Benterki et al. *Boundary Value Problems* (2017) 2017:55
DOI 10.1186/s13661-017-0791-z

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

Open Access

Asymptotic behavior of solutions to a boundary value problem with mixed boundary conditions and friction law

Djamila Benterki[1*], Hamid Benseridi[2] and Mourad Dilmi[2]

[*]Correspondence:
benterkidj@yahoo.fr
[1]Department of Mathematics, Bordj Bou Arreridj University, El Anasser, 34000, Algeria
Full list of author information is available at the end of the article

Abstract

In this paper, we consider a non-linear problem in a stationary regime in a three-dimensional thin domain Ω^ε with Fourier and Tresca boundary conditions. In the first step, we derive a variational formulation of the mechanical problem. We then study the asymptotic behavior in the one dimension case when the domain parameter tends to zero. In the latter case, a specific Reynolds equation associated with variational inequalities is obtained and the uniqueness of the limit velocity and pressure are proved.

MSC: 35R35; 76F10; 78M35

Keywords: asymptotic approach; Bingham fluid; coupled problem; Fourier law; Tresca law; variational inequality; weak solution

1 Introduction

The asymptotic behavior of Bingham fluid has been studied by many authors, see for instance [1], where the analysis of the Bingham fluid flow variational inequality was carried out, and where the authors investigated the existence, uniqueness and regularity of the solution for the steady and in-stationary flows in a reservoir. Existence and extra regularity results for the d-dimensional Bingham fluid flow problem with Dirichlet boundary conditions were studied in [2, 3].

In [4], the author has addressed the asymptotic behavior of a Bingham fluid in a thin domain. Unfortunately in this work a complete characterization was not given because of the difficulty encountered due to the choice of the test functions and of the imposed boundary conditions.

The authors in [5], studied the same problem, in which only the Dirichlet conditions on the boundary have been considered.

Along the same lines, the authors in [6] have proved the asymptotic analysis of an isothermal Bingham fluid in a thin domain with non-linear Tresca boundary conditions.

The asymptotic behavior of a coupled system involving an incompressible Bingham fluid and the equation of the heat energy, in a three-dimensional bounded domain with Tresca free boundary friction conditions were investigated in [7].

The numerical solution of the stationary Bingham fluid flow problem were studied in [8–10]. A variety of work has been done on the mechanical contact with the various laws of behavior and various friction boundary conditions close to our problem; however, these papers were restricted only to the results of existence and uniqueness of the weak solution under several assumptions. Let us mention for example the work by [11] in which the authors worked the contact problems with friction and adhesion for electro-elastic-visco-plastic materials. The existence and stability results of solitary-wave solutions to coupled non-linear Schrödinger equations with power-type non-linearities arising in several models of modern physics were studied in [12]. The authors in [13] have studied a symmetric, non-linear eigenvalue problem arising in earthquake initiation, and proved the existence of infinitely many solutions. The new variational principles for solving extended Dirichlet-Neumann problems is given in [14]. The dynamic evolution with frictional contact of an elastic body was studied in [15]. Other similar problems can be found in monographs such as [16–21], and the literature quoted there.

In our present work, we further the research of [6, 7] on the asymptotic behavior of a stationary problem for the isothermal Bingham fluid. However, this time our operator will be perturbed by a term $(u \cdot \nabla u)$ in a three-dimensional thin domain Ω^ε with Fourier and Tresca boundary conditions. This source term plays an essential role in quantum mechanics, where this non-linear term is used to perturb a linear operator to obtain a non-linear operator which gives applicable results; see [22].

We consider the Dirichlet boundary conditions on $\overline{\Gamma}_1^\varepsilon \cup \overline{\Gamma}_L^\varepsilon$, where $\overline{\Gamma}_L^\varepsilon$ is the lateral one, the Fourier boundary condition at the top surface $\overline{\Gamma}_1^\varepsilon$, finally, a non-linear Tresca interface condition at the bottom one ω. The weak form of the problem is a variational inequality and the main difficulty here is to estimate the velocity, the pressure and its gradient.

Our work is organized as follows. In Section 2, the related weak formulation is given and the existence theorem of weak solutions is proved. In Section 3, we use the change of variable $z = x_3/\varepsilon$, to transform the initial problem posed in the domain Ω^ε into a new problem posed on a fixed domain Ω independent of the parameter ε. We find some estimates on the velocity and pressure which are independent of the parameter and prove the convergence theorem by using different inequalities. The limit problem with a specific weak form of the Reynolds equation, the uniqueness of the limit velocity and pressure are given in Section 4.

2 Problem statement and variational formulation

We consider a mathematical problem governed to the stationary equations for Bingham fluid in a three-dimensional bounded domain $\Omega^\varepsilon \subset \mathbb{R}^3$ with boundary Γ^ε. This boundary of the domain is assumed to be composed of three portions: ω, the bottom of the domain, Γ_1^ε, the upper surface, and Γ_L^ε, the lateral surface. The fluid is supposed to be incompressible. We impose the Fourier boundary condition at the top surface, a non-linear Tresca interface condition at the bottom one and Dirichlet boundary conditions on the top and the lateral parts. The fluid is acted upon by given body forces of density f.

Let ω be a fixed bounded domain of $\mathbb{R}^3$ of equation $x_3 = 0$. We suppose that ω has a Lipschitz continuous boundary and is the bottom of the fluid. The upper surface $\overline{\Gamma}_1^\varepsilon$ is defined by $x_3 = \varepsilon h(x')$, $(x' = (x_1, x_2))$. We introduce a small parameter ε, which will tend to zero, and h a smooth bounded function such that $0 < h_\star \leq h(x') \leq h^\star$, for all $(x', 0) \in \omega$.

We denote by Ω^ε the domain of the flow: $\Omega^\varepsilon = \{(x', x_3) \in \mathbb{R}^3 : (x', 0) \in \omega, 0 < x_3 < \varepsilon h(x')\}$, by σ^ε the deviatoric part and p^ε the pressure. The fluid is supposed to be visco-plastic, and the relation between σ^ε and $D(u^\varepsilon)$ is given:

$$\begin{cases} \sigma_{ij}^\varepsilon = \widetilde{\sigma}_{ij}^\varepsilon - p^\varepsilon \delta_{ij}, \\ \widetilde{\sigma}^\varepsilon = 2\mu D(u^\varepsilon) + \alpha^\varepsilon \frac{D(u^\varepsilon)}{|D(u^\varepsilon)|} \quad \text{if } D(u^\varepsilon) \neq 0, \\ |\widetilde{\sigma}^\varepsilon| \leq \alpha^\varepsilon \quad \text{if } D(u^\varepsilon) \neq 0, \end{cases}$$

here $\alpha^\varepsilon \geq 0$ is the yield stress, $\mu > 0$ is the constant viscosity, u^ε is the velocity field, δ_{ij} is the Kronecker symbol, and $D(u^\varepsilon) = \frac{1}{2}(\nabla u^\varepsilon + (\nabla u^\varepsilon)^T)$.

For any tensor $D = (d_{ij})$, the notation $|D|$ represents the matrix norm: $|D| = \sqrt{\sum_{i,j} d_{ij} d_{ij}}$. We denote by $n = (n_1, n_2, n_3)$ the unit outward normal vector on the boundary Γ^ε. The normal and the tangential velocity on the boundary ω are

$$u_n^\varepsilon = u^\varepsilon \cdot n = u_i^\varepsilon \cdot n_i \quad \text{and} \quad u_{T_i}^\varepsilon = u_i^\varepsilon - u_n^\varepsilon \cdot n_i.$$

Similarly, for a regular stress tensor field σ^ε, we denote by σ_n^ε and σ_T^ε the normal and tangential components of σ^ε on the boundary ω given by

$$\sigma_n^\varepsilon = \left(\sigma^\varepsilon \cdot n\right) \cdot n = \sigma_{ij}^\varepsilon \cdot n_i \cdot n_j \quad \text{and} \quad \sigma_{T_i}^\varepsilon = \sigma_{ij}^\varepsilon \cdot n_j - \sigma_n^\varepsilon \cdot n_i.$$

We consider now the following mechanical problem.

