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Perturbed nonlocal fourth order equations of Kirchhoff type with Navier boundary conditions

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Abstract

We investigate the existence of multiple solutions for perturbed nonlocal fourth-order equations of Kirchhoff type under Navier boundary conditions. We give some new criteria for guaranteeing that the perturbed fourth-order equations of Kirchhoff type have at least three weak solutions by using a variational method and some critical point theorems due to Ricceri. We extend and improve some recent results. Finally, by presenting two examples, we ensure the applicability of our results.

MSC: 35J20; 35J40; 35J35; 31B30; 35G30; 74H20; 35Q35

Keywords: three solutions; Kirchhoff-type equation; perturbed fourth-order boundary value problem; variational methods; critical point theory

1 Introduction

In this paper we study the following perturbed nonlocal fourth-order problem of Kirchhoff type under Navier boundary condition:

$$\begin{cases} T(u) = \lambda f(x,u) + \mu g(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{$P_{\lambda,\mu}^{f,g}$}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded smooth domain,

$$T(u) = \Delta\bigl(|\Delta u|^{p-2}\Delta u\bigr) - \left[M\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right)\right]^{p-1} \Delta_p u + \varrho |u|^{p-2}u,$$

in which $p > \max\{1, \frac{N}{2}\}$, $\varrho > 0$ and $M : [0, +\infty[\to \mathbb{R}$ is a continuous function such that there are two positive constants m_0 and m_1 with $m_0 \leq M(t) \leq m_1$ for all $t \geq 0$, and $\lambda > 0$, $\mu \geq 0$ and $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions.

The problem $(P_{\lambda,\mu}^{f,g})$ is related to the stationary problem of a model introduced by Kirchhoff [1]. More precisely, Kirchhoff introduced a model given by the following equation:

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where ρ is the mass density, ρ_0 is the initial tension, h is the area of the cross section, E is the Young modulus of the material and L is the length of the string, which extends the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Nonlinear Kirchhoff model can also be used for describing the dynamics of an axially moving string. In recent years, axially moving string-like continua such as wires, belts, chains and band-saws, have been subjects of the study of researchers (see [2]).

The problem (1.1) was developed into the form

$$u_{tt} - M\left(\int_\Omega |\nabla u|^2\, \mathrm{d}x\right)\Delta u = f(x,u), \tag{1.2}$$

where $M(s) = as + b$, $a, b > 0$. After that, many authors studied the following nonlocal elliptic boundary value problem:

$$-M\left(\int_\Omega |\nabla u|^2\, \mathrm{d}x\right)\Delta u = f(x,u). \tag{1.3}$$

Problems like (1.3) can be used for modeling several physical and biological systems where u describes a process which depends on the average of itself, such as the population density, see [3]. There are a number of papers concerned with Kirchhoff-type boundary value problem, for instance, see [4–12]. For example, in an interesting paper [12], Ricceri established the existence of at least three weak solutions to a class of Kirchhoff-type doubly eigenvalue boundary value problems. In [5] employing a three critical point theorem due to Ricceri, the authors discussed the existence of at least three weak solutions for Kirchhoff-type problems involving two parameters. In [10] Molica Bisci and Rădulescu, employing mountain pass results, obtained the existence of solutions to nonlocal equations involving the p-Laplacian. More precisely, they proved the existence of at least one nontrivial weak solution and under additional assumptions, the existence of infinitely many weak solutions. In [9], based on an abstract linking theorem for smooth functionals, they also established a multiplicity result on the existence of weak solutions for a nonlocal Neumann problem driven by a nonhomogeneous elliptic differential operator. The existence and multiplicity of stationary problems of Kirchhoff type were also studied in some recent papers via variational methods like the symmetric mountain pass theorem in [13] and via a three critical point theorem in [14]. Moreover, in [15, 16] some evolutionary higher order Kirchhoff problems, mainly focusing on the qualitative properties of solutions, were treated.

We refer to [17–28] for related nonlocal problems concerning the variational analysis of solutions of some classes of boundary value problems.

Fourth-order boundary value problems which describe the deformations of an elastic beam in an equilibrium state whose both ends are simply supported have been extensively studied in the literature. Recently, the existence of solutions to fourth-orderboundary value problems have been studied in many papers, and we refer the reader to the papers [29–37] and the references therein. For example, Liu et al. in [35], employing variational methods, studied the existence and multiplicity of nontrivial solutions for fourth-order elliptic equations. In [30, 32], based on variational methods and critical point theory, the existence of multiple solutions for $(p_1, \dots, p_n)$-biharmonic systems was discussed.

Molica Bisci and Repovš in [37], exploiting variational methods, investigated the existence of multiple weak solutions for a class of elliptic Navier boundary problems involving the p-biharmonic operator and presented a concrete example of an application.

The problem $(P^{f,g}_{\lambda,\mu})$ models the bending equilibrium of simply supported extensible beams on nonlinear foundations. The function f represents the force that the foundation exerts on the beam and $M(\int_\Omega |\nabla u|^p \,dx)$ models the effects of the small changes in the length of the beam. Recently, many researchers have paid their attention to fourth-order Kirchhoff-type problems; we refer the reader to [38–42] and the references therein. In [41], using the mountain pass theorem, Wang and An established the existence and multiplicity of solutions for the following fourth-order nonlocal elliptic problem:

$$\begin{cases} \Delta^2 u - M(\int_\Omega |\nabla u|^2 \,dx)\Delta u = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

In particular, in [38] using variational methods and critical point theory, multiplicity results of nontrivial and nonnegative solutions for the parametric version of the problem $(P^{f,g}_{\lambda,\mu})$ was established. In [39], employing two three critical point theorems, the existence of three distinct weak solutions for the problem $(P^{f,g}_{\lambda,\mu})$ was ensured.

Inspired by the above facts, in the present paper we are interested in looking for the existence of at least three weak solutions for the problem $(P^{f,g}_{\lambda,\mu})$ for appropriate values of the parameters λ and μ belonging to real intervals. Employing variational methods and two three critical point theorems due to Ricceri [43, 44], we establish two existence results for the problem $(P^{f,g}_{\lambda,\mu})$. Two examples are presented to illustrate our main results.

For more details on the subject, we also refer the reader to [45, 46].

2 Preliminaries

Our main tools are two three critical point theorems obtained by Ricceri [43, 44]. They are as follows.

If X is a real Banach space denoted by $\mathcal{W}_X$, the class of all functionals $\Phi : X \to \mathbb{R}$ possesses the following property:

If $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ and $\liminf_{n\to\infty} \Phi(u_n) \le \Phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u.

Remark 2.1 If X is uniformly convex and $g : [0,+\infty) \to \mathbb{R}$ is a continuous and strictly increasing function, then, by a classical result, the functional $u \to g(\|u\|)$ belongs to the class $\mathcal{W}_X$.

Theorem 2.1 ([43]) *Let X be a separable and reflexive real Banach space; let $\Phi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1-functional, belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and whose derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ be a C^1-functional with compact derivative. Assume that Φ has a strict local minimum u_0 with $\Phi(u_0) = J(u_0) = 0$. Finally, setting*

$$\rho = \max\left\{0, \limsup_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\},$$

$$\sigma = \sup_{u\in\Phi^{-1}(0,+\infty)} \frac{J(u)}{\Phi(u)},$$

assume that $\rho < \sigma$. Then, for each compact interval $[c,d] \subset (\frac{1}{\sigma}, \frac{1}{\rho})$ (with the conventions $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$), there exists $R > 0$ with the following property: for every $\lambda \in [c,d]$ and every C^1-functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$,

$$\Phi'(u) = \lambda J'(u) + \mu \Psi'(u)$$

has at least three solutions in X whose norms are less than R.

