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Multiplicity of nontrivial solutions for a class of superquadratic elliptic systems near resonance

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Abstract

Multiple solutions for a class of superquadratic elliptic systems near resonance with high eigenvalues are obtained by using the nabra theorem due to Marino and Saccon in (Topol. Methods Nonlinear Anal. 17:213-237, 2001) and the linking theorem.

Keywords: elliptic systems; superquadratic condition; $(Ce)_c^*$ condition; nabra theorem; linking theorem

1 Introduction and main results

Consider the existence of weak solutions for the following elliptic system:

$$\begin{cases} -\Delta u = au + bv + F_v(x, u, v) & \text{in } \Omega, \\ -\Delta v = bu + av + F_u(x, u, v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded, connected open domain and a, b are real numbers. The nonlinearity $F \in C(\bar{\Omega} \times \mathbb{R}^2, \mathbb{R})$ has continuous derivatives $F_s(x, s, t), F_t(x, s, t)$ with respect to s and t for any $x \in \Omega$ and satisfies the following superquadratic condition:

$$\frac{F(x, s, t)}{|s|^2 + |t|^2} \rightarrow +\infty \quad \text{as } |s| + |t| \rightarrow \infty \text{ uniformly in } x \in \Omega. \quad (2)$$

Denote by $H_0^1(\Omega)$ the usual Hilbert space with the inner product and the norm

$$\langle u, v \rangle = \int_{\Omega} \nabla u \nabla v \, dx, \quad \|u\| = \left(\int_{\Omega} |\nabla u|^2 \, dx \right)^{1/2}$$

for any $u, v \in H_0^1(\Omega)$. By the Sobolev embedding theorem, the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for any $1 \leq p \leq 2^*$, and there exists a positive constant $C = C(p, N, \Omega)$ such that

$$\|u\|_p \leq C \|u\| \quad \text{for all } u \in H_0^1(\Omega), \quad (3)$$

where $2^* := \frac{2N}{N-2}$ is the Sobolev critical exponent and $\|\cdot\|_p$ denotes the norm of $L^p(\Omega)$. Moreover, the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact for any $1 \leq p < 2^*$. Let $H = H_0^1(\Omega) \times H_0^1(\Omega)$ be a Hilbert space with the norm

$$\|(u, v)\| = (\|u\|^2 + \|v\|^2)^{1/2} \quad \text{for any } (u, v) \in H.$$

The spectrum of $-\Delta$ in $H_0^1(\Omega)$ is at most a countable set, which we denote by

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_k < \dots,$$

where each λ_k is an isolated eigenvalue of finite multiplicity of m_k , with φ_k ($k = 1, 2, \dots$) the corresponding eigenfunctions, which will be taken orthogonal and normalized with $\|\varphi_k\| = 1$. It is well known that λ_1 is simple and isolated and φ_1 may be taken positive on Ω .

Define $B : H \times H \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} B((u, v), (\phi, \psi)) &= \int_{\Omega} \nabla u \nabla \psi \, dx + \int_{\Omega} \nabla v \nabla \phi \, dx \\ &\quad - a \int_{\Omega} (u\psi + v\phi) \, dx - b \int_{\Omega} (u\phi + v\psi) \, dx. \end{aligned}$$

As in [2], we define an orthogonal basis for H which diagonalizes B . It follows from [2] that there exist two sequences of eigenvalues for the following eigenvalue problem:

$$B((u, v), (\phi, \psi)) = \mu((u, v), (\phi, \psi)), \quad \forall (\phi, \psi) \in H,$$

that is,

$$\mu_i = \frac{\lambda_i - (a + b)}{\lambda_i} \quad \text{and} \quad \mu_{-i} = \frac{(a - b) - \lambda_i}{\lambda_i}, \quad i = 1, 2, \dots,$$

with the corresponding normalized eigenfunctions

$$\psi_{\pm i} = \frac{(\varphi_i, \pm \varphi_i)}{\sqrt{2}}, \quad i = 1, 2, \dots$$

Moreover, $\{\mu_{\pm i}\}$ and $\{\psi_{\pm i}\}$ have the following simple properties:

- (1) $\lim_{i \rightarrow +\infty} \mu_{\pm i} = \pm 1$;
- (2) If $a + b > 0$, $\mu_i = \frac{\lambda_i - (a+b)}{\lambda_i}$ converges to 1 from left, but if $a + b < 0$, $\mu_i = \frac{\lambda_i - (a+b)}{\lambda_i}$ converges to 1 from right;
- (3) If $a - b > 0$, $\mu_{-i} = \frac{(a-b) - \lambda_i}{\lambda_i}$ converges to -1 from right, but if $a - b < 0$, $\mu_{-i} = \frac{(a-b) - \lambda_i}{\lambda_i}$ converges to -1 from left.
- (4) Let $\mathbf{u} := (u, v) = \sum_{i \in \mathbb{Z} \setminus \{0\}} c_i \psi_i \in H$, then

$$\|\mathbf{u}\|^2 = \sum_{i \in \mathbb{Z} \setminus \{0\}} c_i^2, \quad B(\mathbf{u}, \mathbf{u}) = \sum_{i \in \mathbb{Z} \setminus \{0\}} \mu_i c_i^2, \quad \|\mathbf{u}\|^2 = \sum_{i \in \mathbb{Z} \setminus \{0\}} \lambda_{|i|}^{-1} c_i^2.$$

Multiplicity of nontrivial solutions for elliptic equations or elliptic systems near resonance was considered by many authors (see [1–15] and the references therein). For the

following two point boundary value problem

$$-u'' - \lambda u = f(x, u) + h(x), \quad u(0) = u(\pi) = 0,$$

if f is bounded and satisfies a sign condition, Mawhin and Schmitt [3] obtained the existence of at least three nontrivial solutions under the condition that the parameter λ is sufficiently close to λ_1 from left by using Ekeland’s variational principle and the mountain pass theorem, where λ_1 is the first eigenvalue of the corresponding linear problem. For the semilinear elliptic equation

$$\begin{cases} -\Delta u = \lambda u + f(x, u) + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

if f is sublinear at infinity and the parameter λ approaches λ_1 from the left, Ma et al. [4] proved the existence of at least three nontrivial solutions. Similar results were obtained for the quasilinear problem in [4] and [14], for the (p, q) elliptic systems in [6].