Problem 1 Find a velocity field $u^\varepsilon : \Omega^\varepsilon \to \mathbb{R}^3$ and the pressure p^ε such that

$$-\operatorname{div}\left(\sigma^\varepsilon\right) + u \cdot \nabla u = f^\varepsilon \quad \text{in } \Omega^\varepsilon, \tag{2.1}$$

$$\left.\begin{array}{l} \sigma_{ij}^\varepsilon = \widetilde{\sigma}_{ij}^\varepsilon - p^\varepsilon \delta_{ij}, \\ \widetilde{\sigma}^\varepsilon = 2\mu D(u^\varepsilon) + \alpha^\varepsilon \frac{D(u^\varepsilon)}{|D(u^\varepsilon)|} \quad \text{if } D(u^\varepsilon) \neq 0, \\ |\widetilde{\sigma}^\varepsilon| \leq \alpha^\varepsilon \quad \text{if } D(u^\varepsilon) \neq 0 \end{array}\right\} \quad \text{in } \Omega^\varepsilon, \tag{2.2}$$

$$\operatorname{div}\left(u^\varepsilon\right) = 0 \quad \text{in } \Omega^\varepsilon, \tag{2.3}$$

$$\left.\begin{array}{l} \sigma_T^\varepsilon(u^\varepsilon) + l^\varepsilon u^\varepsilon = 0, \\ u^\varepsilon \cdot n = 0 \end{array}\right\} \quad \text{on } \Gamma_1^\varepsilon, \tag{2.4}$$

$$u^\varepsilon = 0 \quad \text{on } \Gamma_L^\varepsilon, \tag{2.5}$$

$$u^\varepsilon \cdot n = 0 \quad \text{on } \omega, \tag{2.6}$$

$$\left.\begin{array}{lll} |\sigma_\tau^\varepsilon| < k^\varepsilon & \Rightarrow & u_\tau^\varepsilon = s, \\ |\sigma_\tau^\varepsilon| = k^\varepsilon & \Rightarrow & \exists \lambda \geq 0 : u_\tau^\varepsilon = s - \lambda \sigma_\tau^\varepsilon, \end{array}\right\} \quad \text{on } \omega. \tag{2.7}$$

The law of conservation of momentum is given by equation (2.1). The relation between σ^ε and $D(u^\varepsilon)$ is given by (2.2), equation (2.3) represents the incompressibility equation, the boundary condition (2.4) is of Fourier type where $l^\varepsilon > 0$ is a given scalar, the Dirichlet boundary conditions on $\overline{\Gamma}_L^\varepsilon$ is given by (2.5). Since there is a no-flux condition across ω we have equation (2.6). Condition (2.7) represents a Tresca thermal friction law on ω where k^ε is the friction coefficient.

To get a weak formulation, we introduce the closed convex,

$$V^\varepsilon = \left\{v \in \left(H^1\left(\Omega^\varepsilon\right)\right)^3 : v = 0 \text{ on } \Gamma_L^\varepsilon; v \cdot n = 0 \text{ on } \omega \cup \Gamma_1^\varepsilon\right\},$$

$$V^{\varepsilon}_{\mathrm{div}} = \{v \in V^{\varepsilon} : \operatorname{div} v = 0\},$$
$$L^2_0(\Omega^{\varepsilon}) = \left\{q \in L^2(\Omega^{\varepsilon}) : \int_{\Omega^{\varepsilon}} q\,dx = 0\right\}.$$

By standard calculations, the variational formulation of the problem (Pb_1) is given by: Find $u^{\varepsilon} \in V^{\varepsilon}_{\mathrm{div}}$ and $p^{\varepsilon} \in L^2_0(\Omega^{\varepsilon})$ such that, for all $\varphi \in V^{\varepsilon}$, we have

$$\begin{aligned} &a(u^{\varepsilon}, \varphi - u^{\varepsilon}) - (p^{\varepsilon}, \operatorname{div}(\varphi)) + l^{\varepsilon} \int_{\Gamma^{\varepsilon}_1} u^{\varepsilon}(\varphi - u^{\varepsilon})\,d\tau + b(u^{\varepsilon}, \varphi - u^{\varepsilon}) + j^{\varepsilon}(\varphi) - j^{\varepsilon}(u^{\varepsilon}) \\ &\quad \geq (f^{\varepsilon}, \varphi - u^{\varepsilon}), \end{aligned} \tag{2.8}$$

where

$$a(u^{\varepsilon}, \varphi) = 2\mu \int_{\Omega^{\varepsilon}} d_{ij}(u^{\varepsilon}) d_{ij}(\varphi)\,dx, \qquad (p^{\varepsilon}, \operatorname{div}(\varphi)) = \int_{\Omega^{\varepsilon}} p^{\varepsilon} \operatorname{div}(\varphi)\,dx,$$
$$b(u^{\varepsilon}, \varphi) = \int_{\Omega^{\varepsilon}} u^{\varepsilon}_i \frac{\partial u^{\varepsilon}_j}{\partial x_i} \varphi_j\,dx, \qquad (f^{\varepsilon}, \varphi) = \int_{\Omega^{\varepsilon}} f^{\varepsilon}_i \varphi_i\,dx,$$
$$j^{\varepsilon}(\varphi) = \int_{\omega} k^{\varepsilon} |\phi|\,dx' + \alpha^{\varepsilon} \int_{\Omega^{\varepsilon}} |D(\varphi)|\,dx.$$

We introduce some results which will be used in the next. The detailed description can be found in [23]. We have

$$\int_{\Omega^{\varepsilon}} |\nabla(u^{\varepsilon})|^2 \leq 2\int_{\Omega^{\varepsilon}} |D(u^{\varepsilon})|^2\,dx + \mathcal{C}(\Gamma^{\varepsilon}_1) \int_{\Gamma^{\varepsilon}_1} |u^{\varepsilon}|^2\,d\tau, \quad \text{Korn inequality,} \tag{2.9}$$
$$\int_{\Omega^{\varepsilon}} |u^{\varepsilon}|^2 \leq 2h^{\star}\varepsilon \int_{\Gamma^{\varepsilon}_1} |u^{\varepsilon}|^2 + 2h^{\star 2}\varepsilon^2 \int_{\Omega^{\varepsilon}} \left|\frac{\partial u^{\varepsilon}}{\partial x_3}\right|^2, \quad \text{Poincaré inequality,} \tag{2.10}$$
$$ab \leq \eta^2 \frac{a^2}{2} + \eta^{-2} \frac{b^2}{2}, \quad \forall (a,b) \in \mathbb{R}^2, \forall \eta, \quad \text{Young inequality,} \tag{2.11}$$

where $C(\Gamma^{\varepsilon}_1) = 2\|D_2 h^{\varepsilon}\|_{C(\overline{\omega})}(1 + \|D_1 h^{\varepsilon}\|^2_{C(\overline{\omega})})$.