Theorem 2.2 ([44]) *Let X be a reflexive real Banach space and $I \subseteq \mathbb{R}$ be an interval; let $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semi-continuous C^1-functional, bounded on each bounded subset of X, whose derivative admits a continuous inverse on X^*; let $J : X \to \mathbb{R}$ be a C^1-functional with compact derivative. Assume that*

$$\lim_{\|u\| \to +\infty} \bigl(\Phi(u) - \lambda J(u)\bigr) = +\infty$$

for all $\lambda \in I$, and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda \in I} \inf_{u \in X} \bigl(\Phi(u) + \lambda\bigl(\rho - J(u)\bigr)\bigr) < \inf_{u \in X} \sup_{\lambda \in I} \bigl(\Phi(u) + \lambda\bigl(\rho - J(u)\bigr)\bigr).$$

Then there exist a nonempty open set $A \subseteq I$ and a positive number R' with the following property: for every $\lambda \in A$ and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the equation $\Phi'(u) - \lambda J'(u) - \mu \Psi'(u) = 0$ has at least three solutions in X whose norms are less than R'.

Proposition 2.3 ([47]) *Let X be a nonempty set and Φ, J be two real functions on X. Assume that there are $r > 0$ and $u_0, u_1 \in X$ such that*

$$\Phi(u_0) = J(u_0) = 0, \qquad \Phi(u_1) > r, \qquad \sup_{u \in \Phi^{-1}(-\infty,r]} J(u) < r\frac{J(u_1)}{\Phi(u_1)}.$$

Then, for each ρ satisfying

$$\sup_{u \in \Phi^{-1}(-\infty,r]} J(u) < \rho < r\frac{J(u_1)}{\Phi(u_1)},$$

one has

$$\sup_{\lambda \geq 0} \inf_{u \in X} \bigl(\Phi(u) + \lambda\bigl(\rho - J(u)\bigr)\bigr) < \inf_{u \in X} \sup_{\lambda \geq 0} \bigl(\Phi(u) + \lambda\bigl(\rho - J(u)\bigr)\bigr).$$

We refer the reader to the paper [48, 49] in which Theorems 2.1 and 2.2 were successfully employed to ensure the existence of at least three solutions for perturbed second-order Hamiltonian systems with impulsive effects. We also refer the readers to [50] in which Theorems 2.1 and 2.2 were successfully employed to ensure the existence of three solutions for perturbed Kirchhoff-type p-Laplacian discrete problems, and we refer the readers to [51] in which Theorem 2.1 was successfully employed to ensure the existence of three solutions for impulsive perturbed elastic beam fourth-order equations of Kirchhoff type.

Here and in the sequel, E will denote the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega \left(\left| \Delta u(x) \right|^p + \left| \nabla u(x) \right|^p + \left| u(x) \right|^p \right) \mathrm{d}x \right)^{\frac{1}{p}}.$$

Put

$$k = \sup_{u \in X \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u(x)|}{\|u\|}. \tag{2.1}$$

For $p > \max\{1, \frac{N}{2}\}$, since the embedding $\mathrm{E} \hookrightarrow \mathrm{C}^0(\overline{\Omega})$ is compact, one has $k < +\infty$.

Let $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ be two L^1-Carathéodory functions and $M : [0, +\infty[\to \mathbb{R}$ be a continuous function such that there are two positive constants m_0 and m_1 with $m_0 \le M(t) \le m_1$ for all $t \ge 0$.

Put

$$F(x,t) := \int_0^t f(x,\xi)\,\mathrm{d}\xi \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$
$$G(x,t) := \int_0^t g(x,\xi)\,\mathrm{d}\xi \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}$$

and

$$\widehat{M}(t) := \int_0^t \left[M(s) \right]^{p-1} \mathrm{d}s \quad \text{for all } t \ge 0.$$

Set

$$M^- := \min\left\{1, m_0^{p-1}, \varrho\right\}$$

and

$$M^+ := \max\left\{1, m_1^{p-1}, \varrho\right\}.$$

We say that a function $u \in \mathrm{E}$ is a (weak) solution of the problem $(P_{\lambda,\mu}^{f,g})$ if

$$\int_\Omega |\Delta u|^{p-2} \Delta u \Delta v \,\mathrm{d}x + \left[M\left(\int_\Omega |\nabla u|^p \,\mathrm{d}x \right) \right]^{p-1} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \,\mathrm{d}x$$
$$+ \varrho \int_\Omega |u|^{p-2} u v \,\mathrm{d}x - \lambda \int_\Omega f(x,u) v \,\mathrm{d}x - \mu \int_\Omega g(x,u) v \,\mathrm{d}x = 0$$

for every $v \in \mathrm{E}$.

Now, for every $u \in \mathrm{E}$, we define

$$\Phi(u) = \frac{1}{p} \int_\Omega \left| \Delta u(x) \right|^p \mathrm{d}x + \frac{1}{p} \widehat{M}\left[\int_\Omega \left| \nabla u(x) \right|^p \mathrm{d}x \right] + \frac{\varrho}{p} \int_\Omega \left| u(x) \right|^p \mathrm{d}x, \tag{2.2}$$

$$J(u) = \int_\Omega F\big(x, u(x)\big)\,\mathrm{d}x \tag{2.3}$$

and

$$\Psi(u) = \int_\Omega G\bigl(x, u(x)\bigr)\,\mathrm{d}x. \tag{2.4}$$

Standard arguments show that $I =: \Phi - \mu\Psi - \lambda J$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in \mathrm{E}$ is given by

$$\begin{aligned} I'(u)(v) = {} & \int_\Omega |\Delta u|^{p-2}\Delta u\Delta v\,\mathrm{d}x + \biggl[M\biggl(\int_\Omega |\nabla u|^p\,\mathrm{d}x\biggr)\biggr]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\nabla v\,\mathrm{d}x \\ & + \varrho\int_\Omega |u|^{p-2}uv\,\mathrm{d}x - \lambda\int_\Omega f(x,u)v\,\mathrm{d}x - \mu\int_\Omega g(x,u)v\,\mathrm{d}x \end{aligned}$$

for all $u, v \in \mathrm{E}$. We observe that a vector $u \in \mathrm{E}$ is a solution of the problem $(P^{f,g}_{\lambda,\mu})$ if and only if u is a critical point of the function I.

3 Main results

In this section, we formulate our main results.