Many results were obtained for the case that the parameter λ approaches λ_k ($k \geq 2$) from left (see [1, 7–15] and the references therein). de Paiva and Massa [7] obtained the existence of two solutions for problem (4) under the condition that f is sublinear at infinity by using saddle point theory. In [2], Massa and Rossato obtained the existence of at least two solutions for problem (1) by using of the saddle point theorem and Galerkin approximation, where $F_s(x, s, t) = F_s(x, s)$ and $F_t(x, s, t) = F_t(x, t)$ are the sublinear nonlinearities. In [8], under a generalized Landesman-Lazer type condition, Ke and Tang obtained the existence of at least two solutions for a class of asymptotically linear noncooperative elliptic systems by using the saddle point theorem. On the other hand, if $F(x, t) = \int_0^t f(x, s) ds$ satisfies the (AR) condition, Mugnai [9] obtained three nontrivial solutions for problem (4) by using the ∇ -theorem due to Marino and Saccon in [10], and a similar result was obtained in [15] under the following superquadratic condition:

$$\frac{F(x, t)}{|t|^2} \rightarrow +\infty \quad \text{as } |t| \rightarrow \infty \text{ uniformly in } x \in \Omega.$$

The ∇ -theorem was widely applied to consider the multiplicity of nontrivial solutions for several other equations (see [11–14]). In [1], by using the notion of limit relative category, Marino and Saccon developed a ‘limit’ version of the ∇ -theorems and found multiple solutions for some noncooperative elliptic systems where the nonlinearity satisfies the global (AR) condition.

In this paper, influenced by [1] and [2], we consider the existence of multiple solutions for problem (1) near resonance at the higher eigenvalue, where $F(x, s, t)$ satisfies the superquadratic condition (2). Now we state some other conditions on $F(x, s, t)$.

(F1) There are $c_1 > 0, 1 < r < (N + 2)/(N - 2)$ such that

$$|F_s(x, s, t)| + |F_t(x, s, t)| \leq c_1(1 + |s|^r + |t|^r), \quad \forall (x, s, t) \in \Omega \times R^2;$$

(F2) $F(x, s, t) = o(|s|^2 + |t|^2)$ as $|s| + |t| \rightarrow 0$ uniformly in $x \in \Omega$;

(F3) There exist $N(r - 1)/2 < \beta < 2^*$, $c_2 > 0$ and $L > 0$ such that

$$F_s(x, s, t)s + F_t(x, s, t)t - 2F(x, s, t) > 0, \quad \forall x \in \Omega, (s, t) \neq (0, 0), \quad \text{and}$$

$$F_s(x, s, t)s + F_t(x, s, t)t - 2F(x, s, t) \geq c_2(|s|^\beta + |t|^\beta), \quad \forall x \in \Omega, |s|^2 + |t|^2 \geq L;$$

(F4) $F(x, s, t) \geq 0, \forall (x, s, t) \in \Omega \times R^2$.

The main results of the paper are the following theorems.

Theorem 1 *Assume that (2) and (F1), (F2), (F3), (F4) hold. Let λ_k be an eigenvalue of $-\Delta$ and λ_l be the first eigenvalue above $a - b$. $\forall \delta > 0, \exists \varepsilon_0 > 0$, if $\text{dist}(a - b, \sigma(-\Delta)) > \delta$ and $a + b \in (\lambda_k - \varepsilon_0, \lambda_k)$, then problem (1) has at least three nontrivial solutions.*

Theorem 2 *Assume that (2) and (F1), (F2), (F3), (F4) hold. Let λ_k be an eigenvalue of $-\Delta$ and λ_l be the first eigenvalue above $a + b$. $\forall \delta > 0, \exists \varepsilon_1 > 0$, if $\text{dist}(a + b, \sigma(-\Delta)) > \delta$ and $a - b \in (\lambda_k, \lambda_k + \varepsilon_1)$, then problem (1) has at least three nontrivial solutions.*

2 Proofs of theorems

In this section, we first recall two compactness conditions, i.e., the $(PS)_c^*$ condition, which was introduced by Li and Willem in [16], and the $(Ce)_c^*$ condition, which was introduced by Luan and Mao in [17] where they extended the local linking theorem of [16]. The $(Ce)_c^*$ condition is a version of the $(PS)_c^*$ condition and a generalization of the (Ce) condition of [18], and it can ensure a deformation lemma. And then, we recall the ∇ -theorem due to Marino and Saccon in [1], where the $(PS)_c^*$ condition is replaced by the $(Ce)_c^*$ condition.

Let W be a real Banach space with $W = W^1 \oplus W^2$. $W_0^i \subset W_1^i \subset W_2^i \subset \dots \subset W^i$ and $\bigcup_{n \in N} W_n^i$ is dense in $W^i, i = 1, 2$. For every multi-index $\alpha = (\alpha_1, \alpha_2) \in N^2, W_\alpha = W_{\alpha_1}^1 \oplus W_{\alpha_2}^2$, and it is well known that $\alpha \leq \beta \iff \alpha_1 \leq \beta_1, \alpha_2 \leq \beta_2$. A sequence $\{\alpha_n\} \in N^2$ is admissible if for every $\alpha \in N^2$, there is $m \in N$ such that $n \geq m \implies \alpha_n \geq \alpha$.

Definition 1 (see [17]) The functional $I \in C^1(W, R)$ satisfies the $(Ce)_c^*$ condition at the level $c \in R$, if $\{\alpha_n\}$ is admissible and $\{u_{\alpha_n}\}$ satisfies

$$u_{\alpha_n} \in W_{\alpha_n} \quad \text{and} \quad I_{\alpha_n}(u_{\alpha_n}) \rightarrow c, \quad (1 + \|u_{\alpha_n}\|)I'_{\alpha_n}(u_{\alpha_n}) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

$\{u_{\alpha_n}\}$ has a subsequence which converges to a critical point of I , where $I_{\alpha_n} = I|_{W_{\alpha_n}}$.

Let E be a real Hilbert space with $E = E^1 \oplus E^2 \oplus E^3$ and $\dim E^2 \geq 1$. Let $\{E_n\}$ be a sequence of finite dimension closed subspaces of E such that $E^2 \subset E_n, E_n \subset E_{n+1}$, and let $\bigcup_{n \in N} E_n$ be dense in E . Let X be a closed subspace of E and P_X denote the orthogonal projection onto X , and for any $u \in E$, let $[u] = \{tu : t \in R\}$.