Theorem 2.1 *Let u^{ε} be a solution of the variational problem* (2.8), *then*

$$\begin{aligned} &a(u^{\varepsilon}, u^{\varepsilon}) + \frac{l^{\varepsilon}}{2} \int_{\Gamma^{\varepsilon}_1} |u^{\varepsilon}|^2\,d\tau + \int_{\omega} k^{\varepsilon} |u^{\varepsilon}|\,dx' + \alpha^{\varepsilon} \int_{\Omega^{\varepsilon}} |D(u^{\varepsilon})|\,dx \\ &\quad \leq \frac{\mu}{16} \|\nabla u^{\varepsilon}\|^2_{L^2(\Omega^{\varepsilon})} + \left[\frac{8\varepsilon^2 h^{\star 2}}{\mu} + \frac{\varepsilon h^{\star}}{l^{\varepsilon}}\right] \|f^{\varepsilon}\|^2_{L^2(\Omega^{\varepsilon})}. \end{aligned} \tag{2.12}$$

Proof Choosing $\varphi = 0$ as test function in inequality (2.8), we get

$$a(u^{\varepsilon}, u^{\varepsilon}) + l^{\varepsilon} \int_{\Gamma^{\varepsilon}_1} |u^{\varepsilon}|^2\,d\tau + b(u^{\varepsilon}, u^{\varepsilon}) + \int_{\omega} k^{\varepsilon} |u^{\varepsilon}|\,dx' + \alpha^{\varepsilon} \int_{\Omega^{\varepsilon}} |D(u^{\varepsilon})|\,dx \leq (f^{\varepsilon}, u^{\varepsilon}) \tag{2.13}$$

as $(f^{\varepsilon}, u^{\varepsilon}) \leq \|f^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})} \|u^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})}$ and by using the Poincaré inequality (2.10), we obtain

$$(f^{\varepsilon}, u^{\varepsilon}) \leq \sqrt{2\varepsilon h^{\star}} \|f^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})} \left(\int_{\Gamma^{\varepsilon}_1} |u^{\varepsilon}|^2\,d\tau\right)^{\frac{1}{2}} + \sqrt{2}(\varepsilon h^{\star}) \|f^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})} \|\nabla u^{\varepsilon}\|_{L^2(\Omega^{\varepsilon})}.$$

By the inequality (2.11), we have

$$\left|\left(f^\varepsilon,u^\varepsilon\right)\right| \le \frac{\mu}{16}\left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} + \left[\frac{8\varepsilon^2 h^{\star 2}}{\mu} + \frac{\varepsilon h^\star}{l^\varepsilon}\right]\left\|f^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} + \frac{l^\varepsilon}{2}\int_{\Gamma_1^\varepsilon}\left|u^\varepsilon\right|^2 d\tau. \tag{2.14}$$

On the other hand

$$b\left(u^\varepsilon,u^\varepsilon\right) = \int_{\Omega^\varepsilon} u_i^\varepsilon \frac{\partial u_j^\varepsilon}{\partial x_i} u_j^\varepsilon \, dx.$$

Using the fact that $2u_j^\varepsilon \frac{\partial u_j^\varepsilon}{\partial x_i} = \frac{\partial}{\partial x_i}(u_j^\varepsilon)^2$ and by the Green formula, we find

$$b\left(u^\varepsilon,u^\varepsilon\right) = \frac{1}{2}\int_{\Gamma^\varepsilon}\left(u_j^\varepsilon\right)^2 u_i^\varepsilon \cdot n_i \, d\tau - \int_{\Omega^\varepsilon}\left(u_j^\varepsilon\right)^2 \frac{\partial u_i^\varepsilon}{\partial x_i}\, dx.$$

As $u^\varepsilon \in V^\varepsilon_{\mathrm{div}}$, we obtain $b(u^\varepsilon,u^\varepsilon) = 0$, and from (2.13)-(2.14), we deduce (2.12). □

Theorem 2.2 *Let* u^ε *be a solution of the variational inequality* (2.8). *We assume that* $\varepsilon l^\varepsilon = \hat{l}$ *and*

$$\frac{C(\Gamma_1^\varepsilon)}{l^\varepsilon} \le \frac{1}{\mu}, \tag{2.15}$$

then

$$\left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} \le \left(\frac{384\varepsilon^2 h^{\star 2}}{\mu^2} + \frac{48\varepsilon h^\star}{l^\varepsilon \mu}\right)\left\|f^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)}. \tag{2.16}$$

Proof We estimate $a(u^\varepsilon,u^\varepsilon)$ and $\int_{\Gamma_1^\varepsilon}|u^\varepsilon|^2\,d\tau$. According to (2.12), we obtain

$$\begin{aligned}
\left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} &\le \frac{4}{\mu}a\left(u^\varepsilon,u^\varepsilon\right) + 4C\left(\Gamma_1^\varepsilon\right)\int_{\Gamma_1^\varepsilon}\left|u^\varepsilon\right|^2 d\tau \\
&\le \frac{1}{4}\left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} + \left[\frac{32\varepsilon^2 h^{\star 2}}{\mu^2} + \frac{4\varepsilon h^\star}{\mu l^\varepsilon}\right]\left\|f^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} \\
&\quad + \frac{1}{2}\frac{\mu C(\Gamma_1^\varepsilon)}{l^\varepsilon}\left\|\nabla u^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)} + \left[\frac{64C(\Gamma_1^\varepsilon)}{\mu l^\varepsilon}\varepsilon^2 h^{\star 2} + \frac{8C(\Gamma_1^\varepsilon)}{(l^\varepsilon)^2}\varepsilon h^\star\right]\left\|f^\varepsilon\right\|^2_{L^2(\Omega^\varepsilon)}.
\end{aligned}$$

Now, using the inequality (2.15) we deduce (2.16). □

3 Change of the domain and some estimates

In this section, we will use the technique of scaling in Ω^ε on the coordinate x_3, by introducing the change of the variables $z = \frac{x_3}{\varepsilon}$. We obtain a fixed domain Ω which is independent of ε:

$$\Omega = \left\{\left(x',z\right) \in \mathbb{R}^3 : \left(x',0\right) \in \omega, 0 < z < h\left(x'\right)\right\}.$$

We denote by $\Gamma = \overline{\omega} \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_L$ its boundary, then we define the following functions in Ω:

$$\begin{cases} \hat{u}_i^\varepsilon(x',z) = u_i^\varepsilon(x',x_3) \quad \text{for } i = 1,2; \\ \hat{u}_3^\varepsilon(x',z) = \varepsilon^{-1} u_3^\varepsilon(x',x_3); \\ \widehat{p}^\varepsilon(x',z) = \varepsilon^2 p^\varepsilon(x',x_3). \end{cases} \tag{3.1}$$

Let us assume the following:

$$\hat{k} = \varepsilon k^\varepsilon; \qquad \hat{f}(x',z) = \varepsilon^2 f^\varepsilon(x',x_3) \quad \text{and} \quad \widehat{l} = \varepsilon l^\varepsilon; \qquad \widehat{\alpha} = \varepsilon \alpha^\varepsilon.$$

Let

$$\begin{aligned}
&V = \{\widehat{\varphi} \in (H^1(\Omega))^3 : \varphi = 0 \text{ on } \Gamma_L; \varphi \cdot n = 0 \text{ on } \omega \cup \Gamma_1\}, \\
&V_{\mathrm{div}} = \{\widehat{\varphi} \in V : \mathrm{div}(\widehat{\varphi}) = 0\}, \\
&\Pi(V) = \{\varphi^\star = (\varphi_1^\star, \varphi_2^\star) \in (H^1(\Omega))^2 : \varphi^\star = 0 \text{ on } \Gamma_L\}, \\
&V_z = \left\{\varphi^\star = (\varphi_1^\star, \varphi_2^\star) \in (L^2(\Omega))^2 : \frac{\partial \varphi_i^\star}{\partial z} \in L^2(\Omega), i = 1,2 \text{ and } \varphi^\star = 0 \text{ on } \Gamma_L\right\}, \\
&\sum(V) = \{\phi \in \Pi(V) : \text{satisfies the condition } (D')\},
\end{aligned}$$

where the condition (D') is given by

$$\int_\Omega \left(\varphi_1^\star(x',z)\frac{\partial\theta}{\partial x_1}(x') + \varphi_2^\star(x',z)\frac{\partial\theta}{\partial x_2}(x')\right) dx'\,dz = 0, \quad \forall\theta \in C_0^1(\omega).$$