First we give the following application of Theorem 2.1 as our first main result. Let

$$\begin{aligned} \lambda_1 = \inf\biggl\{ & \frac{\int_\Omega |\Delta u(x)|^p\,\mathrm{d}x + \widehat{M}[\int_\Omega |\nabla u(x)|^p\,\mathrm{d}x] + \varrho\int_\Omega |u(x)|^p\,\mathrm{d}x}{p\int_\Omega F(x,u(x))\,\mathrm{d}x} : \\ & u \in \mathrm{E}, \int_\Omega F\bigl(x,u(x)\bigr)\,\mathrm{d}x > 0\biggr\} \end{aligned}$$

and $\lambda_2 = \frac{1}{\max\{0,\lambda_0,\lambda_\infty\}}$, where

$$\lambda_0 = \limsup_{|u|\to 0}\frac{p\int_\Omega F(x,u(x))\,\mathrm{d}x}{\int_\Omega |\Delta u(x)|^p\,\mathrm{d}x + \widehat{M}[\int_\Omega |\nabla u(x)|^p\,\mathrm{d}x] + \varrho\int_\Omega |u(x)|^p\,\mathrm{d}x}$$

and

$$\lambda_\infty = \limsup_{\|u\|\to +\infty}\frac{p\int_\Omega F(x,u(x))\,\mathrm{d}x}{\int_\Omega |\Delta u(x)|^p\,\mathrm{d}x + \widehat{M}[\int_\Omega |\nabla u(x)|^p\,\mathrm{d}x] + \varrho\int_\Omega |u(x)|^p\,\mathrm{d}x}.$$

Theorem 3.1 *Assume that*

$(\mathcal{A}_1)$ *there exists a constant* $\varepsilon > 0$ *such that*

$$\max\biggl\{\limsup_{u\to 0}\frac{\max_{x\in\overline{\Omega}}F(x,u(x))}{|u|^p}, \limsup_{|u|\to +\infty}\frac{\max_{x\in\overline{\Omega}}F(x,u(x))}{|u|^p}\biggr\} < \varepsilon;$$

$(\mathcal{A}_2)$ *there exists a function* $w \in \mathrm{E}$ *such that*

$$K_w := \int_\Omega \bigl|\Delta w(x)\bigr|^p\,\mathrm{d}x + \widehat{M}\biggl[\int_\Omega \bigl|\nabla w(x)\bigr|^p\,\mathrm{d}x\biggr] + \varrho\int_\Omega \bigl|w(x)\bigr|^p\,\mathrm{d}x \neq 0$$

and

$$\varepsilon < \frac{M^-\int_\Omega F(x,w(x))\,\mathrm{d}x}{k^p K_w}.$$

Then, for each compact interval $[c,d] \subset (\lambda_1,\lambda_2)$, *there exists* $R > 0$ *with the following property*: *for every* $\lambda \in [c,d]$ *and every* L^1*-Carathéodory function* $g : \Omega \times \mathbb{R} \to \mathbb{R}$, *there exists* $\gamma > 0$ *such that*, *for each* $\mu \in [0,\gamma]$, *the problem* $(P^{f,g}_{\lambda,\mu})$ *has at least three weak solutions whose norms in* E *are less than* R.

Proof Take $X = \mathrm{E}$. Clearly, E is a separable and uniformly convex Banach space. Let the functionals Φ, J and Ψ be as given in (2.2), (2.3) and (2.4), respectively. The functional Φ is C^1, and by [38], Proposition 2.3, its derivative admits a continuous inverse on X^*. Moreover, since $m_0 \le K(s) \le m_1$ for all $s \in [0,+\infty[$, from (2.2) we have

$$\frac{M^-}{p}\|u\|^p \le \Phi(u) \le \frac{M^+}{p}\|u\|^p \tag{3.1}$$

for all $u \in X$, it follows $\lim_{\|u\|\to+\infty}\Phi(u) = +\infty$, namely, Φ is coercive. Furthermore, Φ is sequentially weakly lower semicontinuous. Moreover, let A be a bounded subset of X. That is, there exists a constant $c > 0$ such that $\|u\| \le c$ for each $u \in A$. Then, by (3.1), we have

$$\left|\Phi(u)\right| \le \frac{M^+}{p}c^p.$$

Hence Φ is bounded on each bounded subset of X. Furthermore, by Remark 2.1, $\Phi \in \mathcal{W}_X$. The functionals J and Ψ are two C^1-functionals with compact derivatives. Moreover, Φ has a strict local minimum 0 with $\Phi(0) = J(0) = 0$. In view of $(\mathcal{A}_1)$, there exist τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(x,u) \le \varepsilon|u|^p \tag{3.2}$$

for every $x \in \Omega$ and every u with $|u| \in [0,\tau_1) \cup (\tau_2,+\infty)$. Since $F(x,u)$ is continuous on $\Omega \times \mathbb{R}$, it is bounded on $x \in \Omega$ and $|u| \in [\tau_1,\tau_2]$. Thus we can choose $\eta > 0$ and $\upsilon > p$ such that

$$F(x,u) \le \varepsilon|u|^p + \eta|u|^\upsilon$$

for all $(k,u) \in \Omega \times \mathbb{R}$. So, by (2.1), we have

$$J(u) \le \varepsilon k^p\|u\|^p + \eta k^\upsilon\|u\|^\upsilon \tag{3.3}$$

for all $u \in X$. Hence, from (3.3) we have

$$\limsup_{|u|\to 0}\frac{J(u)}{\Phi(u)} \le \frac{pk^p\varepsilon}{M^-}. \tag{3.4}$$

Moreover, by using (3.2), for each $u \in \mathrm{E}\setminus\{0\}$, we obtain

$$\begin{aligned}\frac{J(u)}{\Phi(u)} &= \frac{\int_{|u|\le\tau_2}F(x,u)\,\mathrm{d}x}{\Phi(u)} + \frac{\int_{|u|>\tau_2}F(x,u)\,\mathrm{d}x}{\Phi(u)}\\ &\le \frac{\sup_{x\in\Omega,|u|\in[0,\tau_2]}F(x,u)}{\Phi(u)} + \frac{\varepsilon\|u\|^p}{k^p\Phi(u)} \le \frac{p\sup_{x\in\Omega,|u|\in[0,\tau_2]}F(x,u)}{\|u\|^p} + \frac{pk^p\varepsilon}{M^-}.\end{aligned}$$

So, we get

$$\limsup_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)} \le \frac{pk^p\varepsilon}{M^-}. \tag{3.5}$$

In view of (3.4) and (3.5), we have

$$\rho = \max\left\{0, \limsup_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to 0} \frac{J(u)}{\Phi(u)}\right\} \le \frac{pk^p\varepsilon}{M^-}. \tag{3.6}$$

Assumption $(\mathcal{A}_2)$ in conjunction with (3.6) yields

$$\begin{aligned}
\sigma &= \sup_{u\in\Phi^{-1}(0,+\infty)} \frac{J(u)}{\Phi(u)} \\
&= \sup_{X\setminus\{0\}} \frac{J(u)}{\Phi(u)} \ge \frac{\int_\Omega F(x,w(x))}{\Phi(w(x))} \\
&= \frac{p\int_\Omega F(x,w(x))\,\mathrm{d}x}{K_w} > \frac{pk^p\varepsilon}{M^-} \ge \rho.
\end{aligned}$$

Thus, all the hypotheses of Theorem 2.1 are satisfied. Clearly, $\lambda_1 = \frac{1}{\sigma}$ and $\lambda_2 = \frac{1}{\rho}$. Then, using Theorem 2.1, for each compact interval $[c,d] \subset (\lambda_1, \lambda_2)$, there exists $R > 0$ with the following property: for every $\lambda \in [c,d]$ and every L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$, the problem $(P^{f,g}_{\lambda,\mu})$ has at least three solutions whose norms in X are less than R. □

Another announced application of Theorem 2.1 reads as follows.

Theorem 3.2 *Assume that*

$$\max\left\{\limsup_{u\to 0} \frac{\max_{x\in\overline{\Omega}} F(x,u(x))}{|u|^p}, \limsup_{|u|\to+\infty} \frac{\max_{x\in\overline{\Omega}} F(x,u(x))}{|u|^p}\right\} \le 0 \tag{3.7}$$

and

$$\sup_{u\in \mathrm{E}} \frac{M^- \int_\Omega F(x,u(x))\,\mathrm{d}x}{k^p(\int_\Omega |\Delta u(x)|^p\,\mathrm{d}x + \widehat{M}[\int_\Omega |\nabla u(x)|^p\,\mathrm{d}x] + \varrho\int_\Omega |u(x)|^p\,\mathrm{d}x)} > 0. \tag{3.8}$$

Then, for each compact interval $[c,d] \subset (\lambda_1, +\infty)$, there exists $R > 0$ with the following property: for every $\lambda \in [c,d]$ and every L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the problem $(P^{f,g}_{\lambda,\mu})$ has at least three weak solutions whose norms in E are less than R.

