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Some optimal control problems of heat equations with weighted controls

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Abstract

In this paper, the time and norm optimal control problems of controlled heat equations with a weight function are considered. For the time optimal problems, we study the following two cases: one is for equations with multi-domain control under null controllability, and the other is for equations under approximate null controllability. We prove the solvability, and obtain the bang-bang principle of the time optimal controls for aforementioned both cases. For the norm optimal control problems, we focus on equations with multi-time and multi-domain control, and present the solvability of these problems.

MSC: 35K05; 49J20

Keywords: heat equation; weight function; optimal control problem; multi-domain

1 Introduction

Let T be a positive number and Ω be an open bounded domain with smooth boundary in $\mathbb{R}^N$, $N \geq 1$. Let $K \in \mathbb{Z}^+$, $\{E_i\} \equiv \{E_i\}_{i=1}^K$ be a sequence of Lebesgue measurable subsets of $(0,T)$ and $\{\omega_i\} \equiv \{\omega_i\}_{i=1}^K$ be a sequence of positive Lebesgue measurable subsets of Ω with $\omega_i \cap \omega_j = \emptyset$, for all $i,j \in \{1,2,\ldots,K\}$ and $i \neq j$. Denote by χ_{E_i}, χ_{ω_i} the characteristic function of E_i, ω_i, respectively, for each $i \in \{1,2,\ldots,K\}$. Consider the following controlled heat equation with a weight function:

$$\begin{cases} \partial_t y(x,t) - \Delta y(x,t) = \rho(x) \sum_{i=1}^K \chi_{E_i}(t)\chi_{\omega_i}(x)u_i(x,t), & \text{in } \Omega \times (0,T), \\ y(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\rho \in L^2(\Omega)$ is a weight function satisfying $0 < \rho(x) \leq 1$ for a.e. $x \in \Omega$, and $0 \neq y_0 \in L^2(\Omega)$ is a given function. We denote the solution to (1.1) by $y(\cdot,\cdot;\{\chi_{E_i}\chi_{\omega_i}u_i\},y_0)$. For simplicity, when $E_i = (0,T)$ for all $i \in \{1,2,\ldots,K\}$, we write $y(\cdot,\cdot;\{\chi_{\omega_i}u_i\},y_0)$ for $y(\cdot,\cdot;\{\chi_{E_i}\chi_{\omega_i}u_i\},y_0)$; furthermore, when $K=1$, write ω, $y(\cdot,\cdot;\chi_\omega u,y_0)$ for ω_i, $y(\cdot,\cdot;\{\chi_{\omega_i}u_i\}, y_0)$, respectively.

The weight function ρ in equation (1.1) is meaningful, which stands for the different influence of the control function in different location.

As is well known, optimization is one of the most important problems in control theory and there exist some work on this topic (see, *e.g.*, [1–3]). Roughly speaking, the goal of optimization is to improve a variable in order to maximize a benefit (or minimize a cost). The time and norm optimal control problems are important and interesting branches of optimization. For the deterministic systems, the reader can refer [4] to obtain recent results and find open problems. The reader can also refer [5–10] for controlled heat equations. For the stochastic ones, the norm optimal control problems were considered in [11, 12] for controlled stochastic ordinary differential equations, and in [13] for controlled stochastic heat equations.

In this paper, we shall consider the time and norm optimal control problems of heat equations with a weight function. In Section 2, we consider two kind time optimal control problems: one is for equations with multi-domain control under null controllability, and the other is for equations under approximate null controllability. We obtain the bang-bang principle of the time optimal controls for these two problems. In Section 3, we consider the norm optimal problems with multi-time and multi-domain control, and we obtain the solvability of these problems.

2 Time optimal control problems

In this section, we first state two time optimal control problems, and then study the solvability of these problems, obtain the bang-bang property of the time optimal controls. Throughout this section, for all $i \in \{1, 2, \dots, K\}$, $E_i = (0, T)$, and

$$u_i \in \mathcal{U}_{\mathrm{ad}}^i \equiv \{u \in L^\infty(0, +\infty; L^2(\Omega)) \mid \|u(t)\|_{L^2(\Omega)} \le M_i \text{ a.e. } t \in (0, +\infty)\}. \tag{2.1}$$

When $K = 1$, for simplicity, we write $\mathcal{U}_{\mathrm{ad}}$ for $\mathcal{U}_{\mathrm{ad}}^i$.

In the following, we consider the following two time optimal control problems subject to (1.1):

Problem (TP1)

$$T^* = \inf\{T \mid y(\cdot, T; \{\chi_{\omega_i} u_i\}, y_0) = 0, u_i \in \mathcal{U}_{\mathrm{ad}}^i \text{ for all } i \in \{1, 2, \dots, K\}\}.$$

Problem (TP2) For $K = 1$,

$$T_\varepsilon^* = \inf\{T \mid y(\cdot, T; \chi_\omega u, y_0) \in \bar{B}(0, \varepsilon), u \in \mathcal{U}_{\mathrm{ad}}\}.$$

Here and in what follows, we denote by $B(u, r)$ the open ball in $L^2(\Omega)$ with center $u \in L^2(\Omega)$ and radius $r > 0$, and by $\bar{B}(u, r)$ the closed ball in $L^2(\Omega)$ with center $u \in L^2(\Omega)$ and radius $r > 0$.

In order to obtain the solvability of Problem (TP1), we assume that there exists a constant $M > 0$ such that

$$M_i \le M \quad \text{for all } i \in \{1, 2, \dots, K\}. \tag{2.2}$$

Notice that the hypothesis (2.2) is reasonable: for a single control system

$$\begin{cases} \partial_t y(x,t) - \Delta y(x,t) = \rho(x)\chi_\omega(x)u(x,t), & \text{in } \Omega \times (0, +\infty), \\ y(x,t) = 0, & \text{on } \partial\Omega \times (0, +\infty), \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases}$$

its optimal time

$$T^* \equiv \inf\{T \mid y(\cdot, T; \chi_\omega u, y_0) = 0, u \in \mathcal{U}_{ad}\} \to 0$$

as $M \to \infty$.

It is obvious that Problem (TP1) is related to null controllable problem of (1.1), while Problem (TP2) is related to approximately controllable problem of (1.1). It is well known that, when $K = 1$ and $\rho \equiv 1$, the system (1.1) is null controllable for the measurable control domain ω (see [5]), even if the characteristic function χ_ω can be relaxed by a measurable function $\beta \in L^2(\Omega)$ with $0 \le \beta \le 1$ for a.e. $x \in \Omega$ and $\int_\Omega \beta^2(x)\,dx = \alpha|\Omega|$ (see [6]). Here $\alpha \in (0,1)$ is a given constant and $|\Omega|$ is the Lebesgue measure of Ω. It is natural that there exist a positive constant T and a control $u \in L^\infty(0,T;L^2(\Omega))$ such that $y(x,T;u,y_0) = 0$ (see [5, 6]).