Injecting the new data and unknown in (2.8), and after multiplication by ε we deduce that $(\hat{u}^\varepsilon, \widehat{p}^\varepsilon)$ satisfies the following problem:

$$\begin{aligned}
&\sum_{1\le i,j\le 2} \int_\Omega \left[\mu\varepsilon^2\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i}\right) - \delta_{ij}\widehat{p}^\varepsilon\right]\frac{\partial}{\partial x_j}(\hat{\phi}_i - \hat{u}_i^\varepsilon)\,dx'\,dz \\
&\quad + \mu\sum_{i=1}^2 \int_\Omega \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right)\frac{\partial}{\partial z}(\hat{\phi}_i - \hat{u}_i^\varepsilon)\,dx'\,dz \\
&\quad + \sum_{j=1}^2 \int_\Omega \mu\varepsilon^2\left(\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial z}\right)\frac{\partial}{\partial x_j}(\hat{\phi}_3 - \hat{u}_3^\varepsilon)\,dx'\,dz \\
&\quad + \int_\Omega \left(2\mu\varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial z} - \widehat{p}^\varepsilon\right)\frac{\partial}{\partial z}(\hat{\phi}_3 - \hat{u}_3^\varepsilon)\,dx'\,dz \\
&\quad + \sum_{1\le i,j\le 2} \int_\Omega \varepsilon^2\hat{u}_i^\varepsilon\frac{\partial \hat{u}_j^\varepsilon}{\partial x_i}(\hat{\phi}_j - \hat{u}_j^\varepsilon)\,dx'\,dz + \sum_{i=1}^2 \int_\Omega \varepsilon^4\hat{u}_i^\varepsilon\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}(\hat{\phi}_3 - \hat{u}_3^\varepsilon)\,dx'\,dz \\
&\quad + \sum_{i=1}^2 \int_\Omega \varepsilon^2\hat{u}_3^\varepsilon\frac{\partial \hat{u}_i^\varepsilon}{\partial z}(\hat{\phi}_i - \hat{u}_i^\varepsilon)\,dx'\,dz + \int_\Omega \varepsilon^4\hat{u}_3^\varepsilon\frac{\partial \hat{u}_3^\varepsilon}{\partial z}(\hat{\phi}_3 - \hat{u}_3^\varepsilon)\,dx'\,dz \\
&\quad + \sum_{i=1}^2 \int_\omega \widehat{l}\hat{u}_i^\varepsilon(x',h(x')(\hat{\phi}_i(x',h(x')) - \hat{u}_i^\varepsilon(x',h(x')))\sqrt{1 + |\nabla h^\varepsilon(x')|^2}\,dx'
\end{aligned}$$

$$+ \int_{\omega} \widehat{l}\varepsilon^2 \hat{u}_3^\varepsilon(x', h(x'))(\hat{\phi}_3(x', h(x')) - \hat{u}_3^\varepsilon(x', h(x')))\sqrt{1 + |\nabla h^\varepsilon(x')|^2}\, dx'$$

$$+ \int_{\omega} \hat{k}(|\hat{\phi}| - |\hat{u}^\varepsilon|)\, dx' + \widehat{\alpha}\int_{\Omega}|D(\hat{\phi})|\, dx - \widehat{\alpha}\int_{\Omega}|D(\hat{u}^\varepsilon)|\, dx'\, dz$$

$$\geq \sum_{i=1}^{2}\int_{\Omega}\hat{f}_i(\hat{\phi}_i - \hat{u}_i^\varepsilon)\, dx'\, dz + \int_{\Omega}\varepsilon\hat{f}_3(\hat{\phi}_3 - \hat{u}_3^\varepsilon)\, dx'\, dz, \quad \forall\hat{\phi} \in V, \tag{3.2}$$

where

$$|D(v)| = \left[\frac{1}{4}\sum_{1\leq i,j\leq 2}\varepsilon^2\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)^2 + \frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial v_i}{\partial z} + \varepsilon^2\frac{\partial v_3}{\partial x_i}\right)^2 + \varepsilon^2\left(\frac{\partial v_3}{\partial z}\right)^2\right]^{\frac{1}{2}},$$
$$\forall v \in V. \tag{3.3}$$

Now we establish the estimates and convergences for the velocity field $\hat{u}^\varepsilon$ and the pressure $\widehat{p}^\varepsilon$ in the domain Ω.

Theorem 3.1 *Assuming that f^ε in $(L^2(\Omega^\varepsilon))^3$, the friction coefficient k^ε is a non-negative function in $L^\infty(\omega)$, and under the assumption of* (2.15), *there exists a constant $c_l > 0$, $l = 0$ to 4 independent of ε such that*

$$\sum_{1\leq i,j\leq 2}\left\|\varepsilon\frac{\partial\hat{u}_i^\varepsilon}{\partial x_j}\right\|_{L^2(\Omega)}^2 + \left\|\varepsilon\frac{\partial\hat{u}_3^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2 + \sum_{i=1}^{2}\left(\left\|\frac{\partial\hat{u}_i^\varepsilon}{\partial z}\right\|_{L^2(\Omega)}^2 + \left\|\varepsilon^2\frac{\partial\hat{u}_3^\varepsilon}{\partial x_i}\right\|_{L^2(\Omega)}^2\right) \leq c_0, \tag{3.4}$$

$$\|\hat{u}_i^\varepsilon\|_{L^2(\Omega)} \leq c_1, \quad \textit{for } i = 1, 2, \tag{3.5}$$

$$\|\varepsilon\hat{u}_3^\varepsilon\|_{L^2(\Omega)} \leq c_2, \tag{3.6}$$

$$\left\|\frac{\partial\widehat{p}^\varepsilon}{\partial x_i}\right\|_{H^{-1}(\Omega)} \leq c_3, \quad \textit{for } i = 1, 2, \tag{3.7}$$

$$\left\|\frac{\partial\widehat{p}^\varepsilon}{\partial z}\right\|_{H^{-1}(\Omega)} \leq \varepsilon c_4. \tag{3.8}$$

Proof As $\varepsilon^3\|f^\varepsilon\|_{L^2(\Omega^\varepsilon)}^2 = \|\hat{f}\|_{L^2(\Omega)}^2$ and passing to the fixed domain Ω in the right member of the inequality (2.16), we obtain

$$\|\nabla\hat{u}^\varepsilon\|_{L^2(\Omega)}^2 \leq \left[\frac{384h^{\star 2}}{\mu^2} + \frac{48h^\star}{\mu\widehat{l}}\right]\|\hat{f}\|_{L^2(\Omega)}^2; \tag{3.9}$$

from (3.9), we deduce (3.4) with $c_0 = [\frac{384h^{\star 2}}{\mu^2} + \frac{48h^\star}{\mu\widehat{l}}]\|\hat{f}\|_{L^2(\Omega)}^2$.

For getting the estimates (3.5) and (3.6), using the Poincaré inequality in the domain Ω:

$$\int_{\Omega}|\hat{u}_i^\varepsilon|^2\, dx \leq 2h^\star\int_{\Gamma_1}|\hat{u}_i^\varepsilon|^2\, d\tau' + 2h^{\star 2}\int_{\Omega}\left|\frac{\partial\hat{u}_i^\varepsilon}{\partial z}\right|^2 dx'\, dz, \quad i = 1, 2, 3, \tag{3.10}$$

from (3.4), we have

$$2h^{\star 2}\int_{\Omega}\left|\frac{\partial\hat{u}_i^\varepsilon}{\partial z}\right|^2 dx'\, dz \leq c_0 \quad \text{and} \quad 2\varepsilon^2 h^{\star 2}\int_{\Omega}\left|\frac{\partial\hat{u}_3^\varepsilon}{\partial z}\right|^2 dx'\, dz \leq c_0, \quad i = 1, 2.$$

On the other hand

$$\int_{\Gamma_1} \left|\hat{u}_i^\varepsilon\right|^2 d\tau' \le \max_{x'\in\omega} \sqrt{1 + \left|\nabla h(x')\right|^2} \int_{\Gamma_1^\varepsilon} \left|u_i^\varepsilon\right|^2 d\tau, \quad \text{for } i = 1, 2, \tag{3.11}$$

$$\int_{\Gamma_1} \left|\hat{u}_3^\varepsilon\right|^2 d\tau' \le \frac{1}{\varepsilon^2} \max_{x'\in\omega} \sqrt{1 + \left|\nabla h(x')\right|^2} \int_{\Gamma_1^\varepsilon} \left|u_3^\varepsilon\right|^2 d\tau \tag{3.12}$$

from (2.12) and (3.4), we have $\int_{\Gamma_1^\varepsilon} |u^\varepsilon|^2\, d\tau \le \frac{\mu c_0}{8l} + [\frac{16h^{\star 2}}{\mu\hat{l}} + \frac{2h^\star}{\hat{l}^2}]\|\hat{f}\|^2_{L^2(\Omega)}$.