Definition 2 (see [1]) Let X be a closed subspace of E such that $P_X P_{E_n} = P_{E_n} P_X$ for all n , and let c be a real number. The functional $I \in C^1(E, R)$ satisfies the condition $\nabla^*(X, c)$ with respect to $(E_n)_n$ if there exists $\gamma > 0$ such that

$$\liminf_{n \rightarrow \infty} \{ \|P_{E_n} P_{X \oplus [u]} \text{grad } I(u)\| : u \in E_n, \text{dist}(u, X) < \gamma, |I(u) - c| < \gamma \} > 0.$$

The condition $\nabla^*(X, c)$ implies that there are no critical points u in X with $I(u) = c$, with some uniformity. Moreover, the condition $\nabla^*(X, c)$ is equivalent to the following conditions:

- (a) $I|_X$ has no critical point u in X with $I(u) = c$;
- (b) If (h_n) is a sequence in N such that $h_n \rightarrow \infty$, $\{u_n\}$ is a sequence in E such that $u_n \in E_{h_n}$ for all n , $\text{dist}(u_n, X) \rightarrow 0$, $I(u_n) \rightarrow c$ and $P_{E_n} P_{X \oplus [u_n]} \text{grad} I(u_n) \rightarrow 0$ as $n \rightarrow \infty$, then (u_n) has a convergent subsequence.

Theorem 3 (see [1]) *Let E be a Hilbert space and $E^i, i = 1, 2, 3$, be three subspaces of E such that $E = E^1 \oplus E^2 \oplus E^3$ with $1 \leq \dim E^2 < +\infty$, and $I \in C^{1,1}(E, R)$. Define*

$$T = \{u \in E^1 \oplus E^2 : \|P_{E^1} u\| \leq R, R' \leq \|P_{E^2} u\| \leq R''\} \quad \text{and} \quad \Gamma = \partial_{E^1 \oplus E^2} T,$$

$$S_\rho = \{u \in E^2 \oplus E^3 : \|u\| = \rho\} \quad \text{and} \quad B_\rho = \{u \in E^2 \oplus E^3 : \|u\| \leq \rho\},$$

where $R > 0$ and $0 \leq R' < \rho < R''$. Especially, if $R' = 0$, we have

$$\Gamma = \{u \in E^1 : \|u\| \leq R\} \cup \{u \in E^1 \oplus E^2 : \|P_{E^1} u\| \leq R, \|P_{E^2} u\| = R''\}$$

$$\cup \{u \in E^1 \oplus E^2 : \|P_{E^1} u\| = R, \|P_{E^2} u\| \leq R''\}.$$

Assume that

$$\sup I(\Gamma) < \inf I(S_\rho) = \alpha_0 \quad \text{and} \quad \beta_0 = \sup I(T) < +\infty.$$

If the condition $\nabla^*(E^1 \oplus E^3, c)$ and the $(Ce)_c^*$ condition hold at any $c \in [\alpha_0, \beta_0]$, then I has at least two critical points in $I^{-1}([\alpha_0, \beta_0])$.

Define the functional $J : H \rightarrow R$ as follows:

$$J(u, v) = \int_\Omega \nabla u \nabla v \, dx - a \int_\Omega uv \, dx - \frac{b}{2} \int_\Omega (u^2 + v^2) \, dx - \int_\Omega F(x, u, v) \, dx$$

$$= \frac{1}{2} B((u, v), (u, v)) - \int_\Omega F(x, u, v) \, dx. \tag{5}$$

From the subcritical growth condition (F1), it is easy to verify that $J \in C^1(H, R)$ and

$$\langle J'(u, v), (\phi, \psi) \rangle = \int_\Omega \nabla \phi \nabla v \, dx - a \int_\Omega \phi v \, dx - b \int_\Omega u \phi \, dx - \int_\Omega F_u(x, u, v) \phi \, dx$$

$$+ \int_\Omega \nabla u \nabla \psi \, dx - a \int_\Omega u \psi \, dx - b \int_\Omega v \psi \, dx - \int_\Omega F_v(x, u, v) \psi \, dx$$

for all $(\phi, \psi) \in H$. It is well known that finding weak solutions of problem (1) is equivalent to finding critical points of the functional J in H . Since Theorem 2 is similar to Theorem 1, we will be devoted to the proof of Theorem 1. Two solutions are obtained by Theorem 3, and the existence of the third solution is proved by the classical linking theorem (see [19, 20]) and Galerkin approximation. Let λ_l be the first eigenvalue above $a - b$ such that $\text{dist}(a - b, \sigma(-\Delta)) > \delta$ for given $\delta > 0$ and $a + b \in (\lambda_{k-1}, \lambda_k)$. From the definitions of $\mu_{\pm i}$, we obtain that

$$\mu_i < 0, \quad i = 1, 2, \dots, k - 1 \quad \text{and} \quad \mu_i > 0, \quad i = k, k + 1, \dots,$$

and

$$\mu_{-i} > 0, \quad i = 1, 2, \dots, l-1 \quad \text{and} \quad \mu_{-i} < 0, \quad i = l, l+1, \dots$$

Now, define

$$H^- = \overline{\text{span}\{\psi_i : \mu_i < 0, i = \pm 1, \pm 2, \dots\}}, \quad H^+ = \overline{\text{span}\{\psi_i : \mu_i > 0, i = \pm 1, \pm 2, \dots\}},$$

then $H = H^- \oplus H^+$. Especially, set

$$Y = \text{span}\{\psi_i : \mu_i = \mu_k\}, \quad Y^\perp = \overline{\text{span}\{\psi_i : \mu_i > 0 \text{ and } \mu_i \neq \mu_k\}},$$

so that $\dim Y < \infty$ and $H^+ = Y \oplus Y^\perp$. Moreover, for any $\mathbf{u} = \mathbf{u}^- \oplus \mathbf{u}^+ \in H^- \oplus H^+$,

$$B(\mathbf{u}, \mathbf{u}) = B(\mathbf{u}^-, \mathbf{u}^-) + B(\mathbf{u}^+, \mathbf{u}^+). \tag{6}$$