The following result is related to the solvability of Problem (TP1).

Theorem 2.1 *Let $\{M_i\}$ be a given positive real number sequence satisfying* (2.2). *Then there exists $T^* > 0$, such that T^* is the solution to Problem* (TP1). *Moreover, for each $i = 1,2,\dots,K$, there exists a unique $u_i^* \in L^\infty(0,T^*;L^2(\Omega))$, such that*

$$\|u_i^*\|_{L^2(\Omega)} = M_i \quad \text{for a.e. } t \in (0,T^*) \text{ with } M_i \le M \text{ for all } i \in \{1,2,\dots,K\}, \tag{2.3}$$

i.e., the time optimal controls sequence of Problem (TP1) *has the bang-bang property.*

The following lemma is needed in proving Theorem 2.1, which comes from [5, 9].

Lemma 2.2 *Let $E \subset [0,T]$ and $\omega \subset \Omega$ be two positive measurable sets. Then, for each $y_0 \in L^2(\Omega)$, there is a bounded control function $u(\cdot) \in L^\infty(0,T;L^2(\Omega))$ with*

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} \le C\|y_0\|_{L^2(\Omega)},$$

such that the solution to the equation

$$\begin{cases} \partial_t y(x,t) - \Delta y(x,t) = \rho(x)\chi_E(t)\chi_\omega(x)u(x,t), & \text{in } \Omega \times (0,T), \\ y(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases}$$

satisfies $y(\cdot,T;u,y_0) = 0$. Here $C = C(\Omega,T,|E|,|\omega|)$ is a constant.

We are now in the position to prove Theorem 2.1.

Proof of Theorem 2.1 Since the proof is long, we separate it into two steps.

Step 1. For fixed $i_0 \in \{1,2,\dots,K\}$, consider the following system:

$$\begin{cases} \partial_t y(x,t) - \Delta y(x,t) = \rho(x)\chi_{\omega_{i_0}}(x)u_{i_0}(x,t), & \text{in } \Omega \times (0,+\infty), \\ y(x,t) = 0, & \text{on } \partial\Omega \times (0,+\infty), \\ y(x,0) = y_0(x), & \text{in } \Omega. \end{cases}$$

By Lemma 2.1 of [6], we know that there exist a control $u_{i_0} \in \mathcal{U}_{\text{ad}}^{i_0}$ and T, such that $y(\cdot, T, \chi_{\omega_{i_0}} u_{i_0}, y_0) = 0$. we also know that $0 \neq y_0 \in L^2(\Omega)$ is a given function. Therefore,

$$\begin{aligned} 0 < T^* &\equiv \inf\bigl\{T \mid y\bigl(\cdot, T; \{\chi_{\omega_i} u_i\}, y_0\bigr) = 0, u^i \in \mathcal{U}_{\text{ad}}^i \text{ for all } i \in \{1, 2, \dots, K\}\bigr\} \\ &\leq \inf\bigl\{T \mid y(\cdot, T; \chi_{\omega_{i_0}} u_{i_0}, y_0) = 0, u_{i_0} \in \mathcal{U}_{\text{ad}}^{i_0}\bigr\} < \infty. \end{aligned}$$

Hence, there exists a sequence $\{T_n\}$, such that $\{T_n\}$ is a monotone decreasing sequence with $y(\cdot, T_n; \{\chi_{\omega_i} u_i^n\}, y_0) = 0$ and

$$T_n \to T^* \equiv \inf\bigl\{T \mid y\bigl(\cdot, T; \{\chi_{\omega_i} u_i\}, y_0\bigr) = 0, u_i \in \mathcal{U}_{\text{ad}}^i \text{ for all } i \in \{1, 2, \dots, K\}\bigr\}.$$

Without loss of generality, we assume that $T_n \leq T^* + 1$ for all $n \in \mathbb{N}$. Then $y^n \equiv y(\cdot, \cdot; \{\chi_{\omega_i} u_i^n\}, y_0)$ is a solution to the following equation:

$$\begin{cases} \partial_t y^n(x,t) - \Delta y^n(x,t) = \rho(x) \sum_{i=1}^K \chi_{\omega_i}(x) u_i^n(x,t), & \text{in } \Omega \times (0, T_n), \\ y^n(x,t) = 0, & \text{on } \partial\Omega \times (0, T_n), \\ y^n(x,0) = y_0(x), y^n(x, T_n) = 0, & \text{in } \Omega. \end{cases}$$

Now, denote

$$\tilde{u}_i^n(x,t) = \begin{cases} u_i^n(x,t), & (x,t) \in \Omega \times (0, T_n), \\ 0, & (x,t) \in \Omega \times [T_n, +\infty). \end{cases}$$

Then

$$\tilde{y}^n(x,t) = \begin{cases} y^n(x,t), & (x,t) \in \Omega \times (0, T_n), \\ 0, & (x,t) \in \Omega \times [T_n, +\infty), \end{cases}$$

solves the following system:

$$\begin{cases} \partial_t \tilde{y}^n(x,t) - \Delta \tilde{y}^n(x,t) = \rho(x) \sum_{i=1}^K \chi_{\omega_i}(x) \tilde{u}_i^n(x,t), & \text{in } \Omega \times (0, +\infty), \\ \tilde{y}^n(x,t) = 0, & \text{on } \partial\Omega \times (0, +\infty), \\ \tilde{y}^n(x,0) = y_0(x), & \text{in } \Omega, \\ \tilde{y}_n(x,t) = 0, & \text{in } \Omega \times [T_n, +\infty). \end{cases}$$

Moreover, by the definition of $\tilde{y}^n$, it is easy to see that $\tilde{y}^n$ solves the following system:

$$\begin{cases} \partial_t \tilde{y}^n(x,t) - \Delta \tilde{y}^n(x,t) = \rho(x) \sum_{i=1}^K \chi_{\omega_i}(x) \tilde{u}_i^n(x,t), & \text{in } \Omega \times (0, T^* + 1), \\ \tilde{y}^n(x,t) = 0, & \text{on } \partial\Omega \times (0, T^* + 1), \\ \tilde{y}^n(x,0) = y_0(x), & \text{in } \Omega, \\ \tilde{y}_n(x,t) = 0, & \text{in } \Omega \times [T_n, T^* + 1). \end{cases}$$

Note that $\|\tilde{u}_1^n\|_{L^2(\Omega)} \le M_1$ for all $n \in \mathbb{N}$. Then there exist a subsequence $\{\tilde{u}_1^{n_1}\} \subset L^\infty(0, T^* + 1; L^2(\Omega))$ of $\{\tilde{u}_1^n\}$ and $\tilde{u}_1^0 \in L^\infty(0, T^* + 1; L^2(\Omega))$ such that

$$\tilde{u}_1^{n_1} \to \tilde{u}_1^0 \quad \text{weakly}^* \text{ in } L^\infty\big(0, T^* + 1; L^2(\Omega)\big) \text{ as } n_1 \to \infty.$$