Then

$$\int_{\Gamma_1^\varepsilon} \left|u^\varepsilon\right|^2 d\tau \le C'\left(\Omega, \widehat{l}, \mu, \hat{f}, h^\star\right). \tag{3.13}$$

From (3.10)-(3.13), we deduce $\|\hat{u}_i^\varepsilon\|_{L^2(\Omega)} \le c_1, i = 1, 2$ and $\|\varepsilon\hat{u}_3^\varepsilon\|_{L^2(\Omega)} \le c_2$.

Choosing $\hat{\phi} = (\hat{u}_1^\varepsilon + \psi, \hat{u}_2^\varepsilon, \hat{u}_3^\varepsilon)$ in (3.2) and by Cauchy-Schwarz's inequality, we have

$$\begin{aligned}
&\int_\omega \widehat{l}\hat{u}_1^\varepsilon\left(x', h(x')\right)\psi\left(x', h(x')\right)\sqrt{\left|1 + \nabla h(x')\right|^2}\, dx' \\
&\quad\le \widehat{l} \max_{x'\in\omega} \sqrt{\left|1 + \nabla h(x')\right|^2}\left[\int_\omega \left|\hat{u}_1^\varepsilon\left(x', h(x')\right)\right|^2 \sqrt{\left|1 + \nabla h(x')\right|^2}\, dx'\right]^{\frac{1}{2}} \\
&\qquad\times\left[\int_\omega \left|\psi\left(x', h(x')\right)\right|^2 \sqrt{\left|1 + \nabla h(x')\right|^2}\, dx'\right]^{\frac{1}{2}} \\
&\quad\le \widehat{l} \max_{x'\in\omega} \sqrt{\left|1 + \nabla h(x')\right|^2}\left(\int_{\Gamma_1} \left|\hat{u}_1^\varepsilon\right|^2 d\tau'\right)^{\frac{1}{2}}\left(\int_{\Gamma_1} |\psi|^2\, d\tau'\right)^{\frac{1}{2}}.
\end{aligned}$$

As $(\int_{\Gamma_1} |\hat{u}_1^\varepsilon|^2\, d\tau')^{\frac{1}{2}} \le C'$ and according to the continuity of the trace application from $H^1(\Omega)$ in $L^2(\Gamma_1)$, there exists a constant C'' independent of ε such that $(\int_{\Gamma_1} |\psi|^2\, d\tau')^{\frac{1}{2}} \le C''\|\psi\|_{H^1(\Omega)}$.

Therefore

$$\int_\omega \widehat{l}\hat{u}_1^\varepsilon\left(x', h(x')\right)\psi\left(x', h(x')\right)\sqrt{\left|1 + \nabla h(x')\right|^2}\, dx' \le C'''\|\psi\|_{H^1(\Omega)},$$

where $C''' = C'C''\widehat{l}\max_{x'\in\omega}\sqrt{|1 + \nabla h(x')|^2}$.

By the Hölder inequality, we get

$$\left|\varepsilon^2 \int_\Omega \hat{u}_1^\varepsilon \frac{\partial \hat{u}_1^\varepsilon}{\partial x_1}\psi\, dx'\, dz\right| \le \varepsilon^2 \left\|\hat{u}_1^\varepsilon\right\|_{L^4(\Omega)}\left\|\frac{\partial \hat{u}_1^\varepsilon}{\partial x_1}\right\|_{L^2(\Omega)}\|\psi\|_{L^4(\Omega)}.$$

Using a Sobolev imbedding $\|v\|_{L^4(\Omega)} \le c\|v\|_{H^1(\Omega)}$, we obtain

$$\left|\varepsilon^2 \int_\Omega \hat{u}_1^\varepsilon \frac{\partial \hat{u}_1^\varepsilon}{\partial x_1}\psi\, dx'\, dz\right| \le c\varepsilon^2 \left\|\hat{u}_1^\varepsilon\right\|_{H^1(\Omega)}\left\|\frac{\partial \hat{u}_1^\varepsilon}{\partial x_1}\right\|_{L^2(\Omega)}\|\psi\|_{H^1(\Omega)}.$$

In the same way,

$$\left|\varepsilon^2 \int_\Omega \hat{u}_2^\varepsilon \frac{\partial \hat{u}_1^\varepsilon}{\partial x_2}\psi\, dx'\, dz\right| \le c\varepsilon^2 \left\|\hat{u}_2^\varepsilon\right\|_{H^1(\Omega)}\left\|\frac{\partial \hat{u}_1^\varepsilon}{\partial x_2}\right\|_{L^2(\Omega)}\|\psi\|_{H^1(\Omega)},$$

$$\left|\varepsilon^2 \int_\Omega \hat{u}_3^\varepsilon \frac{\partial \hat{u}_1^\varepsilon}{\partial z} \psi \, dx' \, dz\right| \le c\varepsilon^2 \|\hat{u}_3^\varepsilon\|_{H^1(\Omega)} \left\|\frac{\partial \hat{u}_1^\varepsilon}{\partial z}\right\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)}.$$

The Cauchy-Schwarz inequality gives

$$\begin{aligned}
&\int_\Omega \frac{\partial \widehat{p}^\varepsilon}{\partial x_1} \psi \, dx' \, dz \\
&\quad \le \mu\varepsilon^2 \sum_{j=1}^{2} \left(\left\|\frac{\partial \hat{u}_1^\varepsilon}{\partial x_j}\right\|_{L^2(\Omega)} + \left\|\frac{\partial \hat{u}_j^\varepsilon}{\partial x_1}\right\|_{L^2(\Omega)} \right) \left\|\frac{\partial \psi}{\partial x_j}\right\|_{L^2(\Omega)} \\
&\qquad + \mu \left(\left\|\frac{\partial \hat{u}_1^\varepsilon}{\partial z}\right\|_{L^2(\Omega)} + \varepsilon^2 \left\|\frac{\partial \hat{u}_3^\varepsilon}{\partial x_1}\right\|_{L^2(\Omega)} \right) \left\|\frac{\partial \psi}{\partial z}\right\|_{L^2(\Omega)} \\
&\qquad + c\varepsilon^2 \|\hat{u}_1^\varepsilon\|_{H^1(\Omega)} \left\|\frac{\partial \hat{u}_1^\varepsilon}{\partial x_1}\right\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)} \\
&\qquad + c\varepsilon^2 \|\hat{u}_2^\varepsilon\|_{H^1(\Omega)} \left\|\frac{\partial \hat{u}_1^\varepsilon}{\partial x_2}\right\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)} + c\varepsilon^2 \|\hat{u}_3^\varepsilon\|_{H^1(\Omega)} \left\|\frac{\partial \hat{u}_1^\varepsilon}{\partial z}\right\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)} \\
&\qquad + C''' \|\psi\|_{H^1(\Omega)} + \widehat{\alpha}\sqrt{|\Omega|} \|D(\psi)\|_{L^2(\Omega)} + \|\hat{f}\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)}
\end{aligned}$$

as $\|D(\psi)\|_{L^2(\Omega)} \le \|\psi\|_{H^1(\Omega)}$ and from (3.4)-(3.6), we deduce

$$\begin{aligned}
&\int_\Omega \frac{\partial \widehat{p}^\varepsilon}{\partial x_1} \psi \, dx' \, dz \\
&\quad \le \mu C \left(\sum_{j=1}^{2} \left\|\frac{\partial \psi}{\partial x_j}\right\|_{L^2(\Omega)} + \left\|\frac{\partial \psi}{\partial z}\right\|_{L^2(\Omega)} \right) \\
&\qquad + \left(2cc_1C + cc_2C + C'''\right) \|\psi\|_{H^1(\Omega)} + \widehat{\alpha}\sqrt{|\Omega|} \|\psi\|_{H^1(\Omega)} + \|\hat{f}\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)}.
\end{aligned} \tag{3.14}$$