Lemma 1 (see [2]) *Let $a + b \notin \sigma(-\Delta)$ and λ_k be the first eigenvalue above $a + b$, of multiplicity m_k , and λ_l be the first eigenvalue above $a - b$. If $\text{dist}(a - b, \sigma(-\Delta)) > \delta > 0$, then there exists a constant $K = K(a + b, \delta) > 0$ such that*

$$\begin{aligned} B((u, v), (u, v)) &\leq -K(\|u\|^2 + \|v\|^2), \quad \forall (u, v) \in H^-, \\ B((u, v), (u, v)) &\geq K(\|u\|^2 + \|v\|^2), \quad \forall (u, v) \in H^+. \end{aligned}$$

Lemma 2 *Assume that (2), (F1) and (F3) hold. For any real number c , the functional J satisfies the $(Ce)_c^*$ condition.*

Proof Let $W^1 = H^-$, $W^2 = H^+$, then $H = H^- \oplus H^+ = W^1 \oplus W^2$. Let $\{\alpha_n\}$ be admissible and $\{(u_{\alpha_n}, v_{\alpha_n})\}$ be a $(Ce)_c^*$ sequence, that is,

$$(u_{\alpha_n}, v_{\alpha_n}) \in H_{\alpha_n}, \quad J_{\alpha_n}(u_{\alpha_n}, v_{\alpha_n}) \rightarrow c, \quad (1 + \|u_{\alpha_n}\| + \|v_{\alpha_n}\|)J'_{\alpha_n}(u_{\alpha_n}, v_{\alpha_n}) \rightarrow 0 \tag{7}$$

as $n \rightarrow \infty$.

We first proved that $\{(u_{\alpha_n}, v_{\alpha_n})\}$ is bounded in H . Arguing by contradiction, we suppose that $\|(u_{\alpha_n}, v_{\alpha_n})\| = (\|u_{\alpha_n}\|^2 + \|v_{\alpha_n}\|^2)^{1/2} \rightarrow \infty$ as $n \rightarrow \infty$. Define

$$\tilde{u}_{\alpha_n} = \frac{u_{\alpha_n}}{\|(u_{\alpha_n}, v_{\alpha_n})\|}, \quad \tilde{v}_{\alpha_n} = \frac{v_{\alpha_n}}{\|(u_{\alpha_n}, v_{\alpha_n})\|},$$

then $\{(\tilde{u}_{\alpha_n}, \tilde{v}_{\alpha_n})\}$ is bounded in H , i.e.,

$$\|\tilde{u}_{\alpha_n}\|^2 + \|\tilde{v}_{\alpha_n}\|^2 = 1 \quad \text{for all } n.$$

From (F3) and the continuity of F , there exists a constant M_1 such that

$$F_s(x, s, t)s + F_t(x, s, t)t - 2F(x, s, t) \geq c_2(|s|^\beta + |t|^\beta) - M_1 \tag{8}$$

for any $(x, s, t) \in \Omega \times R^2$. Hence, one gets

$$\begin{aligned} & 2J_{\alpha_n}(u_{\alpha_n}, v_{\alpha_n}) - \langle J'_{\alpha_n}(u_{\alpha_n}, v_{\alpha_n}), (u_{\alpha_n}, v_{\alpha_n}) \rangle \\ &= \int_{\Omega} (F_s(x, u_{\alpha_n}, v_{\alpha_n})u_{\alpha_n} + F_t(x, u_{\alpha_n}, v_{\alpha_n})v_{\alpha_n} - 2F(x, u_{\alpha_n}, v_{\alpha_n})) \, dx \\ &\geq c_2 \left(\int_{\Omega} |u_{\alpha_n}|^{\beta} \, dx + \int_{\Omega} |v_{\alpha_n}|^{\beta} \, dx \right) - M_1|\Omega|, \end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of the set Ω . From (7) and the above inequality, there is a positive constant c_3 such that

$$\|u_{\alpha_n}\|_{\beta}^{\beta} + \|v_{\alpha_n}\|_{\beta}^{\beta} \leq c_3 \quad \text{for any } n. \tag{9}$$

Let $(u_{\alpha_n}, v_{\alpha_n}) = (u_{\alpha_n}^-, v_{\alpha_n}^-) + (u_{\alpha_n}^+, v_{\alpha_n}^+) \in H_{\alpha_n} = H_{\alpha_n}^- \oplus H_{\alpha_n}^+$ and note that

$$\frac{N}{2}(r-1) < \beta \quad \text{and} \quad \frac{N}{2}(r-1) < \frac{2N}{N+2}r.$$

We first consider the case $\frac{2N}{N+2}r < \beta$, by Hölder’s inequality, (F1), (3) and (9), we have

$$\begin{aligned} & \left| \int_{\Omega} F_s(x, u_{\alpha_n}, v_{\alpha_n})u_{\alpha_n}^+ \, dx \right| \\ &\leq c_1 \int_{\Omega} (|u_{\alpha_n}^+| + |u_{\alpha_n}|^r |u_{\alpha_n}^+| + |v_{\alpha_n}|^r |u_{\alpha_n}^+|) \, dx \\ &\leq c_1 \|u_{\alpha_n}^+\|_1 + c_1 \left(\int_{\Omega} |u_{\alpha_n}|^{r \frac{\beta}{r}} \, dx \right)^{\frac{r}{\beta}} \cdot \left(\int_{\Omega} |u_{\alpha_n}^+|^{\frac{\beta}{\beta-r}} \, dx \right)^{\frac{\beta-r}{\beta}} \\ &\quad + c_1 \left(\int_{\Omega} |v_{\alpha_n}|^{r \frac{\beta}{r}} \, dx \right)^{\frac{r}{\beta}} \cdot \left(\int_{\Omega} |u_{\alpha_n}^+|^{\frac{\beta}{\beta-r}} \, dx \right)^{\frac{\beta-r}{\beta}} \\ &\leq c_4 \|u_{\alpha_n}^+\|. \end{aligned}$$

Similarly, we obtain

$$\left| \int_{\Omega} F_t(x, u_{\alpha_n}, v_{\alpha_n})v_{\alpha_n}^+ \, dx \right| \leq c_4 \|v_{\alpha_n}^+\|,$$

where c_4 is a positive constant. Hence, it follows from the above two inequalities that

$$\begin{aligned} & \langle J'_{\alpha_n}(u_{\alpha_n}, v_{\alpha_n}), (u_{\alpha_n}^+, v_{\alpha_n}^+) \rangle \\ &= B((u_{\alpha_n}, v_{\alpha_n}), (u_{\alpha_n}^+, v_{\alpha_n}^+)) - \int_{\Omega} (F_s(x, u_{\alpha_n}, v_{\alpha_n})u_{\alpha_n}^+ + F_t(x, u_{\alpha_n}, v_{\alpha_n})v_{\alpha_n}^+) \, dx \\ &\geq K(\|u_{\alpha_n}^+\|^2 + \|v_{\alpha_n}^+\|^2) - c_4(\|u_{\alpha_n}^+\| + \|v_{\alpha_n}^+\|). \end{aligned} \tag{10}$$

