta_t^2\,dx - \int_0^L \theta_t\int_0^\infty g'(s)\eta(s)\,ds\,dx \\ &\le -\frac{(1-l)}{2}\int_0^L \theta_t^2\,dx - \frac{l_1}{2(1-l)\lambda_1}\int_0^\infty g'(s)\left\|\eta_x(s)\right\|^2 ds,\end{aligned}$$

where $l_1 = -\int_0^\infty g'(s)\,ds$. Inserting these estimates into (3.7), we get (3.6) with

$$c_2 = \frac{l_1}{2(1-l)\lambda_1}.$$

The proof is completed. □

We further define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = NE(t) + N_1\phi(t) + N_2\psi(t) + N_3\chi(t),$$

where N, N_1, N_2, N_3 are positive constants to be chosen later. First, it is easy to verify that there exists two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \le \mathcal{L}(t) \le \beta_2 E(t). \tag{3.8}$$

Lemma 3.5 *For suitable constants $N, N_1, N_2, N_3 > 0$, there exist two positive constants c_4 and c_5 such that*

$$\mathcal{L}'(t) \le -c_4 E(t) + c_5\int_0^\infty g(s)\left\|\eta_x(s)\right\|^2 ds. \tag{3.9}$$

Proof Combining (3.1)-(3.2), (3.4), and (3.6) and using (2.1), we can obtain that, for any $t > 0$,

$$\begin{aligned}
\mathcal{L}'(t) &= NE'(t) + N_1\phi'(t) + N_2\psi'(t) + N_3\chi'(t) \\
&\leq -\left[2N\delta_1 - \left(\frac{m_2}{\lambda_1} + c_1\right)N_1 - \frac{\kappa}{2\lambda_1}N_2 - \frac{\kappa^2}{2\lambda_1}N_3\right]\int_0^L u_{xt}^2\,dx \\
&\quad - \frac{N_1 p_1}{2}\int_0^L u_x^2\,dx - \left(\frac{N_2}{2} - \frac{\kappa}{2}N_1\right)\int_0^L \theta_x^2\,dx + c_3\|\eta\|_{\mathcal{M}}^2 \\
&\quad - \left[\frac{1-l}{2}N_3 - \left(\frac{\kappa}{2} + 1\right)N_2\right]\int_0^L \theta_t^2\,dx + (N - c_2N_3)\int_0^\infty g'(s)\left\|\eta_x(s)\right\|^2\,ds,
\end{aligned}$$

where $c_3 = N_1\frac{1-l}{2} + 2N_3(1-l)$.

At this point, we first choose $N_1, N_2, N_3 > 0$ that satisfy

$$N_3 > \frac{\kappa + 2}{2(1-l)}N_2, \qquad N_2 > \kappa N_1,$$

which implies

$$\frac{N_2}{2} - \frac{\kappa}{2}N_1 > 0, \qquad \frac{1-l}{2}N_3 - \left(\frac{\kappa}{2} + 1\right)N_2 > 0.$$

Then we get that there exist constants $\gamma > 0$ and $\mu > 0$ such that

$$\begin{aligned}
\mathcal{L}'(t) &\leq -\left[2N\delta_1 - \left(\frac{m_2}{\lambda_1} + c_1\right)N_1 - \frac{\kappa}{2\lambda_1}N_2 - \frac{\kappa^2}{2\lambda_1}N_3 - \frac{\mu}{\lambda_1}\right]\int_0^L u_{xt}^2\,dx \\
&\quad - \gamma E(t) + c_3\|\eta\|_{\mathcal{M}}^2 + (N - c_2N_3)\int_0^\infty g'(s)\left\|\eta_x(s)\right\|^2\,ds.
\end{aligned} \tag{3.10}$$