Similarly, since $\|u_2^{n_1}\|_{L^2(\Omega)} \le M_2$ for all $n_1 \in \mathbb{N}$, there exist a subsequence $\{\tilde{u}_2^{n_2}\}$ of $\{\tilde{u}_2^{n_1}\}$ and $\tilde{u}_2^0 \in L^\infty(0, T^* + 1; L^2(\Omega))$ such that

$$\tilde{u}_2^{n_2} \to \tilde{u}_2^0 \quad \text{weakly}^* \text{ in } L^\infty\big(0, T^* + 1; L^2(\Omega)\big) \text{ as } n_2 \to \infty.$$

By inductive argument, for each $i \in \{1, 2, \dots, K\}$, there exist a subsequence $\{\tilde{u}_i^{n_i}\}$ of $\{\tilde{u}_i^{n_{i-1}}\}$ and $\tilde{u}_i^0 \in L^\infty(0, T^* + 1; L^2(\Omega))$ such that

$$\tilde{u}_i^{n_i} \to \tilde{u}_i^0 \quad \text{weakly}^* \text{ in } L^\infty\big(0, T^* + 1; L^2(\Omega)\big) \text{ as } n_i \to \infty.$$

By the diagram argument, for all $i \in \{1, 2, \dots, K\}$, we can abstract a subsequence $\{\tilde{u}_i^{n_n}\}$ of $\{\tilde{u}_i^n\}$ such that

$$\tilde{u}_i^{n_n} \to \tilde{u}_i^0 \quad \text{weakly}^* \text{ in } L^\infty\big(0, T^* + 1; L^2(\Omega)\big) \text{ as } n \to \infty.$$

Since $\omega_i \cap \omega_j = \emptyset$ for all $i, j \in \{1, 2, \dots, K\}$ with $i \neq j$, one can get

$$\rho(x) \sum_{i=1}^K \chi_{\omega_i} \tilde{u}_i^{n_n} \to \rho(x) \sum_{i=1}^K \chi_{\omega_i} \tilde{u}_i^0 \quad \text{weakly}^* \text{ in } L^\infty\big(0, T^* + 1; L^2(\Omega)\big) \text{ as } n \to \infty.$$

On the other hand, $\tilde{y}^{n_n}$ is the solution to the following system:

$$\begin{cases} \partial_t \tilde{y}^{n_n}(x,t) - \Delta \tilde{y}^{n_n}(x,t) = \rho(x) \sum_{i=1}^K \chi_{\omega_i}(x) \tilde{u}_i^{n_n}(x,t), & \text{in } \Omega \times (0, T^* + 1), \\ \tilde{y}^{n_n}(x,t) = 0, & \text{on } \partial\Omega \times (0, T^* + 1), \\ \tilde{y}^{n_n}(x,0) = y_0(x), & \text{in } \Omega, \\ \tilde{y}_{n_n}(x,t) = 0, & \text{in } \Omega \times [T_{n_n}, T^* + 1). \end{cases}$$

Then there exist a subsequence of $\{\tilde{y}^{n_n}\}$, still so denoted, and $\tilde{y}^0$ such that

$$\tilde{y}^{n_n} \to \tilde{y}^0 \quad \text{weakly in } L^2\big(\Omega \times \big(0, T^* + 1\big)\big) \text{ as } n \to \infty$$

and

$$\begin{cases} \partial_t \tilde{y}^0(x,t) - \Delta \tilde{y}^0(x,t) = \rho(x) \sum_{i=1}^K \chi_{\omega_i}(x) \tilde{u}_i^0(x,t), & \text{in } \Omega \times (0, T^* + 1), \\ \tilde{y}^0(x,t) = 0, & \text{on } \partial\Omega \times (0, T^* + 1), \\ \tilde{y}^0(x,0) = y_0(x), & \text{in } \Omega. \end{cases}$$

Note that $\tilde{y}^{n_n} = 0$ in $\Omega \times [T_{n_n}, T^* + 1)$. Since for $n \in \mathbb{N}$, one has $\tilde{y}^{n_n} = 0$ in $\Omega \times [T_{n_n}, T^* + 1)$,

$$\tilde{y}^{n_n} \to \tilde{y}^0 \quad \text{in } L^2\big(\Omega \times \big(0, T^* + 1\big)\big) \text{ as } n \to \infty.$$

We get $\tilde{y}^0 = 0$ in $\Omega \times [T_{n_n}, T^* + 1)$. We get $\tilde{y}^0 = 0$ in $[T^*, T^* + 1)$ since $T_{n_n} \to T^*$ as $n \to \infty$. Take

$$u_i^0 = \tilde{u}_i^0|_{\Omega\times(0,T^*)}, \quad \text{and} \quad y^0 = \tilde{y}^0|_{\Omega\times(0,T^*)}.$$

By the fact that $\tilde{y}^0 \in C((0,T];L^2(\Omega))$, y^0 is the solution to the following system:

$$\begin{cases} \partial_t y^0(x,t) - \Delta y^0(x,t) = \rho(x)\sum_{i=1}^K \chi_{\omega_i}(x)u_i^0(x,t), & \text{in } \Omega \times (0,T^*), \\ y^0(x,t) = 0, & \text{on } \partial\Omega \times (0,T^*), \\ y^0(x,0) = y_0(x), \qquad y^0(x,T^*) = 0, & \text{in } \Omega, \end{cases}$$

which implies that $\{u_i^0\}$ are the desired controls respect to the optimal time T^*.

Step 2. In the following, we shall show that the time optimal control of Problem (TP1) has the bang-bang property. Otherwise, we suppose that there exist $i_0 \in \{1,2,\dots,K\}$ and a subset $E^0 \subset [\alpha, T^* - \alpha]$ with positive measure for some $\alpha > 0$ and a positive number ε_0, such that

$$u_{i_0}^* \in \mathcal{U}_{\text{ad}}^{i_0} \quad \text{and} \quad M_{i_0} - \|u_{i_0}^*\|_{L^2(\Omega)} \geq \varepsilon_0, \quad \text{for each } t \text{ in the set } E^0,$$

where $u_{i_0}^*$ is the time optimal control respect to T^*. It is obvious that the solution to (1.1) satisfies

$$y(\cdot, T^*; \{\chi_{\omega_i}u_i^*\}, y_0) = 0,$$

and, for each $t \in E^0$, $B(u_{i_0}^*(t), \frac{\varepsilon_0}{2}) \subset B(0, M_{i_0})$.