For the case $\frac{N}{2}(r-1) < \beta < \frac{N}{2}(r-1)$, let

$$\gamma = \frac{2Nr - (N+2)\beta}{2N - (N-2)\beta} \in (0, 1) \quad \text{and} \quad p = \frac{\beta}{r - \gamma} > 1.$$

By Hölder’s inequality, (F1), (3) and (9), we have

$$\begin{aligned} & \left| \int_{\Omega} F_s(x, u_{\alpha_n}, v_{\alpha_n}) u_{\alpha_n}^+ dx \right| \\ & \leq c_1 \int_{\Omega} (|u_{\alpha_n}^+| + |u_{\alpha_n}|^{\beta/p} |u_{\alpha_n}|^{\gamma} |u_{\alpha_n}^+| + |v_{\alpha_n}|^{\beta/p} |v_{\alpha_n}|^{\gamma} |u_{\alpha_n}^+|) dx \\ & \leq c_1 \|u_{\alpha_n}^+\|_1 + c_1 \left(\int_{\Omega} |u_{\alpha_n}|^{\beta} dx \right)^{\frac{1}{p}} \cdot \left(\int_{\Omega} |u_{\alpha_n}|^{2^*} dx \right)^{\frac{\gamma}{2^*}} \cdot \left(\int_{\Omega} |u_{\alpha_n}^+|^{2^*} dx \right)^{\frac{1}{2^*}} \\ & \quad + c_1 \left(\int_{\Omega} |v_{\alpha_n}|^{\beta} dx \right)^{\frac{1}{p}} \cdot \left(\int_{\Omega} |v_{\alpha_n}|^{2^*} dx \right)^{\frac{\gamma}{2^*}} \cdot \left(\int_{\Omega} |u_{\alpha_n}^+|^{2^*} dx \right)^{\frac{1}{2^*}} \\ & \leq c_5 \|u_{\alpha_n}^+\| (\|u_{\alpha_n}\|^{\gamma} + \|v_{\alpha_n}\|^{\gamma}). \end{aligned}$$

Similarly, we obtain

$$\left| \int_{\Omega} F_t(x, u_{\alpha_n}, v_{\alpha_n}) v_{\alpha_n}^+ dx \right| \leq c_5 \|v_{\alpha_n}^+\| (\|u_{\alpha_n}\|^{\gamma} + \|v_{\alpha_n}\|^{\gamma}),$$

where c_5 is a positive constant. Therefore, from the above two inequalities, one has

$$\begin{aligned} & \langle J'_{\alpha_n}(u_{\alpha_n}, v_{\alpha_n}), (u_{\alpha_n}^+, v_{\alpha_n}^+) \rangle \\ & = B((u_{\alpha_n}, v_{\alpha_n}), (u_{\alpha_n}^+, v_{\alpha_n}^+)) - \int_{\Omega} (F_s(x, u_{\alpha_n}, v_{\alpha_n}) u_{\alpha_n}^+ + F_t(x, u_{\alpha_n}, v_{\alpha_n}) v_{\alpha_n}^+) dx \\ & \geq K(\|u_{\alpha_n}^+\|^2 + \|v_{\alpha_n}^+\|^2) - c_5(\|u_{\alpha_n}^+\| + \|v_{\alpha_n}^+\|)(\|u_{\alpha_n}\|^{\gamma} + \|v_{\alpha_n}\|^{\gamma}). \end{aligned} \tag{11}$$

Dividing (10) (or (11)) by $\|u_{\alpha_n}\|^2 + \|v_{\alpha_n}\|^2$, it follows from (7) that

$$\|\tilde{u}_{\alpha_n}^+\| + \|\tilde{v}_{\alpha_n}^+\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{12}$$

Similarly, we obtain

$$\|\tilde{u}_{\alpha_n}^-\| + \|\tilde{v}_{\alpha_n}^-\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{13}$$

From (12) and (13), we have the following contradiction:

$$1 = \|\tilde{u}_{\alpha_n}\| + \|\tilde{v}_{\alpha_n}\| \leq \|\tilde{u}_{\alpha_n}^-\| + \|\tilde{v}_{\alpha_n}^-\| + \|\tilde{u}_{\alpha_n}^+\| + \|\tilde{v}_{\alpha_n}^+\| \rightarrow 0$$

as $n \rightarrow \infty$. Hence, $\{(u_{\alpha_n}, v_{\alpha_n})\}$ must be bounded.

It follows that there is a subsequence of $\{(u_{\alpha_n}, v_{\alpha_n})\}$, still denoted by $\{(u_{\alpha_n}, v_{\alpha_n})\}$, and $(u, v) \in H$ such that $(u_{\alpha_n}, v_{\alpha_n}) \rightharpoonup (u, v)$ weakly in H , $(u_{\alpha_n}, v_{\alpha_n}) \rightarrow (u, v)$ strongly in $L^p(\Omega) \times L^p(\Omega)$ for any $1 \leq p < 2^*$, and $(u_{\alpha_n}(x), v_{\alpha_n}(x)) \rightarrow (u(x), v(x))$ for a.e. $x \in \Omega$. Therefore, from (7), we have

$$\lim_{n \rightarrow \infty} \langle J'(u_{\alpha_n}, v_{\alpha_n}), (v_{\alpha_n} - v, u_{\alpha_n} - u) \rangle = 0. \tag{14}$$