In the same way, if we choose $\hat{\phi} = (\hat{u}_1^\varepsilon - \psi, \hat{u}_2^\varepsilon, \hat{u}_3^\varepsilon)$ in (3.14), we obtain

$$\begin{aligned}
&-\int_\Omega \frac{\partial \widehat{p}^\varepsilon}{\partial x_1} \psi \, dx' \, dz \\
&\quad \le \mu C \left(\sum_{j=1}^{2} \left\|\frac{\partial \psi}{\partial x_j}\right\|_{L^2(\Omega)} + \left\|\frac{\partial \psi}{\partial z}\right\|_{L^2(\Omega)} \right) \\
&\qquad + \left(2cc_1C + cc_2C + C'''\right) \|\psi\|_{H^1(\Omega)} + \widehat{\alpha}\sqrt{|\Omega|} \|\psi\|_{H^1(\Omega)} \\
&\qquad + \|\hat{f}\|_{L^2(\Omega)} \|\psi\|_{H^1(\Omega)}.
\end{aligned}$$

Then we get (3.7) for $i = 1$.

Choosing now $\hat{\phi} = (\hat{u}_1^\varepsilon, \hat{u}_2^\varepsilon \pm \psi, \hat{u}_3^\varepsilon)$ in (3.14), we obtain (3.7) for $i = 2$, and we choose $\hat{\phi} = (\hat{u}_1^\varepsilon, \hat{u}_2^\varepsilon, \hat{u}_3^\varepsilon \pm \psi)$ for getting (3.8). □

4 Convergence results and limit problem

Theorem 4.1 *Under the same assumptions as in Theorem* 3.1, *there exist* $u^\star = (u_1^\star, u_2^\star) \in V_z$ *and* $p^\star \in L_0^2(\Omega)$, *such that*

$$\hat{u}_i^\varepsilon \rightharpoonup u_i^\star \quad (1 \le i \le 2) \text{ weakly in} V_z, \tag{4.1}$$

$$\varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} \rightharpoonup 0 \quad (1 \leq i,j \leq 2) \text{ weakly in } L^2(\Omega), \tag{4.2}$$

$$\varepsilon \frac{\partial \hat{u}_3^\varepsilon}{\partial z} \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega), \tag{4.3}$$

$$\varepsilon^2 \frac{\partial \hat{u}_3^\varepsilon}{\partial x_i} \rightharpoonup 0 \quad (1 \leq i \leq 2) \text{ weakly in } L^2(\Omega), \tag{4.4}$$

$$\varepsilon \hat{u}_3^\varepsilon \rightharpoonup 0 \quad \text{weakly in } L^2(\Omega), \tag{4.5}$$

$$\widehat{p}^\varepsilon \rightharpoonup p^\star \quad \text{weakly in } L^2(\Omega), \text{ depend only of } x'. \tag{4.6}$$

For the proof of this theorem, we follow the same steps as in [6].

Theorem 4.2 *With the same assumptions of Theorem* 3.1, *the solution* $(u^\star, p^\star)$ *satisfies the following relations*:

$$\begin{aligned}
&\sum_{i=1}^{2} \int_\Omega \frac{\partial u_i^\star}{\partial z} \frac{\partial}{\partial z}\big(\hat{\phi}_i - u_i^\star\big)\,dx'\,dz - \int_\Omega p^\star(x')\left(\frac{\partial \hat{\phi}_1}{\partial x_1} + \frac{\partial \hat{\phi}_2}{\partial x_2}\right)dx'\,dz \\
&\quad - \int_\omega p^\star(x')\left[\hat{\phi}_1\big(x', h(x')\big)\frac{\partial h(x')}{\partial x_1} + \hat{\phi}_2\big(x', h(x')\big)\frac{\partial h(x')}{\partial x_2}\right]dx' \\
&\quad + \sum_{i=1}^{2} \widehat{l} \int_\omega u_i^\star\big(x', h(x')\big)\big[\hat{\phi}_i\big(x', h(x')\big) - u_i^\star\big(x', h(x')\big)\big]\,dx' \\
&\quad + \int_\omega \hat{k}\big(|\hat{\phi}| - |u^\star|\big)\,dx' + \widehat{\alpha} \int_\Omega \left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial \hat{\phi}_i}{\partial z}\right)^2\right)^{\frac{1}{2}} dx'\,dz \\
&\quad - \widehat{\alpha} \int_\Omega \left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^\star}{\partial z}\right)^2\right)^{\frac{1}{2}} dx'\,dz \\
&\geq \sum_{i=1}^{2} \big(\hat{f}_i, \hat{\phi}_i - u_i^\star\big), \quad \forall \hat{\phi} \in \Pi(V).
\end{aligned} \tag{4.7}$$

Proof As $\int_\omega \widehat{l}\varepsilon^2 \hat{u}_3^\varepsilon(x', h(x')\hat{u}_3^\varepsilon(x', h(x')))\sqrt{1 + |\nabla h(x')|^2}\,dx' \geq 0$, then applying $\lim_{\varepsilon\to 0}\inf$ in the left part of (3.2) and the $\lim_{\varepsilon\to 0}$ in the right part of (3.2) and from the convergence results of Theorem 4.1, we obtain

$$\begin{aligned}
&\sum_{i=1}^{2} \int_\Omega \frac{\partial u_i^\star}{\partial z} \frac{\partial u_i^\star}{\partial z}\,dx'\,dz + \sum_{i=1}^{2} \widehat{l} \int_\omega u_i^\star\big(x', h(x')\big) u_i^\star\big(x', h(x')\big)\,dx' \\
&\quad + \int_\omega \hat{k}\big|u^\star\big|\,dx' + \widehat{\alpha} \int_\Omega \left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial u_i^\star}{\partial z}\right)^2\right)^{\frac{1}{2}} dx'\,dz \\
&\leq \sum_{i=1}^{2} \int_\Omega \frac{\partial u_i^\star}{\partial z} \frac{\partial \hat{\phi}_i}{\partial z}\,dx'\,dz \\
&\quad + \sum_{i=1}^{2} \widehat{l} \int_\omega u_i^\star\big(x', h(x')\big)\hat{\phi}_i\big(x', h(x')\big)\,dx' + \int_\Omega p^\star(x')\frac{\partial \hat{\phi}_3}{\partial z}\,dx'\,dz
\end{aligned}$$

$$+ \int_{\omega} \hat{k}|\hat{\phi}|\,dx' + \widehat{\alpha} \int_{\Omega} \left(\frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial \hat{\phi}_i}{\partial z} \right)^2 \right)^{\frac{1}{2}} dx'\,dz$$

$$+ \int_{\Omega} p^{\star}(x') \left(\frac{\partial \hat{\phi}_1}{\partial x_1} + \frac{\partial \hat{\phi}_2}{\partial x_2} \right) dx'\,dz + \sum_{i=1}^{2} \int_{\Omega} \hat{f}_i(\hat{\phi}_i - u_i^{\star})\,dx'\,dz \tag{4.8}$$

as $\int_{\Omega} p^{\star}(x') \frac{\partial \hat{\phi}_3}{\partial z}\,dx'\,dz = \int_{\omega} p^{\star}(x')\hat{\phi}_3(x', h(x'))\,dx'$ and $\hat{\phi}_3 = \hat{\phi}_1 \frac{\partial h}{\partial x_1} + \hat{\phi}_2 \frac{\partial h}{\partial x_2}$, we have