We denote by $e^{\Delta t}$ the semigroup generated by Δ with the Dirichlet boundary condition. Set

$$h_\delta = \int_0^\delta e^{\Delta(\delta-\sigma)}\rho\sum_{i=1}^K \chi_{\omega_i}u_i^*(\sigma)\,d\sigma + (e^{\Delta\delta} - I)y_0. \tag{2.4}$$

Considering the following system:

$$\begin{cases} z_t^\delta(x,t) - \Delta z^\delta(x,t) = \rho(x)\chi_{E_\delta^0}(t)\chi_{\omega_{i_0}}(x)w_\delta(x,t), & \text{in } \Omega \times (0,T^*-\delta), \\ z(x,t) = 0, & \text{on } \partial\Omega \times (0,T^*-\delta), \\ z(x,0) = -h_\delta(x), & \text{in } \Omega, \end{cases} \tag{2.5}$$

where E_δ^0 is the set $\{t \mid t + \delta \in E^0\}$. By Lemma 2.2, there exist positive constants δ_1 and $L = L(\Omega, T, |E_0|, |\omega_{i_0}|)$, such that, for each δ with $0 < \delta \leq \delta_1$, there is a control w_δ in the space $L^\infty(0, T^* - \delta; L^2(\Omega))$ with the estimate

$$\|w_\delta\|_{L^\infty(0,T^*-\delta;L^2(\Omega))} \leq L\|h_\delta\|_{L^2(\Omega)},$$

and the solution to (2.5) satisfies

$$z^\delta(\cdot, T^* - \delta; \omega_\delta, -h_\delta) = 0. \tag{2.6}$$

On the other hand, by (2.4), there exists a positive number δ_2 such that, for each positive number δ with $\delta \leq \delta_2$, one has

$$\|h_\delta\|_{L^2(\Omega)} \leq \frac{\varepsilon_0}{2L}.$$

Therefore, for each $\delta \leq \delta_0 \equiv \min\{\delta_1, \delta_2\}$, there exists a control w_δ satisfying

$$\|w_\delta\|_{L^\infty(0,T^*-\delta;L^2(\Omega))} \leq \frac{\varepsilon_0}{2}, \tag{2.7}$$

and the corresponding solution to (2.5) satisfies (2.6).

Set

$$v_i(x,t) = \begin{cases} u_i^*(x,\delta+t) + w_\delta(x,t), & \text{for } i = i_0 \text{ and } (x,t) \in \Omega \times E_\delta^0, \\ u_i^*(x,\delta+t), & \text{for } i = i_0 \text{ and } (x,t) \in \Omega \times ((0,T^*-\delta) - E_\delta^0), \\ u_i^*(x,\delta+t), & \text{for } i \neq i_0 \text{ and } (x,t) \in \Omega \times (0,T^*-\delta), \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that $v_i \in \mathcal{U}_{\text{ad}}^i$ for all $i \in \{1,2,\dots,K\}$. Consider the following system:

$$\begin{cases} y_t(x,t) - \Delta y(x,t) = \rho(x) \sum_{i=1}^K \chi_{\omega_i}(x) v_i(x,t), & \text{in } \Omega \times (0,+\infty), \\ y(x,t) = 0, & \text{on } \partial\Omega \times (0,+\infty), \\ y(x,0) = y_0(x), & \text{in } \Omega. \end{cases} \tag{2.8}$$

For any $0 < \delta < \min\{\frac{T^*}{2}, \delta_0\}$, it is easy to check that the solution to above equation satisfies

$$\begin{aligned} y(\cdot, T^*-\delta; \{v_i\}, y_0) &= e^{\Delta(T^*-\delta)} y_0 + \rho(x) \sum_{i=1}^K \int_0^{T^*-\delta} e^{\Delta(T^*-\delta-\sigma)} \chi_{\omega_i} v_i(\sigma)\, d\sigma \\ &= e^{\Delta(T^*-\delta)} y_0 + \sum_{i=1}^K \int_0^{T^*-\delta} e^{\Delta(T^*-\delta-\sigma)} \rho(x) \chi_{\omega_i} u_i^*(\delta+\sigma)\, d\sigma \\ &\quad + \int_0^{T^*-\delta} e^{\Delta(T^*-\delta-\sigma)} \rho(x) \chi_{E_\delta^0} \chi_{\omega_{i_0}} w_\delta(\sigma)\, d\sigma \\ &= e^{\Delta(T^*-\delta)} y_0 + \sum_{i=1}^K \int_\delta^{T^*} e^{\Delta(T^*-\sigma)} \rho(x) \chi_{\omega_i} u_i^*(\sigma)\, d\sigma + e^{\Delta(T^*-\delta)} h_\delta \\ &= e^{\Delta(T^*-\delta)} y_0 + \sum_{i=1}^K \int_\delta^{T^*} e^{\Delta(T^*-\sigma)} \rho(x) \chi_{\omega_i} u_i^*(\sigma)\, d\sigma \\ &\quad + \int_0^\delta e^{\Delta(T^*-\sigma)} \rho(x) \sum_{i=1}^K \chi_{\omega_i} u_i^*(\sigma)\, d\sigma + e^{\Delta T^*} y_0 - e^{\Delta(T^*-\delta)} y_0 \\ &= e^{\Delta T^*} y_0 + \int_0^{T^*} e^{\Delta(T^*-\sigma)} \rho(x) \sum_{i=1}^K \chi_{\omega_i} u_i^*(\sigma)\, d\sigma \\ &= 0, \end{aligned} \tag{2.9}$$

which shows that $\{v_i\}$ are the desired controls such that $y(\cdot, T^* - \delta; \{\chi_{\omega_i} v_i\}, y_0) = 0$. It contradicts the definition of T^* and we have proved the theorem. □

Remark 2.3 There are some relations between the optimal control problem of (1.1) and shape design problem. For more about shape design problem see [8, 14–16].

Before stating the results on Problem (TP2), we define the following reachable set:

$$\mathcal{R}(T) = \{y(\cdot, T; \chi_\omega u, y_0) \mid u \in \mathcal{U}_{\text{ad}}\} \tag{2.10}$$

for each $T \in (0, +\infty)$.

Theorem 2.4 *For any given positive constant* ε, *Problem* (TP2) *has a solution* T_ε^*, *and* $\mathcal{R}(T_\varepsilon^*) \cap \bar{B}(0, \varepsilon)$ *has only one point belonging to the boundary of* $B(0, \varepsilon)$. *Moreover, the corresponding time optimal control* u_ε^* *is unique and has the bang-bang property.*

Proof Since the proof is long, we separate it into the following several steps.

Step 1. We shall show that there exists at least one time optimal control, *i.e.*, there exists at least one $u^* \in \mathcal{U}_{\text{ad}}$ such that $y(\cdot, T_\varepsilon^*; u^*, y_0) \in \bar{B}(0, \varepsilon)$.