Recalling the boundedness of $\{(u_{\alpha_n}, v_{\alpha_n})\}$, Hölder’s inequality and (F1), we obtain

$$\begin{aligned} \left| \int_{\Omega} (v_{\alpha_n}(v_{\alpha_n} - v) + u_{\alpha_n}(u_{\alpha_n} - u)) \, dx \right| &\leq \|v_{\alpha_n}\|_2 \|v_{\alpha_n} - v\|_2 + \|u_{\alpha_n}\|_2 \|u_{\alpha_n} - u\|_2 \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \tag{15}$$

$$\begin{aligned} \left| \int_{\Omega} (u_{\alpha_n}(v_{\alpha_n} - v) + v_{\alpha_n}(u_{\alpha_n} - u)) \, dx \right| &\leq \|u_{\alpha_n}\|_2 \|v_{\alpha_n} - v\|_2 + \|v_{\alpha_n}\|_2 \|u_{\alpha_n} - u\|_2 \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty, \end{aligned} \tag{16}$$

$$\begin{aligned} \left| \int_{\Omega} F_s(x, u_{\alpha_n}, v_{\alpha_n})(v_{\alpha_n} - v) \, dx \right| &\leq c_1 \int_{\Omega} (1 + |u_{\alpha_n}|^r + |v_{\alpha_n}|^r) |v_{\alpha_n} - v| \, dx \\ &\leq c_1 \|v_{\alpha_n} - v\|_1 + c_1 (\|u_{\alpha_n}\|_{\beta}^r + \|v_{\alpha_n}\|_{\beta}^r) \|v_{\alpha_n} - v\|_{\frac{\beta}{\beta-r}} \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \tag{17}$$

Similarly, we have

$$\left| \int_{\Omega} F_t(x, u_{\alpha_n}, v_{\alpha_n})(u_{\alpha_n} - u) \, dx \right| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{18}$$

From (14), (15), (16), (17) and (18), it follows that

$$\lim_{n \rightarrow \infty} \int_{\Omega} (\nabla u_{\alpha_n} \nabla(u_{\alpha_n} - u) + \nabla v_{\alpha_n} \nabla(v_{\alpha_n} - v)) \, dx = 0. \tag{19}$$

Similarly, we obtain

$$\lim_{n \rightarrow \infty} \int_{\Omega} (\nabla u \nabla(u_{\alpha_n} - u) + \nabla v \nabla(v_{\alpha_n} - v)) \, dx = 0. \tag{20}$$

From (19) and (20), we get

$$\lim_{n \rightarrow \infty} \|u_{\alpha_n} - u\| + \|v_{\alpha_n} - v\| = 0,$$

which implies that $(u_{\alpha_n}, v_{\alpha_n}) \rightarrow (u, v)$ strongly in H . □

Let $H_n^- \subset H^-$, $H_n^+ \subset Y^\perp$ for any n , $H_n = H_n^- \oplus Y \oplus H_n^+$ with $\dim H_n < \infty$, $H_n \subset H_{n+1}$, $\bigcup_{n \in \mathbb{N}} H_n$ is dense in H . We will prove that the condition $\nabla^*(H^- \oplus Y^\perp, c)$ holds for any real number c .

Lemma 3 *Assume that (2), (F1) and (F3) hold. For any real number c and any sequence $\{(u_n, v_n)\}$ such that*

$$\begin{aligned} (u_n, v_n) \in H_n \quad \text{for any } n, \quad J(u_n, v_n) \rightarrow c, \\ P_{H^- \oplus Y^\perp \oplus \{(u_n, v_n)\}} P_{H_n} J'(u_n, v_n) \rightarrow 0 \end{aligned} \tag{21}$$

as $n \rightarrow \infty$, there is a subsequence of $\{(u_n, v_n)\}$ which converges to a point (u, v) such that $J(u, v) = c$ and $P_{H^- \oplus Y^\perp \oplus \{(u, v)\}} J'(u, v) = 0$.

Proof We first claim that $\{(u_n, v_n)\}$ is bounded in H . Arguing by contradiction, we suppose that $\|(u_n, v_n)\| \rightarrow \infty$ as $n \rightarrow \infty$, define $\tilde{u}_n = u_n / \|(u_n, v_n)\|$, $\tilde{v}_n = v_n / \|(u_n, v_n)\|$, then $\{(\tilde{u}_n, \tilde{v}_n)\}$ is bounded in H , that is,

$$\|\tilde{u}_n\|^2 + \|\tilde{v}_n\|^2 = 1 \quad \text{for all } n.$$

Hence there is a subsequence of $\{(\tilde{u}_n, \tilde{v}_n)\}$, still denoted by $\{(\tilde{u}_n, \tilde{v}_n)\}$, and $(\tilde{u}, \tilde{v}) \in H$ such that $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (\tilde{u}, \tilde{v})$ weakly in H , $(\tilde{u}_n, \tilde{v}_n) \rightarrow (\tilde{u}, \tilde{v})$ strongly in $L^p(\Omega) \times L^p(\Omega)$, and $(\tilde{u}_n(x), \tilde{v}_n(x)) \rightarrow (\tilde{u}(x), \tilde{v}(x))$ for a.e. $x \in \Omega$. From (8), we obtain

$$\begin{aligned} & 2J(u_n, v_n) - \langle J'(u_n, v_n), (u_n, v_n) \rangle \\ &= \int_{\Omega} (F_s(x, u_n, v_n)u_n + F_t(x, u_n, v_n)v_n - 2F(x, u_n, v_n)) \, dx \\ &\geq c_2 \left(\int_{\Omega} |u_n|^\beta \, dx + \int_{\Omega} |v_n|^\beta \, dx \right) - M_1|\Omega|. \end{aligned}$$

Dividing the above expression by $\|(u_n, v_n)\|^\beta$, it follows from (21) that

$$\int_{\Omega} (|\tilde{u}_n|^\beta + |\tilde{v}_n|^\beta) \, dx \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

which implies that $(\tilde{u}, \tilde{v}) = (0, 0)$, that is, $(\tilde{u}_n, \tilde{v}_n) \rightharpoonup (0, 0)$ weakly in H . Similar to the proof of Lemma 2, we can obtain that $(\tilde{u}_n, \tilde{v}_n) \rightarrow (0, 0)$ strongly in H , which is a contradiction to $\|\tilde{u}_n\|^2 + \|\tilde{v}_n\|^2 = 1$. Hence $\{(u_n, v_n)\}$ is bounded. As was already noted in the proof of Lemma 2, the same proof implies that (u_n, v_n) converges in H . \square

Lemma 4 *If (F1), (F2) and (F3) hold, then there is $\varepsilon' > 0$ such that J has no critical point (u, v) with $0 < |J(u, v)| < \varepsilon'$.*