Proof In view of (3.7), there exist an arbitrary $\varepsilon > 0$ and τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(x,u) \le \varepsilon|u|^p$$

for every $x \in \Omega$ and every u with $|u| \in [0,\tau_1) \cup (\tau_2, +\infty)$. Since $F(x,u)$ is continuous on $\Omega \times \mathbb{R}$, it is bounded on $x \in \Omega$ and $|u| \in [\tau_1, \tau_2]$. Thus we can choose $\eta > 0$ and $\upsilon > p$ in a manner that

$$F(x,u) \le \varepsilon|u|^p + \eta|u|^\upsilon$$

for all $(x,u) \in \Omega \times \mathbb{R}$. So, by the same process as that in the proof of Theorem 3.1, we have Relations (3.4) and (3.5). Since ε is arbitrary, (3.4) and (3.5) give

$$\max\left\{0, \limsup_{\|u\|\to+\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to 0} \frac{J(u)}{\Phi(u)}\right\} \le 0.$$

Then, with the notation of Theorem 2.1, we have $\rho = 0$. By (3.8), we also have $\sigma > 0$. In this case, clearly $\lambda_1 = \frac{1}{\sigma}$ and $\lambda_2 = +\infty$. Thus, by using Theorem 2.1, the result is achieved. □

Now we formulate the following applications of Theorem 2.2 as our second main result.

Theorem 3.3 *Assume that*

($\mathcal{B}_1$) *there exist two positive functions $\nu, \xi \in \mathrm{L}^1(\Omega,\mathbb{R})$ and $\alpha \in [0,p)$ such that*

$$\left|F(x,u)\right| \le \nu(x)|u|^\alpha + \xi(x) \quad \textit{for all } u \in \mathbb{R} \textit{ and } x \in \Omega;$$

($\mathcal{B}_2$) *there exist a positive constant r and $w \in \mathrm{E}$ such that $K_w > r$, where K_w is as given in Assumption ($\mathcal{A}_2$) in Theorem 3.1, and*

$$\max_{x\in\overline{\Omega}, |u|\le \sqrt[p]{\frac{pr}{M^-}}} F(x,u) < \frac{r\int_\Omega F(x,w(x))\,\mathrm{d}x}{\mathrm{meas}(\Omega)K_w}.$$

Then there exist a nonempty open set $A \subset [0,+\infty)$ and a positive number R' with the following property: for every $\lambda \in A$ and every continuous function $g: \Omega \times \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the problem $(P^{f,g}_{\lambda,\mu})$ has at least three weak solutions whose norms in E *are less than R'.*

Proof Take $X = \mathrm{E}$. Let the functionals Φ and J be as given in (2.2) and (2.3), respectively. For any $\lambda \ge 0$ and $u \in \mathrm{E}$, by ($\mathcal{B}_1$), we have

$$\begin{aligned}
\Phi(u) - \lambda J(u) &= \frac{1}{p}\int_\Omega \left|\Delta u(x)\right|^p \mathrm{d}x + \frac{1}{p}\widehat{M}\left[\int_\Omega \left|\nabla u(x)\right|^p \mathrm{d}x\right] \\
&\quad + \frac{\varrho}{p}\int_\Omega \left|u(x)\right|^p \mathrm{d}x - \lambda\int_\Omega F\bigl(x,u(x)\bigr)\,\mathrm{d}x \\
&\ge \frac{M^-}{p}\|u\|^p - \lambda\int_\Omega \bigl(\nu(x)\left|u(x)\right|^\alpha + \xi(x)\bigr)\,\mathrm{d}x \\
&\ge \frac{M^-}{p}\|u\|^p - \lambda k^\alpha\|u\|^\alpha\|\nu\|_{\mathrm{L}^1} - \lambda\|\xi\|_{\mathrm{L}^1}.
\end{aligned}$$

Since $\alpha < p$, one has $\lim_{\|u\|\to+\infty}(\Phi(u) - \lambda J(u)) = +\infty$ for all $\lambda \ge 0$. If $\Phi(u) \le r$, we have $\|u\| \le \sqrt[p]{\frac{pr}{M^-}}$, that is,

$$\Phi^{-1}(-\infty,r] \subseteq \left\{u \in X : \max_{x\in\overline{\Omega}}\left|u(x)\right| \le \sqrt[p]{\frac{pr}{M^-}}\right\}.$$

Therefore,

$$\begin{aligned}\sup_{u\in\Phi^{-1}(-\infty,r]} J(u) &\le \max_{|u|\le \sqrt[p]{\frac{pr}{M^-}}} J(u)\\ &= \max_{|u|\le \sqrt[p]{\frac{pr}{M^-}}} \int_\Omega F\big(x,u(x)\big)\,\mathrm{d}x\\ &\le \operatorname{meas}(\Omega) \max_{x\in\overline{\Omega},|u|\le \sqrt[p]{\frac{pr}{M^-}}} F(x,u).\end{aligned}\tag{3.9}$$

It is clear that $\Phi(0)=J(0)=0$ and owing to $(\mathcal{B}_2)$ and (3.9), $\Phi(w)>r$ and

$$\sup_{u\in\Phi^{-1}(-\infty,r]} J(u) < r\frac{J(w)}{\Phi(w)}.$$

Thus we can fix ρ such that

$$\sup_{u\in\Phi^{-1}(-\infty,r]} J(u) < \rho < r\frac{J(w)}{\Phi(w)}.$$

Now, from Proposition 2.3, we obtain

$$\sup_{\lambda\ge 0}\inf_{u\in \mathrm{E}}\big(\Phi(u)+\lambda\big(\rho-J(u)\big)\big) < \inf_{u\in \mathrm{E}}\sup_{\lambda\ge 0}\big(\Phi(u)+\lambda\big(\rho-J(u)\big)\big).$$

Therefore, by Theorem 2.2, for each compact interval $[a,b]\subseteq(\lambda_1,\lambda_2)$, there exists $R'>0$ with the following property: for every $\lambda\in[a,b]$ and every L^1-Carathéodory function $g:\Omega\times\mathbb{R}\to\mathbb{R}$, there exists $\delta>0$ such that, for each $\mu\in[0,\delta]$, $\Phi'(u)-\lambda J'(u)-\mu\Psi'(u)=0$ has at least three solutions in E. Hence, the problem $(P^{f,g}_{\lambda,\mu})$ has at least three weak solutions whose norms are less than R'. □

Now fix $x^0\in\Omega$ and pick $s>0$ such that $B(x^0,s)\subset\Omega$, where $B(x^0,s)$ denotes the ball with center at x^0 and radius of s. Put

$$\begin{aligned}\vartheta_1 &:= \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\int_{\frac{s}{2}}^{s}\left|\frac{12(N+1)}{s^3}r-\frac{24N}{s^2}+\frac{9(N-1)}{s}\frac{1}{r}\right|^p r^{N-1}\,\mathrm{d}r,\\ \vartheta_2 &:= \int_{B(x^0,s)\setminus B(x^0,\frac{s}{2})}\left[\sum_{i=1}^{N}\left(\frac{12(x_i-x_i^0)}{s^3}-\frac{24(x_i-x_i^0)}{s^2}+\frac{9(x_i-x_i^0)}{s\ell}\right)^2\right]^{\frac{p}{2}}\mathrm{d}x,\\ \vartheta_3 &:= \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\left[\frac{(\frac{s}{2})^N}{N}+\int_{\frac{s}{2}}^{s}\left|\frac{4}{s^3}r^3-\frac{12}{s^2}r^2+\frac{9}{s}r-1\right|^p r^{N-1}\,\mathrm{d}r\right],\end{aligned}$$

where Γ denotes the gamma function, and

$$L:=\vartheta_1+\vartheta_2+\vartheta_3.\tag{3.10}$$

The next two theorems provide sufficient conditions for applying Theorems 3.1 and 3.3, which does not require to know a test function w satisfying $(\mathcal{A}_2)$ and $(\mathcal{B}_2)$, respectively.