Let $\{T_n\}$ be a monotone decreasing sequence such that $T_n \to T_\varepsilon^*$ as $n \to +\infty$, then there exists a sequence $\{u_n\} \subset \mathcal{U}_{\text{ad}}$ such that $y(\cdot, T_n; \chi_\omega u_n, y_0) \in \bar{B}(0, \varepsilon)$. Set

$$\tilde{u}_n(t) = \begin{cases} u_n(t), & t \in (0, T_n), \\ 0, & t \in [T_n, T_1). \end{cases}$$

Since $M \in L^\infty(0, T_1; L^2(\Omega))$, $\{\tilde{u}_n\}$ is a bounded sequence in $L^\infty(0, T_1; L^2(\Omega))$. Then there exist $\tilde{u}^* \in L^\infty(0, T_1; L^2(\Omega))$ and a subsequence of $\{u_n\}$, still so denoted, such that $u_n \to u^*$ weakly* in $L^\infty(0, T_1; L^2(\Omega))$. Moreover,

$$\rho \chi_\omega \tilde{u}_n \to \rho \chi_\omega \tilde{u}^* \quad \text{weakly* in } L^\infty(0, T_1; L^2(\Omega)) \text{ as } n \to \infty.$$

Therefore the solution $y_n(\cdot, \cdot; \chi_\omega \tilde{u}_n, y_0)$ to the following system:

$$\begin{cases} \partial_t y_n(x,t) - \Delta y_n(x,t) = \rho(x)\chi_\omega(x)\tilde{u}_n(x,t), & \text{in } \Omega \times (0, T_1), \\ y_n(x,t) = 0, & \text{on } \partial\Omega \times (0, T_1), \\ y_n(x,0) = y_0(x), & \text{in } \Omega, \end{cases}$$

satisfies $y_n(\cdot, t; \chi_\omega \tilde{u}_n, y_0) \in \bar{B}(0, \varepsilon)$ for $t \geq T_n$. Denote by y^* the solution to the following system:

$$\begin{cases} \partial_t y^*(x,t) - \Delta y^*(x,t) = \rho(x)\chi_\omega(x)\tilde{u}^*(x,t), & \text{in } \Omega \times (0, T_1), \\ y^*(x,t) = 0, & \text{on } \partial\Omega \times (0, T_1), \\ y^*(x,0) = y_0(x), & \text{in } \Omega. \end{cases}$$

Then

$$y_n \to y^* \quad \text{weakly in } L^2\big(0,T;H_0^1(\Omega)\big) \cap H^1\big(0,T;L^2(\Omega)\big),$$
$$\text{strongly in } C\big([0,T_1];L^2(\Omega)\big) \text{ as } n \to \infty,$$

for any $\delta > 0$. Since $y_n(\cdot,t;\chi_\omega \tilde{u}^n, y_0) \in \bar{B}(0,\varepsilon)$ for $t \ge T_n$, $y^*(\cdot,t;\chi_\omega \tilde{u}^*, y_0) \in \bar{B}(0,\varepsilon)$ for all $t \ge T_n$ and $n \in \mathbb{N}$. Hence $y^*(\cdot,T_\varepsilon^*;\chi_\omega \tilde{u}^*, y_0) \in \bar{B}(0,\varepsilon)$. Set

$$u^*(t) = \tilde{u}^*(t), \quad t \in \big[0, T_\varepsilon^*\big].$$

Then y^* satisfies the following system:

$$\begin{cases} \partial_t y^*(x,t) - \Delta y^*(x,t) = \rho(x)\chi_\omega(x)u^*(x,t), & \text{in } \Omega \times (0,T_1), \\ y^*(x,t) = 0, & \text{on } \partial\Omega \times (0,T_1), \\ y^*(x,0) = y_0(x), & \text{in } \Omega, \end{cases}$$

and $y^*(\cdot,T_\varepsilon^*;\chi_\omega u^*, y_0) \in \bar{B}(0,\varepsilon)$.

Claim: $\|u^*(t)\|_{L^2(\Omega)} \le M(t)$ for a.e. $t \in [0,T_\varepsilon^*]$. Indeed, let $\{\zeta_k\}_{k\in\mathbb{N}}$ be the countable density subset of $L^2(\Omega)$. Denote by $\mathcal{L}$ the Lebesgue point of $\langle u_n(t),\zeta_k\rangle$, $t \in [0,T_\varepsilon^*]$, where $\langle\cdot,\cdot\rangle$ is the inner product of $u_n(t)$ and ζ_k in $L^2(\Omega)$. Since $\langle u_n(t),\zeta_k\rangle, \langle u^*(t),\zeta_k\rangle \in L^\infty(0,T_\varepsilon^*)$, for each

$$t_0 \in E_0 \equiv \bigcap_{n,k=1}^{\infty} \mathcal{L}\big(\langle u_n,\zeta_k\rangle\big) \cap \bigcap_{k=1}^{\infty} \mathcal{L}\big(\langle u^*,\zeta_k\rangle\big),$$

we have

$$\lim_{\delta\to 0} \frac{1}{2\delta} \int_{t_0-\delta}^{t_0+\delta} \big\langle u_n(t),\zeta_k\big\rangle\,dt = \big\langle u_n(t_0),\zeta_k\big\rangle,$$
$$\lim_{\delta\to 0} \frac{1}{2\delta} \int_{t_0-\delta}^{t_0+\delta} \big\langle u^*(t),\zeta_k\big\rangle\,dt = \big\langle u^*(t_0),\zeta_k\big\rangle.$$

By virtue of

$$\frac{1}{2\delta} \int_{t_0-\delta}^{t_0+\delta} \big\langle u_n(t),\zeta_k\big\rangle\,dt \to \frac{1}{2\delta} \int_{t_0-\delta}^{t_0+\delta} \big\langle u^*(t),\zeta_k\big\rangle\,dt \quad \text{as } n \to \infty,$$

$u_n \to u^*$ weakly* in $L^\infty(0,T_\varepsilon^*;L^2(\Omega))$, and the arbitrary of $\delta > 0$, we get

$$\big\langle u_n(t_0),\zeta_k\big\rangle = \lim_{\delta\to 0} \frac{1}{2\delta} \int_{t_0-\delta}^{t_0+\delta} \big\langle u_n(t),\zeta_k\big\rangle\,dt \quad \to$$
$$\lim_{\delta\to 0} \frac{1}{2\delta} \int_{t_0-\delta}^{t_0+\delta} \big\langle u^*(t),\zeta_k\big\rangle\,dt = \big\langle u^*(t_0),\zeta_k\big\rangle \quad \text{as } n \to \infty.$$

Since $\{\zeta_k\}$ is dense in $L^2(\Omega)$, we have

$$\big\langle u_n(t_0),\zeta\big\rangle \to \big\langle u^*(t_0),\zeta\big\rangle \quad \text{as } n \to \infty$$

for each $\zeta \in L^2(\Omega)$. That implies $u_n(t_0) \to u^*(t_0)$ weakly in $L^2(\Omega)$, and hence $\|u^*(t_0)\|_{L^2(\Omega)} \leq \liminf_{n\to\infty} \|u_n(t_0)\|_{L^2(\Omega)} \leq M$. Applying

$$|E_0| = T_\varepsilon^*,$$

we obtain $\|u^*(t)\|_{L^2(\Omega)} \leq M$ for a.e. $t \in [0, T_\varepsilon^*]$. That proves the claim.