$$\int_{\Omega} p^{\star}(x') \frac{\partial \hat{\phi}_3}{\partial z}\,dx'\,dz = \int_{\omega} p^{\star}(x') \left(\hat{\phi}_1 \frac{\partial h}{\partial x_1} + \hat{\phi}_2 \frac{\partial h}{\partial x_2} \right) dx' \tag{4.9}$$

and from (4.8)-(4.9), we deduce (4.7). □

Remark 4.1 If $\hat{\phi}$ satisfies the condition (D'), the inequality (4.7) is reduced as follows:

$$\sum_{i=1}^{2} \int_{\Omega} \frac{\partial u_i^{\star}}{\partial z} \left(\frac{\partial \hat{\phi}_i}{\partial z} - \frac{\partial u_i^{\star}}{\partial z} \right) dx'\,dz + \sum_{i=1}^{2} \widehat{l} \int_{\omega} u_i^{\star}(x', h(x')) \left(\hat{\phi}_i(x', h(x')) - u_i^{\star}(x', h(x')) \right) dx'$$

$$+ \int_{\omega} \hat{k}\left(|\hat{\phi}| - |u^{\star}|\right) dx' + \widehat{\alpha} \int_{\Omega} \left(\frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial \hat{\phi}_i}{\partial z} \right)^2 \right)^{\frac{1}{2}} dx'\,dz$$

$$- \widehat{\alpha} \int_{\Omega} \left(\frac{1}{2} \sum_{i=1}^{2} \left(\frac{\partial u_i^{\star}}{\partial z} \right)^2 \right)^{\frac{1}{2}} dx'\,dz$$

$$\geq \sum_{i=1}^{2} \int_{\Omega} \hat{f}_i(\hat{\phi}_i - u_i^{\star})\,dx'\,dz, \quad \forall \hat{\phi} \in \sum(V). \tag{4.10}$$

Theorem 4.3 *The limit functions* $u^{\star}$ *and* $p^{\star}$ *satisfy*

$$p^{\star}(x_1, x_2, z) = p^{\star}(x_1, x_2) \quad \text{in } \Omega \quad \text{and} \quad p^{\star} \in H^1(\omega), \tag{4.11}$$

$$-\mu \frac{\partial^2 u_i^{\star}}{\partial z^2} + \frac{\partial p^{\star}}{\partial x_i} = \hat{f}_i, \quad i = 1, 2 \text{ in } L^2(\Omega). \tag{4.12}$$

Proof Choosing in (3.2) $\hat{\phi}_i = \hat{u}_i^{\varepsilon}$ for $i = 1, 2$, $\hat{\phi}_3 = \hat{u}_3^{\varepsilon} \pm \psi$ with $\psi \in H_0^1(\Omega)$, we obtain

$$\sum_{j=1}^{2} \int_{\Omega} \mu\varepsilon^2 \left(\varepsilon^2 \frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_j} + \frac{\partial \hat{u}_j^{\varepsilon}}{\partial z} \right) \frac{\partial \psi}{\partial x_j}\,dx'\,dz + \int_{\Omega} \left(2\mu\varepsilon^2 \frac{\partial \hat{u}_3^{\varepsilon}}{\partial z} - \widehat{p}^{\varepsilon} \right) \frac{\partial \psi}{\partial z}\,dx'\,dz$$

$$+ \sum_{i=1}^{2} \int_{\Omega} \varepsilon^4 \hat{u}_i^{\varepsilon} \frac{\partial \hat{u}_3^{\varepsilon}}{\partial x_i} \psi\,dx'\,dz + \int_{\Omega} \varepsilon^4 \hat{u}_3^{\varepsilon} \frac{\partial \hat{u}_3^{\varepsilon}}{\partial z} \psi\,dx'\,dz$$

$$= \int_{\Omega} \varepsilon \hat{f}_3 \psi\,dx'\,dz.$$

By the results of convergence (4.1)-(4.7), we deduce

$$\int_{\Omega} p^{\star} \frac{\partial \psi}{\partial z}\,dx'\,dz = 0, \quad \forall \psi \in H_0^1(\Omega), \tag{4.13}$$

then

$$\frac{\partial p^\star}{\partial z} = 0 \quad \text{in } H^{-1}(\Omega). \tag{4.14}$$

Choosing now $\hat{\phi}_i = \hat{u}_i^\varepsilon \pm \psi_i$ for $i = 1, 2$, with $\psi_i \in H_0^1(\Omega)$ and $\hat{\phi}_3 = \hat{u}_3^\varepsilon$ in (3.2) we obtain

$$\begin{aligned}
&\sum_{1\le i,j\le 2} \int_\Omega \left[\mu\varepsilon^2\left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_j} + \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i}\right) - \delta_{ij}\widehat{p}^\varepsilon\right]\frac{\partial \psi_i}{\partial x_j}\,dx'\,dz \\
&\quad + \mu\sum_{i=1}^2 \int_\Omega \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial z} + \varepsilon^2\frac{\partial \hat{u}_3^\varepsilon}{\partial x_i}\right)\frac{\partial \psi_i}{\partial z}\,dx'\,dz + \sum_{1\le i,j\le 2}\int_\Omega \varepsilon^2 \hat{u}_i^\varepsilon \frac{\partial \hat{u}_j^\varepsilon}{\partial x_i}\psi_j\,dx'\,dz \\
&\quad + \sum_{i=1}^2 \int_\Omega \varepsilon^2 \hat{u}_3^\varepsilon \frac{\partial \hat{u}_i^\varepsilon}{\partial z}\psi_i\,dx'\,dz \\
&= \sum_{i=1}^2 \int_\Omega \hat{f}_i\psi_i\,dx'\,dz.
\end{aligned}$$

Using (4.1)-(4.7), and choosing $\psi_1 = 0$ and $\psi_2 \in H_0^1(\Omega)$, then $\psi_2 = 0$ and $\psi_1 \in H_0^1(\Omega)$, we obtain

$$\sum_{i=1}^{-2} \int_\Omega p^\star \frac{\partial \psi_i}{\partial x_i}\,dx'\,dz + \mu\sum_{i=1}^2 \int_\Omega \frac{\partial u_i^\star}{\partial z}\frac{\partial \psi_i}{\partial z}\,dx'\,dz = \sum_{i=1}^2 \int_\Omega \hat{f}_i\psi_i\,dx'\,dz. \tag{4.15}$$

By utilizing the Green formula, we obtain (4.12) but in $H^{-1}(\Omega)$.

To prove $p^\star \in H^1(\omega)$, we see from (4.14) that $p^\star$ does not depend on z, then following [23], we choose ψ_i in (4.15) such that $\psi_i(x', z) = z(z - h(x'))\theta(x')$ with $\theta \in H_0^1(\omega)$, and using the Green formula, we obtain

$$\frac{1}{6}\int_\omega p^\star \frac{\partial (h^3\theta)}{\partial x_i}\,dx' - 2\int_\omega h\tilde{u}_i^\star\theta\,dx' + \int_\omega h(x')\left[u_i^\star(x', h(x')) + u_i^\star(x', 0)\right]\theta\,dx' = \int_\omega \widetilde{f}_i\theta\,dx',$$

where

$$\tilde{u}_i^\star = \frac{1}{h(x')}\int_0^{h(x')} u_i^\star(x', z)\,dz \quad \text{and} \quad \widetilde{f}_i = \int_0^{h(x')} z(z - h(x'))\hat{f}_i(x', z)\,dz;$$

so

$$h(x')\left[u_i^\star(x', h(x')) + u_i^\star(x', 0)\right] - 2h\tilde{u}_i^\star - \frac{1}{6}h^3\frac{\partial p^\star}{\partial x_i} = \widetilde{f}_i, \quad i = 1, 2 \text{ in } H^{-1}(\omega) \tag{4.16}$$

as $\hat{f}_i \in L^2(\Omega)$, $u_i^\star \in V_z$ in particular in $L^2(\omega)$, therefore $\tilde{u}_i^\star$ and $\widetilde{f}_i$ are in $L^2(\omega)$. Then from (4.16) we get (4.11). As $\hat{f}_i \in L^2(\omega)$, from (4.16) we have $\frac{\partial^2 u_i^\star}{\partial z^2}$ in $L^2(\Omega)$; then (4.12) holds. □