Theorem 3.4 *Assume that Assumption* $(\mathcal{A}_1)$ *in Theorem* 3.1 *holds and there exists a positive constant d such that*

$(\mathcal{A}_3)$ $F(x,t) \geq 0$ *for each* $x \in B(x^0,s) \setminus B(x^0,\frac{s}{2})$, $t \in [0,d]$;

$(\mathcal{A}_4)$ $\vartheta_1 d^p + \widehat{M}(\vartheta_2 d^p) + \varrho\vartheta_3 d^p \neq 0$ *and* $\varepsilon < \frac{pM^- \int_{B(x^0,\frac{s}{2})} F(x,d)\,\mathrm{d}x}{k^p(\vartheta_1 d^p + \widehat{M}(\vartheta_2 d^p) + \varrho\vartheta_3 d^p)}$.

Then, for each compact interval $[c,d] \subset (\lambda_1,\lambda_2)$, *there exists* $R > 0$ *with the following property: for every* $\lambda \in [c,d]$ *and every* L^1*-Carathéodory function* $g : \Omega \times \mathbb{R} \to \mathbb{R}$, *there exists* $\gamma > 0$ *such that, for each* $\mu \in [0,\gamma]$, *the problem* $(P^{f,g}_{\lambda,\mu})$ *has at least three weak solutions whose norms in* E *are less than R.*

Proof We claim that all the assumptions of Theorem 3.1 are fulfilled by choosing w as follows:

$$w(x) := \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x^0,s), \\ d(\frac{4}{s^3}\ell^3 - \frac{12}{s^2}\ell^2 + \frac{9}{s}\ell - 1) & \text{if } x \in B(x^0,s) \setminus B(x^0,\frac{s}{2}), \\ d & \text{if } x \in B(x^0,\frac{s}{2}) \end{cases} \tag{3.11}$$

where $\ell = \operatorname{dist}(x,x^0) = \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}$ (see [31, 36]). We have

$$\frac{\partial w(x)}{\partial x_i} = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x^0,s) \cup B(x^0,\frac{s}{2}), \\ d(\frac{12\ell(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9}{s}\frac{(x_i - x_i^0)}{\ell}) & \text{if } x \in B(x^0,s) \setminus B(x^0,\frac{s}{2}) \end{cases}$$

and

$$\frac{\partial^2 w(x)}{\partial x_i^2} = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x^0,s) \cup B(x^0,\frac{s}{2}), \\ d(\frac{12}{s^3}\frac{(x_i - x_i^0)^2 + \ell^2}{\ell} - \frac{24}{s^2} + \frac{9}{s}\frac{\ell^2 - (x_i - x_i^0)^2}{\ell^3}) & \text{if } x \in B(x^0,s) \setminus B(x^0,\frac{s}{2}), \end{cases}$$

and so that

$$\sum_{i=1}^N \frac{\partial^2 w(x)}{\partial x_i^2} = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x^0,s) \cup B(x^0,\frac{s}{2}), \\ d(\frac{12l(N+1)}{s^3} - \frac{24N}{s^2} + \frac{9}{s}\frac{N-1}{\ell}) & \text{if } x \in B(x^0,s) \setminus B(x^0,\frac{s}{2}). \end{cases}$$

In particular, since

$$\begin{aligned} &\int_\Omega \bigl|\Delta w(x)\bigr|^p \,\mathrm{d}x = d^p \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \int_{\frac{s}{2}}^s \left|\frac{12(N+1)}{s^3}r - \frac{24N}{s^2} + \frac{9(N-1)}{s}\frac{1}{r}\right|^p r^{N-1}\,\mathrm{d}r, \\ &\int_\Omega \bigl|\nabla w(x)\bigr|^p \,\mathrm{d}x \\ &\quad = \int_{B(x^0,s)\setminus B(x^0,\frac{s}{2})} \left[\sum_{i=1}^N d^2\left(\frac{12l(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9}{s}\frac{(x_i - x_i^0)}{l}\right)^2\right]^{\frac{p}{2}} \mathrm{d}x \\ &\quad = d^p \int_{B(x^0,s)\setminus B(x^0,\frac{s}{2})} \left[\sum_{i=1}^N \left(\frac{12l(x_i - x_i^0)}{s^3} - \frac{24(x_i - x_i^0)}{s^2} + \frac{9}{s}\frac{(x_i - x_i^0)}{l}\right)^2\right]^{\frac{p}{2}} \mathrm{d}x \end{aligned}$$

and

$$\int_{\Omega}\left|w(x)\right|^{p}\,\mathrm{d}x = d^{p}\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\left(\frac{(\frac{s}{2})^{N}}{N}+\int_{\frac{s}{2}}^{s}\left|\frac{4}{s^{3}}r^{3}-\frac{12}{s^{2}}r^{2}+\frac{9}{s}r-1\right|^{p}r^{N-1}\,\mathrm{d}r\right).$$

It is easy to see that $w \in \mathrm{E}$, and one has

$$\begin{aligned}\frac{d^{p}}{p}M^{-}L &\leq \frac{1}{p}\left(\vartheta_{1}d^{p}+m_{0}^{p-1}\vartheta_{2}d^{p}+\varrho\vartheta_{3}d^{p}\right)\\ &\leq \Phi(w)\\ &= \frac{1}{p}\left(\vartheta_{1}d^{p}+\widehat{M}\left(\vartheta_{2}d^{p}\right)+\varrho\vartheta_{3}d^{p}\right)\\ &\leq \frac{1}{p}\left(\vartheta_{1}d^{p}+m_{1}^{p-1}\vartheta_{2}d^{p}+\varrho\vartheta_{3}d^{p}\right)\\ &\leq \frac{d^{p}}{p}M^{+}L. \end{aligned}\tag{3.12}$$

From Assumptions $(\mathcal{A}_3)$ and $(\mathcal{A}_4)$ we observe that Assumption $(\mathcal{A}_2)$ in Theorem 3.1 is satisfied. Hence, Theorem 3.1 follows the result. □

Theorem 3.5 *Assume that Assumption* $(\mathcal{B}_1)$ *in Theorem* 3.3 *and Assumption* $(\mathcal{A}_3)$ *in Theorem* 3.4 *hold and there exist three positive constants* c, d *and* α *with* $\sqrt[p]{L}d < c$, *where* L *is given as in* (3.10), *and* $\alpha \in [0,p)$ *such that*

$(\mathcal{B}_3)$ $\max_{x\in\overline{\Omega},|u|\leq c}F(x,u) < \frac{M^{-}}{M^{+}\mathrm{meas}(\Omega)}\int_{B(x^{0},\frac{s}{2})}F(x,d)\,\mathrm{d}x.$

Then, there exist a nonempty open set $A \subset [0,+\infty)$ *and a positive number* R' *with the following property: for every* $\lambda \in A$ *and every continuous function* $g : \Omega \times \mathbb{R} \to \mathbb{R}$, *there exists* $\delta > 0$ *such that, for each* $\mu \in [0,\delta]$, *the problem* $(P^{f,g}_{\lambda,\mu})$ *has at least three weak solutions whose norms in* E *are less than* R'.