Step 2. We show that $\mathcal{R}(T_\varepsilon^*) \cap \bar{B}(0,\varepsilon)$ has only one point. If so, it is obvious that this point belongs to the boundary of $B(0,\varepsilon)$.

In Step 1 we get $\mathcal{R}(T_\varepsilon^*) \cap \bar{B}(0,\varepsilon) \neq \emptyset$. By contradiction, we assume that $\mathcal{R}(T_\varepsilon^*) \cap \bar{B}(0,\varepsilon)$ has at least two points, *i.e.*, there exist $y_1 \equiv y(\cdot, T_\varepsilon^*; \chi_\omega u_1^*, y_0)$, $y_2 \equiv y(\cdot, T_\varepsilon^*; \chi_\omega u_2^*, y_0) \in \mathcal{R}(T_\varepsilon^*) \cap \bar{B}(0,\varepsilon)$ with $y_1 \neq y_2$. It is obvious that $u_1^* \neq u_2^*$ in $\mathcal{U}_{\mathrm{ad}}$. Define

$$\hat{u} \equiv \frac{u_1^* + u_2^*}{2}.$$

Since $\hat{u} \in \mathcal{U}_{\mathrm{ad}}$, and $B(0,\varepsilon)$ is strongly convex in $L^2(\Omega)$, we get $\hat{y} \equiv y(\cdot, T_\varepsilon^*; \chi_\omega \hat{u}, y_0) = \frac{1}{2}y_1 + \frac{1}{2}y_2$ is an inner point of $B(0,\varepsilon)$, *i.e.*, there exists $\gamma > 0$ such that $B(\hat{y}, \gamma) \subset B(0,\varepsilon)$.

For any $\xi > 0$, define

$$h_\xi \equiv \hat{y} - y\bigl(\cdot, T_\varepsilon^* - \xi; \chi_\omega \hat{u}, y_0\bigr).$$

It is easy to check that

$$\begin{aligned} h_\xi = {}& e^{\Delta(T_\varepsilon^*-\xi)}\bigl[e^{\Delta\xi} - I\bigr]y_0 + \int_0^{T_\varepsilon^*-\xi} \bigl[e^{\Delta\xi} - I\bigr] e^{\Delta(T_\varepsilon^*-\xi-\sigma)} \rho\chi_\omega \hat{u}(\sigma)\,d\sigma \\ & + \int_{T_\varepsilon^*-\xi}^{T_\varepsilon^*} e^{\Delta(T_\varepsilon^*-\sigma)} \rho\chi_\omega \hat{u}(\sigma)\,d\sigma. \end{aligned}$$

Hence, ξ can be chosen small enough such that $\|h_\xi\|_{L^2(\Omega)} \leq \gamma$. Therefore, we can get $y(\cdot, T_\varepsilon^* - \xi; \chi_\omega \hat{u}, y_0) \in \bar{B}(0,\varepsilon)$. This contradicts the optimal time T_ε^*. Subsequently, the set $\mathcal{R}(T_\varepsilon^*) \cap \bar{B}(0,\varepsilon)$ has only one point, and this point belongs to the boundary of $B(0,\varepsilon)$.

Step 3. The time optimal control u^* has the bang-bang property.

Since $\mathcal{R}(T_\varepsilon^*) \cap \bar{B}(0,\varepsilon)$ has only one point (denote this point by $y^* = y(\cdot, T_\varepsilon^*; \chi_\omega u^*, y_0)$), and $\mathcal{R}(T_\varepsilon^*)$ and $\bar{B}(0,\varepsilon)$ are two convex sets, by hyperplane separation theorem, there exists $\eta^* \in L^2(\Omega)$ such that

$$\sup_{y\in\mathcal{R}(T_\varepsilon^*)} \langle y, \eta^* \rangle \leq \inf_{z\in\bar{B}(0,\varepsilon)} \langle z, \eta^* \rangle \leq \langle y^*, \eta^* \rangle. \tag{2.11}$$

Notice that $y \in \mathcal{R}(T_\varepsilon^*)$ can be written by

$$y\bigl(\cdot, T_\varepsilon^*; \chi_\omega u, y_0\bigr) = e^{\Delta T_\varepsilon^*} y_0 + \int_0^{T_\varepsilon^*} e^{\Delta(T_\varepsilon^*-\sigma)} \rho\chi_\omega u(\sigma)\,d\sigma.$$

Then (2.11) can be written as

$$\sup_{\bar{u}\in\mathcal{U}_1} \int_0^{T_\varepsilon^*} \bigl\langle e^{\Delta(T_\varepsilon^*-\sigma)} \rho\chi_\omega M\bar{u}(\sigma), \eta^* \bigr\rangle\,d\sigma \leq \int_0^{T_\varepsilon^*} \bigl\langle e^{\Delta(T_\varepsilon^*-\sigma)} \rho\chi_\omega M\bar{u}^*(\sigma), \eta^* \bigr\rangle\,d\sigma.$$

Here

$$\bar{u}^* \in \mathcal{U}_1 \equiv \{\bar{u} \in L^\infty(0, T_\varepsilon^*; L^2(\Omega)) \mid \|\bar{u}(t)\|_{L^2(\Omega)} \leq 1 \text{ for a.e. } t \in [0, T_\varepsilon^*]\},$$

and

$$u^*(t) = M\bar{u}^*(t) \quad \text{for all } t \in [0, T_\varepsilon^*]. \tag{2.12}$$

Hence, we have

$$\sup_{\bar{u}\in\mathcal{U}_1} \int_0^{T_\varepsilon^*} \langle \bar{u}(\sigma), e^{\Delta(T_\varepsilon^*-\sigma)}\rho\chi_\omega M\eta^*\rangle \, d\sigma \leq \int_0^{T_\varepsilon^*} \langle \bar{u}^*(\sigma), e^{\Delta(T_\varepsilon^*-\sigma)}\rho\chi_\omega M\eta^*\rangle \, d\sigma.$$