Proof We first claim that $(0, 0)$ is an isolated critical point for J . Actually, from (F1) and (F2), for any $\varepsilon > 0$, there is $M_2 = M_2(\varepsilon) > 0$ such that

$$F(x, s, t) \leq \frac{\varepsilon}{2}(s^2 + t^2) + M_2(|s|^{r+1} + |t|^{r+1}), \quad \forall (x, s, t) \in \Omega \times \mathbb{R}^2,$$

hence, we have

$$\left| \int_{\Omega} F(x, u, v) \, dx \right| \leq \frac{\varepsilon}{2}(\|u\|_2^2 + \|v\|_2^2) + M_2(\|u\|_{r+1}^{r+1} + \|v\|_{r+1}^{r+1}), \quad \forall (u, v) \in H,$$

which together with Lemma 1 and (3) implies that for any $(u, v) \in H^+$,

$$\begin{aligned} J(u, v) &= \frac{1}{2}B((u, v), (u, v)) - \int_{\Omega} F(x, u, v) \, dx \\ &\geq \frac{K - C^2\varepsilon}{2}(\|u\|^2 + \|v\|^2) - C^{r+1}M_2(\|u\|^{r+1} + \|v\|^{r+1}), \end{aligned} \tag{22}$$

and for any $(u, v) \in H^-$,

$$\begin{aligned} J(u, v) &= \frac{1}{2}B((u, v), (u, v)) - \int_{\Omega} F(x, u, v) \, dx \\ &\leq \frac{-K + C^2\varepsilon}{2}(\|u\|^2 + \|v\|^2) + C^{r+1}M_2(\|u\|^{r+1} + \|v\|^{r+1}). \end{aligned}$$

Since $r + 1 > 2$, let $\varepsilon < \frac{K}{2C^2}$, there exists a positive constant R_0 such that

$$J(u, v) > 0 \quad \text{for any } (u, v) \in H^+ \text{ and } 0 < \|(u, v)\| \leq R_0,$$

$$J(u, v) < 0 \quad \text{for any } (u, v) \in H^- \text{ and } 0 < \|(u, v)\| \leq R_0,$$

which implies that $(0, 0)$ is an isolated critical point for J .

Moreover, $(0, 0)$ is the only critical point (u, v) of J with $J(u, v) = 0$. In fact, if (u, v) is such a point

$$0 = \langle J'(u, v), (u, v) \rangle = 2J(u, v) - \int_{\Omega} (F_s(x, u, v)u + F_t(x, u, v)v - 2F(x, u, v)) \, dx,$$

we obtain

$$\int_{\Omega} (F_s(x, u, v)u + F_t(x, u, v)v - 2F(x, u, v)) \, dx = 0,$$

from (F3), it follows that $(u, v) = (0, 0)$.

Now we will finish the argument by contradiction. If there is a sequence $\{(u_n, v_n)\}$ such that $J'(u_n, v_n) = 0$, $J(u_n, v_n) \neq 0$ for any n and $J(u_n, v_n) \rightarrow 0$ as $n \rightarrow \infty$. Similar to the proof of Lemma 2, we have that (u_n, v_n) converges to a critical point (u, v) with $J(u, v) = 0$, which implies that $(u_n, v_n) \rightarrow (0, 0)$ as $n \rightarrow \infty$, but $(0, 0)$ is an isolated critical point for J . Hence we obtain a contradiction. \square

Next, we will prove that the functional J satisfies the geometry of Theorem 3. We introduce some notations for the later convenience. For $R, \rho > 0$, let

$$\Gamma_R = \{ \mathbf{u} \in H^- : \|\mathbf{u}\| \leq R \} \cup \{ \mathbf{u} \in H^- \oplus Y : \|\mathbf{u}\| = R \},$$

$$T_R = \{ \mathbf{u} \in H^- \oplus Y : \|\mathbf{u}\| \leq R \} \quad \text{and} \quad \Gamma_R = \partial T_R,$$

$$S_\rho = \{ \mathbf{u} \in H^+ : \|\mathbf{u}\| = \rho \}, \quad B_\rho = \{ \mathbf{u} \in H^+ : \|\mathbf{u}\| \leq \rho \}.$$

Lemma 5 *Suppose that (2), (F1), (F2), (F3) and (F4) are satisfied. For any $a + b \in (\lambda_{k-1}, \lambda_k)$, there exist $R > \rho > 0$ such that*

$$0 = \sup J(\Gamma_R) < \inf J(S_\rho).$$

Proof From (5), Lemma 1 and (F4), for any $(u, v) \in H^-$, we have

$$\begin{aligned} J(u, v) &= \frac{1}{2}B((u, v), (u, v)) - \int_{\Omega} F(x, u, v) \, dx \\ &\leq -\frac{K}{2}(\|u\|^2 + \|v\|^2) \\ &\leq 0. \end{aligned} \tag{23}$$

In view of (2) and the continuity of F , for any $c_6 > 0$, there is $M_3 > 0$ such that

$$F(x, s, t) \geq \frac{c_6}{2}(s^2 + t^2) - M_3, \quad \forall (x, s, t) \in \Omega \times \mathbb{R}^2,$$

which implies that

$$\int_{\Omega} F(x, u, v) \, dx \geq \frac{c_6}{2} (\|u\|_2^2 + \|v\|_2^2) - M_3|\Omega|, \quad \forall (u, v) \in H, \tag{24}$$

which together with (6) implies that for any $(u, v) = (u^-, v^-) \oplus (u^{\sim}, v^{\sim}) \in H^- \oplus Y$ and $a + b \in (\lambda_{k-1}, \lambda_k)$,

$$\begin{aligned} J(u, v) &\leq \frac{1}{2}B((u^-, v^-), (u^-, v^-)) + \frac{1}{2}B((u^{\sim}, v^{\sim}), (u^{\sim}, v^{\sim})) \\ &\quad - \frac{c_6}{2} (\|u\|_2^2 + \|v\|_2^2) + M_3|\Omega| \\ &\leq -\frac{K}{2} (\|u^-\|^2 + \|v^-\|^2) + \frac{1}{2}\mu_k (\|u^{\sim}\|^2 + \|v^{\sim}\|^2) \\ &\quad - \frac{c_6}{2} (\|u^-\|^2 + \|v^-\|^2) - \frac{c_6}{2\lambda_k} (\|u^{\sim}\|^2 + \|v^{\sim}\|^2) + M_3|\Omega| \\ &\leq -c_7 (\|u^-\|^2 + \|v^-\|^2) + \frac{\lambda_k - (a + b) - c_6}{2\lambda_k} (\|u^{\sim}\|^2 + \|v^{\sim}\|^2) + M_3|\Omega|. \end{aligned}$$