Finally, for fixed $N_1, N_2, N_3 > 0$, we take $N > 0$ large enough so that

$$2N\delta_1 - \left(\frac{m_2}{\lambda_1} + c_1\right)N_1 - \frac{\kappa}{2\lambda_1}N_2 - \frac{\kappa^2}{2\lambda_1}N_3 - \frac{\mu}{\lambda_1} > 0, \qquad N - c_2N_3 > 0,$$

which, together with (3.10), gives us (3.9). □

Lemma 3.6 *Under the assumptions of Theorem* 2.2, *there exists a positive constant* $\gamma_1 > 0$ *such that, for any* $\epsilon_0 > 0$,

$$G'\big(\epsilon_0 E(t)\big)\int_0^\infty g(s)\left\|\eta_x(s)\right\|^2\,ds \leq -\gamma_1 E'(t) + \gamma_1\epsilon_0 E(t)G'\big(\epsilon_0 E(t)\big). \tag{3.11}$$

Proof We prove this lemma by using the method developed by Guesmia [31]. It follows from (2.3), (2.7), and (3.1) that

$$\begin{aligned}
\int_0^L \eta_x^2\,dx &\leq 2\int_0^L \big(u_x^2(x,t) + u_x^2(x,t-s)\big)\,dx \\
&\leq 4\sup_{t\geq 0}\int_0^L u_x^2(t)\,dx + 2\sup_{s>0}\int_0^L u_{0x}^2(x,s)\,dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{8}{a_0} E(0) + 4 \sup_{s>0} \int_0^L \big((\eta_x^0)^2(x,s) + u_{0x}^2(x,0) \big)\, dx \\
&\leq \frac{16}{a_0} E(0) + 4C := C_0,
\end{aligned}
\tag{3.12}
$$

where C_0 is a positive constant depending on $E(0)$, a_0, and C.

First, if $g'(s_0) = 0$ for some $s_0 \geq 0$, then from (2.5) we infer that $g(s_0) = 0$. Since $g(t)$ is nonincreasing and nonnegative, we know that $g(s) = 0$ for any $s \geq s_0$. Therefore

$$
\int_0^\infty g(s)\eta_x^2(s)\, ds = \int_0^{s_0} g(s)\eta_x^2(s)\, ds.
$$

Without loss of generality, we can assume that $g'(s) < 0$ for $s \in \mathbb{R}^+$.

For $s \in \mathbb{R}^+$, we define $K(s) = \frac{s}{G^{-1}(s)}$. By the properties of G we know that $K(0) = G'(0) = 0$, and hence the function $K(s)$ is nondecreasing. Taking into account (3.12), we obtain that, for any $s_1 > 0$,

$$
K\big(-s_1 g'(s)\|\eta_x\|^2\big) \leq K\big(-C_0 s_1 g'(s)\big).
$$

Therefore, for $\epsilon_0, \tau_1 > 0$,

$$
\begin{aligned}
&\int_0^L g(s)\big\|\eta_x(s)\big\|^2\, ds \\
&\quad = \frac{1}{\tau_1 G'(\epsilon_0 E(t))} \int_0^\infty G^{-1}\big(-s_1 g(s)\|\eta_x\|^2\big) \frac{\tau_1 G'(\epsilon_0 E(t)) g(s)}{-s_1 g(s)} K\big(-s_1 g(s)\|\eta_x\|^2\big)\, ds \\
&\quad \leq \frac{1}{\tau_1 G'(\epsilon_0 E(t))} \int_0^\infty G^{-1}\big(-s_1 g(s)\|\eta_x\|^2\big) \frac{\tau_1 G'(\epsilon_0 E(t)) g(s)}{-s_1 g(s)} K\big(-C_0 s_1 g'(s)\big)\, ds \\
&\quad \leq \frac{1}{\tau_1 G'(\epsilon_0 E(t))} \int_0^\infty G^{-1}\big(-s_1 g(s)\|\eta_x\|^2\big) \frac{C_0 \tau_1 G'(\epsilon_0 E(t)) g(s)}{G^{-1}(-C_0 s_1 g'(s))}\, ds.
\end{aligned}
\tag{3.13}
$$