For given $t_0 \in E_0$, choosing

$$\bar{u}(t) = \begin{cases} \bar{u}^*(t), & \text{for } t \in (0, T_\varepsilon^*) \setminus (t_0 - \lambda, t_0 + \lambda), \\ \zeta, & \text{for } t \in (t_0 - \lambda, t_0 + \lambda) \subset (0, T_\varepsilon^*), \end{cases}$$

where $\zeta \in L^2(\Omega)$, we get

$$\sup_{\zeta\in L^2(\Omega)} \langle \zeta, e^{\Delta(T_\varepsilon^*-t_0)}\rho\chi_\omega M\eta^*\rangle \leq \langle \bar{u}^*(t_0), e^{\Delta(T_\varepsilon^*-t_0)}\rho\chi_\omega M\eta^*\rangle,$$

i.e.,

$$\begin{aligned} \|e^{\Delta(T_\varepsilon^*-t_0)}\rho\chi_\omega\eta^*\|_{L^2(\Omega)} &= \sup_{\zeta\in L^2(\Omega)} \langle \zeta, e^{\Delta(T_\varepsilon^*-t_0)}\rho\chi_\omega\eta^*\rangle \\ &\leq \langle \bar{u}^*(t), e^{\Delta(T_\varepsilon^*-t_0)}\rho\chi_\omega\eta^*\rangle \\ &\leq \|\bar{u}^*(t)\|_{L^2(\Omega)} \|e^{\Delta(T_\varepsilon^*-t_0)}\rho\chi_\omega\eta^*\|_{L^2(\Omega)}. \end{aligned}$$

This implies that

$$\|\bar{u}^*(t_0)\|_{L^2(\Omega)} = 1 \tag{2.13}$$

by $\bar{u}^* \in \mathcal{U}_1$. Equation (2.13), together with (2.12) and $|E_0| = T_\varepsilon^*$, yields

$$\|u^*(t)\|_{L^2(\Omega)} = M \quad \text{for a.e. } t \in [0, T_\varepsilon^*].$$

From the above, we get the time optimal control's bang-bang property. That completes the proof. □

3 Norm optimal control problems with multi-time and multi-domain controls

In this section, let $T \in \mathbb{R}^+$, $K \in \mathbb{Z}^+$ be given finite constants, and Π^K be time partition of $[0, T]$ defined by

$$\Pi_K : 0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_K = T. \tag{3.1}$$

For any $i \in \{1, 2, \dots, K\}$, set $I_i = (t_{i-1}, t_i]$. Taking $E_i = I_i$, we can rewrite (1.1) as

$$\begin{cases} \partial_t y(x,t) - \Delta y(x,t) = \rho(x) \sum_{i=1}^{K} \chi_{I_i}(t)\chi_{\omega_i}(x)u_i(x,t), & \text{in } \Omega \times (0,T), \\ y(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ y(x,0) = y_0(x), & \text{in } \Omega. \end{cases} \tag{3.2}$$

It is obvious that the system is null controllable (see [5, 17]). For any given partition Π^K, by standard minimizing sequence method, there exists a solution to the following norm optimal control problem:

$$N(\Pi_K) \equiv \inf\left\{ \sum_{i=1}^{K} \|\rho\chi_{I_i}\chi_{\omega_i}u_i\|_{L^\infty(0,T;L^2(\Omega))} \,\middle|\, y\big(\cdot, T; \{\chi_{I_i}\chi_{\omega_i}u_i\}, y_0\big) = 0, \right.$$
$$\left. \{u_i\}_{i=1}^{K} \subset L^\infty\big(0,T;L^2(\Omega)\big) \right\}. \tag{3.3}$$

We are interested in the partition's existence of the following norm optimal control problem:

Problem (NP)

$$N_K^* \equiv \inf\big\{N(\Pi_K) \mid \Pi_K \text{ is defined by partition (3.1)}\big\}.$$

We have the following solvability result on Problem (NP).

Theorem 3.1 *For any $K > 1$, there exists at least one solution to Problem* (NP).

Proof It is obviously that $N_K^* < \infty$. Let $\{\Pi_K^n\}$ be the partition sequence such that

$$N\big(\Pi_K^n\big) \to N_K^* \quad \text{as } n \to \infty.$$

Then there exists a control sequence $\{u_1^n, u_2^n, \dots, u_K^n\}_{n=1}^{\infty}$, such that

$$\begin{cases} \partial_t y^n(x,t) - \Delta y^n(x,t) = \rho(x) \sum_{i=1}^{K} \chi_{I_i^n}(t)\chi_{\omega_i}(x)u_i^n(x,t), & \text{in } \Omega \times (0,T), \\ y^n(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ y^n(x,0) = y_0(x), y^n(x,T) = 0, & \text{in } \Omega, \end{cases}$$

with

$$\left\| \rho \sum_{i=1}^{K} \chi_{I_i^n}\chi_{\omega_i}u_i^n \right\|_{L^\infty(0,T;L^2(\Omega))} = N\big(\Pi_K^n\big) \quad \text{for all } n \in \mathbb{N}.$$

Since $\{t_1^n\} \subset [0,T]$ is a bounded sequence, there exist a subsequence of $\{t_1^n\}$, still so denoted, and t_1^0 such that $t_1^n \to t_1^0$ as $n \to \infty$. Also, there exist a subsequence of $\{t_2^n\}$, still so denoted, and t_2^0 satisfying $t_2^n \to t_2^0$ as $n \to \infty$. Moreover, since $t_1^n \le t_2^n$ for all $n \in \mathbb{N}$, we have $t_1^0 \le t_2^0$. By the same line, we can get $\{t_i^0\}_{i=1}^{K}$ satisfying

$$0 = t_0^0 \le t_1^0 \le t_2^0 \le \cdots \le t_K^0 = T. \tag{3.4}$$

Denote the above partition by Π_K^0. We also can define I_i^0 for each $i \in \{1, 2, \dots, K\}$. Naturally, one has

$$\left|I_i^n \triangle I_i^0\right| \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

Here and in what follows, we define $I_i^n \triangle I_i^0 \equiv (I_i^n - I_i^0) \cup (I_i^0 - I_i^n)$ for all i, n.