Proof We claim that all the hypotheses of Theorem 3.1 are satisfied by choosing w as given in (3.11) and $r < \frac{LM^{-}}{p}d^{p}$. We observe that

$$K_{w} = \frac{1}{p}\left(\vartheta_{1}d^{p}+\widehat{M}\left(\vartheta_{2}d^{p}\right)+\varrho\vartheta_{3}d^{p}\right) \geq \frac{LM^{-}}{p}d^{p} > r,$$

where K_w is as given in Assumption $(\mathcal{A}_1)$. Owing to $(\mathcal{B}_3)$ and $F(t,0) = 0$, one has $\int_{B(x^{0},\frac{s}{2})}F(x,w(x))\,\mathrm{d}x > 0$. So, by $(\mathcal{A}_3)$, $(\mathcal{B}_3)$ and (3.12), we have

$$\begin{aligned} r\frac{\int_{\Omega}F(x,w(x))\,\mathrm{d}x}{\mathrm{meas}(\Omega)K_{w}} &= \frac{LM^{-}d^{p}}{p}\frac{p\int_{\Omega}F(x,w(x))\,\mathrm{d}x}{\mathrm{meas}(\Omega)(\theta_{1}d^{p}+\widehat{M}(\theta_{2}d^{p})+\varrho\theta_{3}d^{p})}\\ &> \frac{M^{-}}{\mathrm{meas}(\Omega)M^{+}}\int_{B(x^{0},\frac{s}{2})}F(x,d)\,\mathrm{d}x\\ &> \max_{x\in\overline{\Omega},|u|\leq c}F(x,u)\\ &> \max_{x\in\overline{\Omega},|u|\leq\sqrt[p]{\frac{pr}{M^{-}}}}F(x,u). \end{aligned}$$

Thus, Assumption ($\mathcal{B}_2$) in Theorem 3.3 holds. Therefore, by Theorem 3.3, for each compact interval $[a,b] \subseteq (\lambda_1, \lambda_2)$, there exists $R' > 0$ with the following property: for every $\lambda \in [a,b]$ and every L^1-Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, $\Phi'(u) - \lambda J'(u) - \mu \Psi'(u) = 0$ has at least three solutions in E. Hence, the problem ($P^{f,g}_{\lambda,\mu}$) has at least three weak solutions whose norms are less than R'. □

Remark 3.1 Clearly, Theorem 3.5 gives the result of at least three solutions for the problem ($P^{f,g}_{\lambda,\mu}$) with $F(x,u)$ being of subquadratic growth.

Remark 3.2 The statements of Theorems 3.4 and 3.5 depend upon the test function w defined by (3.11). If we take the other choices of w, we have another statement. For example, if $x^0 \in \Omega$ and we pick $s > 0$ such that $B(x^0,s) \subset \Omega$, where $B(x^0,s)$ denotes the ball with center at x^0 and radius of s, and

$$\vartheta_1' := \frac{2^{5P+1}\pi^{N/2}d^p}{s^{4p}\Gamma(N/2)} \int_{s/2}^{s} \left|2(N+2)r^2 - 3(N+1)sr + Nr^2\right|^p r^{N+1}\,\mathrm{d}r,$$

$$\vartheta_2' := \left(\frac{32}{s^4}\right)^p \int_{B(x^0,s)\setminus B(x^0,\frac{s}{2})} \left[\sum_{i=1}^{N} \left((s-\ell)(s-3\ell)\left(x_i - x_i^0\right)\right)^2\right]^{\frac{p}{2}} \mathrm{d}x,$$

$$\vartheta_3' := \left(\frac{16}{s^4}\right)^p \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\left(\frac{(\frac{s}{2})^N}{N} + \int_{\frac{s}{2}}^{s} \left|r^2(s-r)\right|^p r^{N-1}\,\mathrm{d}r\right)$$

and where Γ denotes the gamma function,

$$L' := \vartheta_1' + \vartheta_2' + \vartheta_3',$$

and we take

$$w(x) := \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x^0,s), \\ 16\frac{\ell^2}{s^4}(s-\ell)^2 d & \text{if } x \in B(x^0,s) \setminus B(x^0,\frac{s}{2}), \\ d & \text{if } x \in B(x^0,\frac{s}{2}) \end{cases} \tag{3.13}$$

with $\ell = \operatorname{dist}(x,x^0) = \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}$ (see [37]), then we have

$$\frac{\partial w(x)}{\partial x_i} = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x^0,s) \cup B(x^0,\frac{s}{2}), \\ \frac{32d}{s^4}(s-\ell)(s-3\ell)(x_i - x_i^0) & \text{if } x \in B(x^0,s) \setminus B(x^0,\frac{s}{2}), \end{cases}$$

$$\frac{\partial^2 w(x)}{\partial x_i^2} = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x^0,s) \cup B(x^0,\frac{s}{2}), \\ \frac{32(x_i - x_i^0)d}{s^4}\left(\frac{6(x_i - x_i^0)(\ell - 4s)}{\ell} + s^2 - 4ls + 3l^2\right) & \text{if } x \in B(x^0,s) \setminus B(x^0,\frac{s}{2}) \end{cases}$$

and

$$\sum_{i=1}^N \frac{\partial^2 w(x)}{\partial x_i^2} = \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x^0,s) \cup B(x^0,\frac{s}{2}), \\ 32d\left(\frac{2(N+2)\ell^2 - 3s(N+1)\ell + Ns^2}{s^4}\right) & \text{if } x \in B(x^0,s) \setminus B(x^0,\frac{s}{2}). \end{cases}$$

It is easy to see that $w \in \mathrm{E}$ and, in particular, since

$$\int_{\Omega} \left|\Delta w(x)\right|^p \mathrm{d}x = \frac{2^{5P+1}\pi^{N/2}d^p}{s^{4p}\Gamma(N/2)} \int_{s/2}^{s} \left|2(N+2)r^2 - 3(N+1)rs + Nr^2\right|^p r^{N+1}\,\mathrm{d}r,$$

$$\begin{aligned}
&\int_\Omega \left|\nabla w(x)\right|^p \,\mathrm{d}x \\
&\quad = \int_{B(x^0,s)\setminus B(x^0,\frac{s}{2})} \left[\sum_{i=1}^N \left(\frac{32d}{s^4}(s-\ell)(s-3\ell)\left(x_i - x_i^0\right)\right)^2\right]^{\frac{p}{2}} \mathrm{d}x \\
&\quad = \left(\frac{32d}{s^4}\right)^p \int_{B(x^0,s)\setminus B(x^0,\frac{s}{2})} \left[\sum_{i=1}^N \left((s-\ell)(s-3\ell)\left(x_i - x_i^0\right)\right)^2\right]^{\frac{p}{2}} \mathrm{d}x
\end{aligned}$$

and

$$\int_\Omega \left|w(x)\right|^p \,\mathrm{d}x = \left(\frac{16d}{s^4}\right)^p \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\left(\frac{(\frac{s}{2})^N}{N} + \int_{\frac{s}{2}}^s \left|r^2(s-r)\right|^p r^{N-1}\,\mathrm{d}r\right),$$

one has

$$\begin{aligned}
\frac{d^p}{p}M^- L' &\le \frac{1}{p}\left(\vartheta_1' d^p + m_0^{p-1}\vartheta_2' d^p + \varrho\vartheta_3' d^p\right) \\
&\le \Phi(w) \\
&= \frac{1}{p}\left(\vartheta_1' d^p + \widehat{M}\left(\vartheta_2' d^p\right) + \varrho\vartheta_3' d^p\right) \\
&\le \frac{1}{p}\left(\vartheta_1' d^p + m_1^{p-1}\vartheta_2' d^p + \varrho\vartheta_3' d^p\right) \\
&\le \frac{d^p}{p}M^+ L'.
\end{aligned}$$