Now we denote the conjugate function of the convex function G by G^* (see, e.g., Arnold [32], Daoulatli et al. [33], Lasiecka and Doundykov [34], and Lasiecka and Tataru [35]), that is,

$$
G^*(s) = \sup_{t \in \mathbb{R}^+} \big(st - G(t)\big).
$$

Then,

$$
G^*(s) = s\big(G'\big)^{-1}(s) - G\big[\big(G'\big)^{-1}(s)\big]
$$

for $s \geq 0$ is the Legendre transform of G, which satisfies

$$
t_1 t_2 \leq G(t_1) + G^*(t_2).
$$

for $t_1, t_2 \geq 0$. Denoting

$$
t_1 = G^{-1}\big(-s_1 g(s)\|\eta_x\|^2\big), \qquad t_2 = \frac{C_0 \tau_1 G'(\epsilon_0 E(t)) g(s)}{G^{-1}(-C_0 s_1 g'(s))},
$$

from (3.13) we see that

$$\int_0^\infty g(s)\|\eta_x(s)\|^2\,ds \le \frac{-s_1}{\tau_1 G'(\epsilon_0 E(t))}\int_0^\infty g'(s)\|\eta_x(s)\|^2\,ds + \frac{1}{\tau_1 G'(\epsilon_0 E(t))}\int_0^\infty G^*\left(\frac{C_0\tau_1 G'(\epsilon_0 E(t))g(s)}{G^{-1}(-C_0 s_1 g'(s))}\right)ds,$$

which, together with (3.1) and the inequality $G^*(s)\le s(G')^{-1}(s)$, gives

$$\begin{aligned}\int_0^\infty g(s)\|\eta_x(s)\|^2\,ds &\le \frac{-2s_1}{\tau_1 G'(\epsilon_0 E(t))}E'(t)\\ &\quad + C_0\int_0^\infty \frac{g(s)}{G^{-1}(-C_0 s_1 g'(s))}(G')^{-1}\left(\frac{C_0\tau_1 G'(\epsilon_0 E(t))g(s)}{G^{-1}(-C_0 s_1 g'(s))}\right)ds.\end{aligned}\tag{3.14}$$

Using (2.5), we denote

$$\sup_{s>0}\frac{g(s)}{G^{-1}(-g'(s))} = b < +\infty.$$

By the properties of G we get the function $(G')^{-1}$ is nondecreasing. Then taking $s_1 = \frac{1}{C_0}$, we conclude from (3.14) that

$$\int_0^\infty g(s)\|\eta_x(s)\|^2\,ds \le \frac{-2}{C_0\tau_1 G'(\epsilon_0 E(t))}E'(t) + C_0(G')^{-1}\big(C_0 b\tau_1 G'(\epsilon_0 E(t))\big)\int_0^\infty \frac{g(s)}{G^{-1}(-g'(s))}\,ds.\tag{3.15}$$

Similarly, denoting

$$\int_0^\infty \frac{g(s)}{G^{-1}(-g'(s))}\,ds = d < +\infty$$

and picking $\tau_1 = \frac{1}{C_0 b}$, we infer from (3.15) that

$$\int_0^\infty g(s)\|\eta_x(s)\|^2\,ds \le \frac{-2b}{G'(\epsilon_0 E(t))}E'(t) + C_0\,d\epsilon_0 E(t),$$

which gives (3.11). □

Proof of Theorem 2.2 Multiplying (3.9) by $G'(\epsilon_0 E(t))$ and using (3.11), we obtain

$$G'\big(\epsilon_0 E(t)\big)\mathcal{L}'(t) \le -(c_4 - c_5\gamma_1\epsilon_0)E(t)G'\big(\epsilon_0 E(t)\big) - \gamma_1 c_5 E'(t).$$

Taking ϵ_0 small enough so that $c_4 - c_5\gamma_1\epsilon_0 > 0$, we get that there exists a constant $c_6 > 0$ such that

$$G'\big(\epsilon_0 E(t)\big)\mathcal{L}'(t) + \gamma_1 c_5 E'(t) \le -c_6 E(t)G'\big(\epsilon_0 E(t)\big).\tag{3.16}$$