Let $c_6 > \lambda_k - (a + b)$, the above expression implies that

$$J(u, v) \rightarrow -\infty \quad \text{as } \|(u, v)\| \rightarrow \infty. \tag{25}$$

From (22), (23) and (25), there are two constants $R > \rho > 0$ such that

$$\sup J(\Gamma_R) < \inf J(S_\rho). \tag{26}$$

Lemma 6 *If (F4) holds, then for $R > 0$ in Lemma 5 and for any $\varepsilon'' > 0$, there exists $\varepsilon_0 > 0$ such that for any $a + b \in (\lambda_k - \varepsilon_0, \lambda_k)$, we have*

$$\sup J(T_R) < \varepsilon''.$$

Proof For any $(u, v) = (u^-, v^-) \oplus (u^{\sim}, v^{\sim}) \in H^- \oplus Y$, in view of Lemma 1, (F4) and the fact that $a + b < \lambda_k$, we have

$$\begin{aligned} J(u, v) &\leq \frac{1}{2}B((u^-, v^-), (u^-, v^-)) + \frac{1}{2}B((u^{\sim}, v^{\sim}), (u^{\sim}, v^{\sim})) \\ &\leq -\frac{K}{2} (\|u^-\|^2 + \|v^-\|^2) + \frac{\lambda_k - (a + b)}{2\lambda_k} (\|u^{\sim}\|^2 + \|v^{\sim}\|^2). \end{aligned}$$

Let $\varepsilon_0 = 2\lambda_k\varepsilon''/R^2$, it is easy to see that the conclusion holds. □

Proof of Theorem 1 The argument is finished by two steps.

(a) There are two critical points (u_1, v_1) and (u_2, v_2) of the functional J , which satisfy

$$\inf_{(u,v) \in S_\rho} J(u, v) \leq J(u_i, v_i) \leq \sup_{(u,v) \in T_R} J(u, v) \quad \text{for } i = 1, 2.$$

Let $\varepsilon \in (0, \varepsilon')$, $\alpha_0 = \inf J(S_\rho)$ and $\beta_0 = \sup J(T_R)$, where ε' is defined in Lemma 4. It follows from Lemmas 3 and 6 that $J|_{H^- \oplus Y^\perp}$ has no critical levels between α_0 and β_0 , hence the

functional J satisfies the condition $\nabla^*(H^- \oplus Y^\perp, c)$ for any $c \in [\alpha_0, \beta_0]$. Lemma 2 shows that the $(Ce)_c^*$ condition holds true for any $c \in [\alpha_0, \beta_0]$. Hence two critical points are obtained from Theorem 3.

(b) There is the third critical point, which is different from (u_1, v_1) and (u_2, v_2) . Let $(\varphi_k, \varphi_k) \in H^+$ be fixed, define

$$\begin{aligned} S_{\rho_1} &= \{(u, v) \in H^+ : \|(u, v)\| = \rho_1\}, \\ B_{R_1} &= \{(u, v) + r(\varphi_k, \varphi_k) : r \geq 0, (u, v) \in H^-, \|(u + r\varphi_k, v + r\varphi_k)\| \leq R_1\}, \\ \partial B_{R_1} &= \{(u, v) + r(\varphi_k, \varphi_k) : r \geq 0, (u, v) \in H^-, \|(u + r\varphi_k, v + r\varphi_k)\| = R_1\} \\ &\quad \cup \{(u, v) : (u, v) \in H^-, \|(u, v)\| \leq R_1\}. \end{aligned}$$

For any $(u, v) = (u^-, v^-) + (r\varphi_k, r\varphi_k)$, $r \geq 0$, from (24), we get

$$\begin{aligned} J(u, v) &\leq \frac{1}{2}B((u^-, v^-), (u^-, v^-)) + \frac{r^2}{2}B((\varphi_k, \varphi_k), (\varphi_k, \varphi_k)) \\ &\quad - \frac{c_6}{2}(\|u + r\varphi_k\|_2^2 + \|v + r\varphi_k\|_2^2) + M_3|\Omega| \\ &\leq -\frac{K}{2}(\|u^-\|^2 + \|v^-\|^2) + r^2\mu_k\|\varphi_k\|^2 \\ &\quad - \frac{c_6}{2}(\|u^-\|_2^2 + \|v^-\|_2^2) - \frac{r^2c_6}{\lambda_k}\|\varphi_k\|^2 + M_3|\Omega| \\ &\leq -c_7(\|u^-\|^2 + \|v^-\|^2) + \frac{\lambda_k\mu_k - c_6}{\lambda_k}r^2\|\varphi_k\|^2 + M_3|\Omega|. \end{aligned} \tag{26}$$

Picking $c_6 > \lambda_k\mu_k$, it follows from (22), (26) and $r + 1 > 2$ that there exists $R_1 > \rho_1 > 0$ such that

$$\sup J(\partial B_{R_1}) < \inf J(S_{\rho_1}).$$

Let $\{H_n\}$ be a sequence of subspaces of H , which is defined as before, then $(\varphi_k, \varphi_k) \in H_n$ for all n , we can prove that for any n ,

$$\sup J(\partial B_{R_1} \cap H_n) < \inf J(S_{\rho_1} \cap H_n).$$

Similar to the proof of Lemma 2, the functional $J|_{H_n}$ satisfies the $(Ce)_c$ condition for any $c \in R$. Hence, by linking argument, there exists a critical point (u_n, v_n) of $J|_{H_n}$ with

$$\inf J(S_{\rho_1} \cap H_n) \leq J(u_n, v_n) \leq \sup J(B_{R_1} \cap H_n).$$

From Lemma 2, there is a subsequence of $\{(u_n, v_n)\}$, also denoted by $\{(u_n, v_n)\}$, and $(u, v) \in H$ such that $(u_n, v_n) \rightarrow (u, v)$ as $n \rightarrow \infty$, which implies that (u, v) is a critical point for J with

$$\inf J(S_{\rho_1}) \leq J(u, v) \leq \sup J(B_{R_1}). \quad \square$$

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Authors' contributions

The authors contributed equally to this paper. All authors read and approved the final manuscript.

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