Therefore, condition $(\mathcal{A}_4)$ in Theorem 3.4 takes the following form:

$(\mathcal{A}_5)$ there exists a positive constant d such that

$$\vartheta_1' d^p + \widehat{M}\left(\vartheta_2' d^p\right) + \varrho\vartheta_3' d^p \neq 0 \quad \text{and} \quad \varepsilon < \frac{pM^- \int_\Omega F(x, w(x))\,\mathrm{d}x}{k^p(\vartheta_1' d^p + \widehat{M}(\vartheta_2' d^p) + \varrho\vartheta_3' d^p)},$$

where w is given by (3.13).

Moreover, the condition $\sqrt[p]{L}d < c$ in Theorem 3.5 becomes the condition $\sqrt[p]{L'}d < c$. Also, by choosing w as given in [37], Remark 3.4, which is as follows:

$$w(x) := \begin{cases} 0 & \text{if } x \in \overline{\Omega} \setminus B(x^0, r_2), \\ \frac{3(\ell^4 - r^4) - 4(r_1+r_2)(\ell^3 - r_2^3) + 6r_1 r_2(\ell^2 - r_2^2)}{(r_2 - r_1)^3(r_1 + r_2)} d & \text{if } x \in B(x^0, r_2) \setminus B(x^0, r_1), \\ d & \text{if } x \in B(x^0, r_1), \end{cases}$$

where $\ell = \operatorname{dist}(x, x^0) = \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}$ and $r_1, r_2 \in \mathbb{R}$ with $r_2 > r_1 > 0$ (see [33, 34]), we have other forms of conditions $(\mathcal{A}_4)$ and $\sqrt[p]{L}d < c$.

Now, we point out some results in which the function f has separated variables. To be precise, consider the following problem:

$$\begin{cases} T(u) = \lambda\theta(x)f(u) + \mu g(x,u), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \qquad (P_{\lambda,\mu}^{f,\theta,g})$$

where $\theta : \Omega \to \mathbb{R}$ is a nonnegative and nonzero function, $\theta \in L^1(\Omega)$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is as introduced in the problem $(P^{f,g}_{\lambda,\mu})$ in Introduction.

Set $F(x,t) = \theta(x)F(t)$ for every $(x,t) \in \Omega \times \mathbb{R}$, where

$$F(t) = \int_0^t f(\xi)\,\mathrm{d}\xi$$

for all $t \in \mathbb{R}$. The following existence results are consequences of Theorem 3.4.

Theorem 3.6 *Assume that*

$(\mathcal{A}_6)$ *there exists a constant* $\varepsilon > 0$ *such that*

$$\sup_{x\in\Omega} \theta(x) \cdot \max\left\{ \limsup_{u\to 0} \frac{F(u)}{|u|^p}, \limsup_{|u|\to\infty} \frac{F(u)}{|u|^p} \right\} < \varepsilon;$$

$(\mathcal{A}_7)$ *there exists a positive constant d such that*

$$\vartheta_1 d^p + \widehat{M}\big(\vartheta_2 d^p\big) + \varrho\vartheta_3 d^p \neq 0 \quad \textit{and} \quad \varepsilon < \frac{pM^- \int_\Omega F(x, w(x))\,\mathrm{d}x}{k^p(\vartheta_1 d^p + \widehat{M}(\vartheta_2 d^p) + \varrho\vartheta_3 d^p)},$$

where w is given by (3.11).

Then, for each compact interval $[c,d] \subset (\lambda_3, \lambda_4)$, *where* λ_3 *and* λ_4 *are the same as* λ_1 *and* λ_2, *but* $\int_\Omega F(x,u(x))\,\mathrm{d}x$ *is replaced by* $\int_\Omega \theta(x)F(u(x))\,\mathrm{d}x$, *respectively, there exists* $R > 0$ *with the following property*: *for every* $\lambda \in [c,d]$ *and every* L^1-*Carathéodory function* $g : \Omega \times \mathbb{R} \to \mathbb{R}$, *there exists* $\gamma > 0$ *such that for each* $\mu \in [0,\gamma]$, *the problem* $(P^{f,\theta,g}_{\lambda,\mu})$ *has at least three weak solutions whose norms in* E *are less than R.*

Theorem 3.7 *Assume that there exists a positive constant d such that*

$$\vartheta_1 d^p + \widehat{M}\big(\vartheta_2 d^p\big) + \varrho\vartheta_3 d^p > 0 \quad \textit{and} \quad F(d) > 0. \tag{3.14}$$

Moreover, suppose that

$$\limsup_{u\to 0} \frac{f(u)}{|u|^{p-1}} = \limsup_{|u|\to\infty} \frac{f(u)}{|u|^{p-1}} = 0. \tag{3.15}$$

Then, for each compact interval $[c,d] \subset (\lambda_3, \infty)$, *where* λ_3 *is the same as* λ_1 *but* $\int_\Omega F(x, u(x))\,\mathrm{d}x$ *is replaced by* $\int_\Omega \theta(x)F(u(x))\,\mathrm{d}x$, *there exists* $R > 0$ *with the following property*: *for every* $\lambda \in [c,d]$ *and every* L^1-*Carathéodory function* $g : \Omega \times \mathbb{R} \to \mathbb{R}$, *there exists* $\gamma > 0$ *such that for each* $\mu \in [0,\gamma]$, *the problem* $(P^{f,\theta,g}_{\lambda,\mu})$ *has at least three weak solutions whose norms in* E *are less than R.*

Proof We easily observe that from (3.15) Assumption $(\mathcal{A}_6)$ is satisfied for every $\varepsilon > 0$. Moreover, using (3.14), by choosing $\varepsilon > 0$ small enough, one can derive Assumption $(\mathcal{A}_7)$. Hence, the conclusion follows from Theorem 3.6. □

Remark 3.3 Our results show that no asymptotic conditions on f and g are required, and merely the algebraic conditions on f are supposed to guarantee the existence of solutions.

Now, we present the following example to illustrate Theorem 3.7.