Define

$$\mathcal{E}(t) = \tau\bigl(G'\bigl(\epsilon_0 E(t)\bigr)\mathcal{L}(t) + \gamma_1 c_5 E(t)\bigr)$$

with $\tau > 0$. Noting that $G'(\epsilon_0 E(t))$ is nonincreasing, we can easily get from (3.8) that

$$\mathcal{E}'(t) \le -c_6 \tau G'\bigl(\epsilon_0 E(t)\bigr)E(t). \tag{3.17}$$

Using (3.8) and the inequality $G'(\epsilon_0 E(t)) \le G'(\epsilon_0 E(0))$, we can get that there exist two positive constants β_3 and β_4 such that

$$\beta_3 E(t) \le \mathcal{E}(t) \le \beta_4 E(t).$$

Choose $\tau > 0$ small enough so that

$$\mathcal{E}(t) \le E(t), \qquad \mathcal{E}(0) \le 1. \tag{3.18}$$

Noting that $s \mapsto sG'(\epsilon_0 s)$ is nondecreasing, from (3.17) we obtain

$$\mathcal{E}'(t) \le -c_7 \mathcal{E}(t) G'\bigl(\epsilon_0 \mathcal{E}(t)\bigr) \tag{3.19}$$

with $c_7 = c_6 \tau$. This shows that $(H(\mathcal{E}(t)))' \ge c_7$, where

$$H(t) = \int_t^1 \frac{1}{sG'(\epsilon_0 s)}\, ds.$$

Integrating it over $[0, t]$, we see that

$$H\bigl(\mathcal{E}(t)\bigr) \ge c_7 t + H\bigl(\mathcal{E}(0)\bigr),$$

which, together with (3.18) and $H(1) = 0$, implies

$$H\bigl(\mathcal{E}(t)\bigr) \ge c_7 t.$$

Since H^{-1} is decreasing, we get

$$\mathcal{E}(t) \le H^{-1}(c_7 t).$$

Then (2.8) follows from the equivalence $\mathcal{E}(t) \sim E(t)$. □

Remark 3.7 Condition (2.5) allows g to have a decay rate close to $\frac{1}{t}$, and the rate of energy decay (2.8) depends on g.

Finally, we give three examples to illustrate several rates of energy decay, which can be found in Guesmia [31].

Example 1 Let $g(t) = \frac{\mu}{(1+t)^p}$ with $p > 1$ and $\mu > 0$ small enough so that (2.3) holds. Condition (2.5) is satisfied for $G(t) = t^{1+\frac{1}{q}}$ with $q \in (0, \frac{p-1}{2})$. Then from (2.8) we get that there exists a constant $\beta_5 > 0$ such that, for any $q \in (0, \frac{p-1}{2})$,

$$E(t) \le \frac{\beta_5}{(1+t)^q}.$$

Example 2 Let $g(t) = \mu e^{-(\ln(2+t))^p}$ with $p > 1$ and $\mu > 0$ small enough so that (2.3) holds. For

$$G(t) = \int_0^t (-\ln s)^{1-\frac{1}{q}} e^{-(-\ln s)^{\frac{1}{q}}}\, ds,$$

when t is near zero, (2.5) holds with with $q \in (1, p)$. Then from (2.8) we get that there exist two constants $\beta_5 > 0$ and $\beta_6 > 0$ such that, for any $q \in (1, p)$,

$$E(t) \le \beta_5 e^{-\beta_6 (\ln(1+t))^q}.$$

Example 3 Let $g(t) = \mu e^{-(1+t)^p}$ with $p \in (0, 1)$ and $\mu > 0$ small enough so that (2.3) holds. For

$$G(t) = \int_0^t (-\ln s)^{1-\frac{1}{q}}\, ds,$$

when t is near zero, (2.5) holds with $q \in (1, \frac{p}{2})$. Then from (2.8) we get that there exist two constants $\beta_5 > 0$ and $\beta_6 > 0$ such that, for any $q \in (1, \frac{p}{2})$,

$$E(t) \le \beta_5 e^{-\beta_6 t^q}.$$

4 Conclusions

In this work, we consider a model to the longitudinal vibrations of a beam equation with thermoviscoelastic damping. Motivated by Dafermos [28], we introduce a new variable, and the system is transformed into a new system. We give the global existence of solutions without proof. The main result is the energy decay of solutions. Under suitable assumptions, we established a general decay result of energy for the initial value problem by using the energy perturbation method and some properties of convex functions. Finally, we give three examples, which can be found in Guesmia [31], to illustrate several rates of energy decay.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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