On the other hand, since $\{u_1^n, u_2^n, \dots, u_K^n\}_{n=1}^\infty$ is bounded, $\{u_i^n\}_{n=1}^\infty$ is also a bounded sequence in $L^\infty(0, T; L^2(\Omega))$ for all $i = 1, \dots, K$. Then, for each $i = 1, \dots, K$, there exist a subsequence of $\{u_i^n\}_{n=1}^\infty$, still so denoted, and $u_i^0 \in L^\infty(0, T; L^2(\Omega))$ such that

$$u_i^n \to u_i^0 \quad \text{weakly}^* \text{ in } L^\infty\bigl(0, T; L^2(\Omega)\bigr) \text{ as } n \to \infty. \tag{3.6}$$

For each $v \in L^1(0, T; L^2(\Omega))$, we have

$$\begin{aligned}
&\int_0^T \int_\Omega \rho\bigl(\chi_{I_i^n}\chi_{\omega_i}u_i^n - \chi_{I_i^0}\chi_{\omega_i}u_i^0\bigr)v\,dx\,dt \\
&\quad = \int_0^T \int_\Omega \rho\bigl(\chi_{I_i^n}\chi_{\omega_i}u_i^n - \chi_{I_i^0}\chi_{\omega_i}u_i^n\bigr)v\,dx\,dt \\
&\qquad + \int_0^T \int_\Omega \rho\bigl(\chi_{I_i^0}\chi_{\omega_i}u_i^n - \chi_{I_i^0}\chi_{\omega_i}u_i^0\bigr)v\,dx\,dt \\
&\quad = \int_0^T \int_\Omega \rho(\chi_{I_i^n} - \chi_{I_i^0})\chi_{\omega_i}u_i^n v\,dx\,dt \\
&\qquad + \int_0^T \int_\Omega \rho\bigl(u_i^n - u_i^0\bigr)\chi_{I_i^0}\chi_{\omega_i}v\,dx\,dt.
\end{aligned} \tag{3.7}$$

By (3.6), we get

$$\int_0^T \int_\Omega \bigl(u_i^n - u_i^0\bigr)\rho\chi_{I_i^0}\chi_{\omega_i}v\,dx\,dt \to 0 \quad \text{as } n \to \infty. \tag{3.8}$$

By (3.5) and the absolutely continuity of $v \in L^1(0, T; L^2(\Omega))$, we have

$$\begin{aligned}
&\left|\int_0^T \int_\Omega \rho(\chi_{I_i^n} - \chi_{I_i^0})\chi_{\omega_i}u_i^n v\,dx\,dt\right| \\
&\quad \le \|u_i^n\|_{L^\infty(0,T;L^2(\Omega))}\|\rho(\chi_{I_i^n} - \chi_{I_i^0})\chi_{\omega_i}v\|_{L^1(0,T;L^2(\Omega))} \\
&\quad = \|u_i^n\|_{L^\infty(0,T;L^2(\Omega))}\int_{I_i^n \triangle I_i^0}\left(\int_\Omega |\rho\chi_{\omega_i}v|^2\,dx\right)^{\frac{1}{2}} dt \\
&\quad \le \|u_i^n\|_{L^\infty(0,T;L^2(\Omega))}\int_{I_i^n \triangle I_i^0}\left(\int_\Omega |v|^2\,dx\right)^{\frac{1}{2}} dt \\
&\quad \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{3.9}$$

Equation (3.7), together with (3.8) and (3.9), yields

$$\int_0^T \int_\Omega \rho\bigl(\chi_{I_i^n}\chi_{\omega_i}u_i^n - \chi_{I_i^0}\chi_{\omega_i}u_i^0\bigr)v\,dx\,dt \to 0 \quad \text{as } n \to \infty.$$

That implies, for all $i \in \{1, \ldots, K\}$,

$$\rho \chi_{I_i^n} \chi_{\omega_i} u_i^n \to \rho \chi_{I_i^0} \chi_{\omega_i} u_i^0 \quad \text{weakly}^* \text{ in } L^\infty\big(0, T; L^2(\Omega)\big) \text{ as } n \to \infty.$$

Therefore, one gets

$$\rho \sum_{i=1}^{K} \chi_{I_i^n} \chi_{\omega_i} u_i^n \to \rho \sum_{i=1}^{K} \chi_{I_i^0} \chi_{\omega_i} u_i^0 \quad \text{weakly}^* \text{ in } L^\infty\big(0, T; L^2(\Omega)\big) \text{ as } n \to \infty \tag{3.10}$$

and

$$\begin{aligned} &\left\| \rho \sum_{i=1}^{K} \chi_{I_i^0} \chi_{\omega_i} u_i^0 \right\|_{L^\infty(0,T;L^2(\Omega))} \\ &\quad \le \liminf_{n\to\infty} \left\| \rho \sum_{i=1}^{K} \chi_{I_i^n} \chi_{\omega_i} u_i^n \right\|_{L^\infty(0,T;L^2(\Omega))} \\ &\quad = N_K^*. \end{aligned} \tag{3.11}$$

Let y^0 be the solution to the following system:

$$\begin{cases} \partial_t y^0(x,t) - \Delta y^0(x,t) = \rho(x) \sum_{i=1}^{K} \chi_{I_i^0}(t) \chi_{\omega_i}(x) u_i^0(x,t), & \text{in } \Omega \times (0,T), \\ y^0(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ y^0(x,0) = y_0(x), & \text{in } \Omega. \end{cases} \tag{3.12}$$

By (3.10) we get

$$\begin{aligned} y^n \to y^0 \quad & \text{weakly in } L^2\big(0,T;H_0^1(\Omega)\big) \cap H^1\big(0,T;L^2(\Omega)\big), \\ & \text{strongly in } C\big([\delta, T]; L^2(\Omega)\big), \end{aligned}$$

for every $\delta > 0$. Since $y^n(T) = 0$, immediately we have

$$y^0(T) = 0. \tag{3.13}$$

Combining (3.12), (3.13) and (3.11), we complete the proof. □

Now, let us consider an alteration of system (3.2):

$$\begin{cases} \partial_t y(x,t) - \Delta y(x,t) = \rho(x) \sum_{i=1}^{K} \chi_{I_i}(t) \chi_{\omega_{\sigma(i)}}(x) u_i(x,t), & \text{in } \Omega \times (0,T), \\ y(x,t) = 0, & \text{on } \partial\Omega \times (0,T), \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases}$$

where

$$\sigma : \{1, \ldots, K\} \to \{1, \ldots, K\} \quad \text{is a map.} \tag{3.14}$$

By the aforementioned discussion, this system is null controllable, and for given partition Π_K and map σ, there exists at least a solution to the following norm optimal control problem:

$$N(\sigma) \equiv \inf\left\{ \sum_{i=1}^{K} \|\rho \chi_{I_i} \chi_{\omega_{\sigma(i)}} u_i\|_{L^\infty(0,T;L^2(\Omega))} \;\middle|\; y\big(\cdot, T; \{\chi_{I_i} \chi_{\omega_{\sigma(i)}} u_i\}, y_0\big) = 0, \right.$$
$$\left. \{u_i\}_{i=1}^{K} \subset L^\infty\big(0, T; L^2(\Omega)\big) \right\}.$$

Let us consider the solvability of the following norm optimal control problem:

$$N \equiv \inf\big\{N(\sigma) \mid \sigma \text{ is defined by (3.14)}\big\}. \tag{3.15}$$

Since the number of σ is finite, it is obvious that there exists σ^* satisfying $N^* = N(\sigma^*)$. Hence, we obtain the following result.

Proposition 3.2 *Let $\{I_i\}_{i=1}^{K}$, $\{\omega_k\}_{k=1}^{K}$ be defined as before. Then there exists at least a solution to the problem* (3.15).

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