Example 3.1 Let $N = 2$, $p = 6$, $\varrho = 2$, $\Omega = \{(x_1, x_2) \in \mathbb{R}^2; x_1^2 + x_2^2 < 5\} \subset \mathbb{R}^2$, $x^0 = (0,0)$, $s = 2$,

$$M(t) = 1 + \frac{1}{\cosh t} \quad \text{for all } t \in \mathbb{R},$$

$\vartheta(x) = 1$ for all $x \in \Omega$ and

$$f(t) = \begin{cases} 2(t + \sin t)^2, & \text{if } t < \pi, \\ 2\pi^2 + \tanh(t - \pi), & \text{if } t \geq \pi. \end{cases}$$

Thus,

$$S(x^0, s) = S((0,0), 2) = \{(x_1, x_2) \in \mathbb{R}^2; x_1^2 + x_2^2 < 4\} \subset \Omega,$$

$m_0 = 1$, $m_1 = 2$ and f is a nonnegative and continuous function. By choosing $d = 1$, we have

$$w(x_1, x_2) = \begin{cases} 0 & \text{if } (x_1, x_2) \in \overline{\Omega} \setminus S((0,0), 2), \\ 1 & \text{if } (x_1, x_2) \in S((0,0), 1), \\ \frac{1}{2}\sqrt{(x_1^2 + x_2^2)^3} & \\ \quad - 3(x_1^2 + x_2^2) + \frac{9}{2}\sqrt{x_1^2 + x_2^2} - 1 & \text{if } (x_1, x_2) \in S((0,0), 2) \setminus S((0,0), 1). \end{cases}$$

Therefore,

$$\vartheta_1 = 2\pi \int_1^2 \left(\frac{9}{2}\xi - 12 + \frac{9}{2\xi^3} \right)^6 \xi \, d\xi,$$

$$\vartheta_2 = \iint_{S((0,0),2)\setminus S((0,0),1)} \left[\sum_{i=1}^{2} \left(\frac{3}{2} x_i \sqrt{x_1^2 + x_2^2} - 6x_i + \frac{9x_i}{2\sqrt{x_1^2 + x_2^2}} \right)^2 \right]^4 dx_1 \, dx_2$$

and

$$\vartheta_3 = 2\pi \left(\frac{1}{2} + \int_1^2 \left(\frac{1}{2}\xi^3 - 3\xi^2 + \frac{9}{2}\xi - 1 \right)^6 \xi \, d\xi \right)$$

are positive. So,

$$\vartheta_1 d^p + \widehat{M}(\vartheta_2 d^p) + \varrho \vartheta_3 d^p > 0.$$

Moreover, we have

$$F(d) = F(1) = \int_0^1 2(t + \sin t)^2 \, dt > 0$$

and

$$\lim_{u \to 0} \frac{f(u)}{|u|} = \lim_{u \to \infty} \frac{f(u)}{|u|} = 0.$$

Hence, by applying Theorem 3.7 for each compact interval $[c,d] \subset (0,\infty)$, there exists $R > 0$ with the following property: for every $\lambda \in [c,d]$ and every continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that, for each $\mu \in [0,\gamma]$, the problem

$$\begin{cases} \Delta(|\Delta u|^4 \Delta u) - (1 + \frac{1}{\cosh(\int_\Omega |\nabla u|^6 \, dx)})^5 \Delta_6 u + 2|u|^4 u = \lambda f(u) + \mu g(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions whose norms in the space $W^{2,6}(\Omega) \cap W_0^{1,6}(\Omega)$ are less than R.

The following existence result is a consequence of Theorem 3.5.

Theorem 3.8 *Assume that there exist five positive constants c, d, α, ν and ξ with $c > \sqrt[p]{L}d$, where L is given by* (3.10), *and $\alpha \in [0,p)$ such that*

($\mathcal{B}_4$) $\max_{|u|\le c} F(u) < \frac{M^- F(d) \pi^{N/2}}{M^+ \operatorname{meas}(\Omega) \Gamma(\frac{N}{2})}$;
($\mathcal{B}_5$) $F(u) > 0$ *for each* $u \in \mathbb{R}$;
($\mathcal{B}_6$) $|F(u)| \le \nu |u|^\alpha + \xi$ *for all* $u \in \mathbb{R}$.

Then there exist a nonempty open set $A \subset [0,\infty)$ and a positive number R' with the following property: *for every $\lambda \in A$ and every continuous function $g : \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in [0,\delta]$, the problem* $(P^{f,\theta,g}_{\lambda,\mu})$ *has at least three weak solutions whose norms in* E *are less than R'.*

Finally, we present the following example to illustrate Theorem 3.8.

Example 3.2 Let $N = 2$, $p = 3$, $\varrho = 1$, $\Omega = \{(x_1,x_2) \in \mathbb{R}^2; x_1^2 + x_2^2 < \frac{1}{100}\} \subset \mathbb{R}^2$, $x^0 = (0,0)$, $s = \frac{1}{20}$,

$$M(t) = 2 + \tanh t \quad \text{for all } t \in [0,+\infty[$$

and

$$f(t) = \frac{1}{1+t^2} \quad \text{for all } t \in \mathbb{R}.$$

Thus,

$$S(x^0,s) = S\left((0,0), \frac{1}{20}\right) = \left\{(x_1,x_2) \in \mathbb{R}^2; x_1^2 + x_2^2 < \frac{1}{400}\right\} \subset \Omega,$$

$m_0 = 1$, $m_1 = 3$, and f is a nonnegative and continuous function. By choosing $c = 100$, $d = 1$, $\alpha = 2$, $\nu = 1$ and $\xi = \pi$, we have $\alpha = 2 \in [0,3) = [0,p)$, $M^- = 1$, $M^+ = 3$ and $L = 58.18309\pi$. Therefore,

$$c = 100 > \sqrt[3]{58.18309\pi} > \sqrt[p]{L}d,$$

$$\max_{|u|\le c} F(u) = \max_{|u|\le 100} F(u) = \arctan(100) < \frac{\pi}{2} < \frac{100\pi}{36} = \frac{M^- F(d)\pi}{M^+ \operatorname{meas}(\Omega)},$$

$$F(d) = F(1) = \arctan(1) = \frac{\pi}{4} > 0$$

and

$$\left|F(u)\right| \leq |u|^2 + \pi \quad \text{for all } u \in \mathbb{R}.$$

Hence, by applying Theorem 3.8, there exist a nonempty open set $A \subset [0, +\infty)$ and a positive number R' with the following property: for every $\lambda \in A$ and every nonnegative continuous function $g : \Omega \times \mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the problem

$$\begin{cases} \Delta(|\Delta u|\Delta u) - (2 + \tanh(\int_\Omega |\nabla u|^3 \,dx))^2 \Delta_3 u + |u|u = \lambda \frac{1}{1+u^2} + \mu g(u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions whose norms in the space $\mathrm{W}^{2,3}(\Omega) \cap \mathrm{W}_0^{1,3}(\Omega)$ are less than R.

Remark 3.4 We point out that the same statements of the above given results can be obtained by considering the special case

$$M(t) = b_1 + b_2 t \quad \text{for } t \in [\iota, \kappa],$$

where b_1, b_2, ι and κ are positive numbers. In fact, we have

$$\widehat{M}(t) = \int_0^t (b_1 + b_2\xi)\,d\xi = \frac{(b_1 + b_2 t)^2}{2b_2} - \frac{b_1^2}{2b_2} \quad \text{for } t \in [\iota, \kappa],$$
$$m_1 = b_1 + b_2\iota \quad \text{and} \quad m_2 = b_1 + b_2\kappa.$$

Arguing as in the proof of Theorems 3.1 and 3.5, three weak solutions can be obtained.

4 Concluding remarks

Kirchhoff's model as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings takes into account the changes in length of the string produced by transverse vibrations. It received great attention only after Lions had proposed an abstract framework for the problem. On the other hand, fourth-order boundary value problems which describe the deformations of an elastic beam in an equilibrium state whose both ends are simply supported have been extensively studied in the literature. Here we have investigated the existence of multiple solutions for perturbed nonlocal fourth-order equations of Kirchhoff type under Navier boundary conditions. We have given some new criteria for guaranteeing that the perturbed fourth-order equations of Kirchhoff type possess at least three weak solutions by using a variational method and some critical point theorems due to Ricceri.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors contributed to each part of this study equally and read and approved the final version of the manuscript.

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