Theorem 4.4 *Under the assumptions of the preceding theorems, the traces $s^\star$, $\tau^\star$ satisfy the following equality*:

$$\int_\omega \left(\frac{h^3}{12\mu}\nabla p^\star - \frac{h}{2}s^\star - \frac{h}{2}s_h^\star + \widetilde{F}\right)\nabla\hat{\phi}\,dx' + \int_{\partial\omega} h\widetilde{u}^\star\phi\cdot n\,d\sigma = 0, \quad \forall\hat{\phi} \in H^1(\omega), \tag{4.17}$$

where

$$s^\star(x') = u^\star(x',0), \qquad s_h^\star(x') = u^\star(x',h(x')),$$
$$\tau^\star = \frac{\partial u^\star}{\partial z}(x',0), \qquad \tau_h^\star = \frac{\partial u^\star}{\partial z}(x',h(x')),$$
$$\widetilde{g}(x') = \int_0^{h(x')} \widehat{g}(x',z)\,dz, \quad \forall x' \in \partial\omega,$$
$$F_i(x',z) = \int_0^z \int_0^\zeta \hat{f}_i(x',z)\,dt\,d\zeta, \qquad \widetilde{F}_i(x') = \frac{1}{\mu}\int_0^{h(x')} F_i(x',z)\,dz - \frac{h}{2\mu}F(x',h(x')).$$

Proof Integrating twice (4.12) between 0 and z, we obtain

$$\mu u^\star(x',z) = \mu s^\star(x') + \frac{z^2}{2}\nabla p^\star + \mu z\tau^\star - F(x',z). \tag{4.18}$$

Replacing z by h, we obtain

$$\mu u^\star(x',h(x')) = \mu s^\star(x') + \frac{h^2}{2}\nabla p^\star + \mu h\tau^\star - F(x',h)$$

so

$$h\tau^\star = \frac{1}{\mu}F(x',h(x')) + s_h^\star(x') - s^\star(x') - \frac{h^2}{2\mu}\nabla p^\star; \tag{4.19}$$

integrating (4.18) with respect to z, in the interval $(0,h(x'))$, we obtain

$$h\widetilde{u}^\star(x',h(x')) = hs^\star(x') + \frac{h^3}{6\mu}\nabla p^\star + \frac{h^2}{2}\tau^\star - \frac{1}{\mu}\int_0^z F(x',z)\,dz \tag{4.20}$$

we set for any function φ

$$\widetilde{\varphi}(x') = \frac{1}{h(x')}\int_0^{h(x')} \varphi(x',z)\,dz, \quad \forall x' \in \omega,$$

on the other hand, $\forall \phi \in H^1(\omega)$,

$$\begin{aligned}\int_\Omega \phi \operatorname{div}(\hat{u}^\varepsilon)\,dx'\,dz = 0 &= \int_\omega \phi(x')\int_0^{h(x')} \sum_{i=1}^2 \left(\frac{\partial \hat{u}_i^\varepsilon}{\partial x_i} + \frac{\partial \hat{u}_3^\varepsilon}{\partial z}\right) dz\,dx' \\ &= \int_\omega \phi(x')\left[\sum_{i=1}^2 \frac{\partial(h\widetilde{\hat{u}}_i^\varepsilon)}{\partial x_i} + \hat{u}_3^\varepsilon(x',h(x')) - \hat{u}_3^\varepsilon(x',0)\right] dx',\end{aligned}$$

we have $\hat{u}_3^\varepsilon(x',0) = 0$ in $\partial\Omega$. Then

$$\int_\omega \phi(x')\sum_{i=1}^2 \frac{\partial(h\widetilde{\hat{u}}_i^\varepsilon)}{\partial x_i}\,dx' = 0.$$

Using the Green formula, we get

$$\sum_{i=1}^{2}\int_{\omega}\frac{\partial\phi}{\partial x_i}\widetilde{\hat{u}}_i^{\varepsilon}h\,dx' = \sum_{i=1}^{2}\int_{\partial\omega}h\widetilde{\hat{u}}_i^{\varepsilon}\phi\cdot n_i\,d\sigma.$$

As $\hat{u}_i^{\varepsilon} \rightharpoonup u^{\star}$ in V_z and consequently in $L^2(\omega)$, therefore $\widetilde{\hat{u}}_i^{\varepsilon} \rightharpoonup \widetilde{u}_i^{\star}$ in $L^2(\omega)$, we deduce

$$\sum_{i=1}^{2}\int_{\omega}\frac{\partial\phi}{\partial x_i}\widetilde{u}_i^{\star}h\,dx' = \sum_{i=1}^{2}\int_{\partial\omega}h\widetilde{u}_i^{\star}\phi\cdot n_i\,d\sigma, \quad \forall\phi\in H^1(\omega), \tag{4.21}$$

using (4.19) to eliminate the term containing $\tau^{\star}$ from (4.20). Multiplying now (4.20) by $\nabla\phi$, then integrating it in ω and using (4.21), we obtain (4.16). □

Theorem 4.5 *The solution* $(u^{\star},p^{\star})$ *of our limit problem is unique.*

Proof Let (U^1,p^1), (U^2,p^2) be two solutions of the limit problem (4.10), then

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega}\frac{\partial U_i^1}{\partial z}\left(\frac{\partial\hat{\phi}_i}{\partial z}-\frac{\partial U_i^1}{\partial z}\right)dx'\,dz \\
&\quad+\sum_{i=1}^{2}\widehat{l}\int_{\omega}U_i^1\big(x',h(x')\big)\big(\hat{\phi}_i\big(x',h(x')\big)-U_i^1\big(x',h(x')\big)\big)\,dx' \\
&\quad+\int_{\omega}\hat{k}\big(|\hat{\phi}|-\big|U^1\big|\big)\,dx'+\widehat{\alpha}\int_{\Omega}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\hat{\phi}_i}{\partial z}\right)^2\right)^{\frac{1}{2}}dx'\,dz \\
&\quad-\widehat{\alpha}\int_{\Omega}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial U_i^1}{\partial z}\right)^2\right)^{\frac{1}{2}}dx'\,dz \\
&\quad\geq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\big(\hat{\phi}_i-U_i^1\big)\,dx'\,dz, \quad \forall\hat{\phi}\in\sum(V),
\end{aligned}\tag{4.22}$$

and

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega}\frac{\partial U_i^2}{\partial z}\left(\frac{\partial\hat{\phi}_i}{\partial z}-\frac{\partial U_i^2}{\partial z}\right)dx'\,dz \\
&\quad+\sum_{i=1}^{2}\widehat{l}\int_{\omega}U_i^2\big(x',h(x')\big)\big(\hat{\phi}_i\big(x',h(x')\big)-U_i^2\big(x',h(x')\big)\big)\,dx' \\
&\quad+\int_{\omega}\hat{k}\big(|\hat{\phi}|-\big|U^2\big|\big)\,dx'+\widehat{\alpha}\int_{\Omega}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial\hat{\phi}_i}{\partial z}\right)^2\right)^{\frac{1}{2}}dx'\,dz \\
&\quad-\widehat{\alpha}\int_{\Omega}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial U_i^2}{\partial z}\right)^2\right)^{\frac{1}{2}}dx'\,dz \\
&\quad\geq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\big(\hat{\phi}_i-U_i^2\big)\,dx'\,dz, \quad \forall\hat{\phi}\in\sum(V).
\end{aligned}\tag{4.23}$$

Taking $\hat{\phi} = U^2$ in (4.22) and $\hat{\phi} = U^1$ in (4.23), we obtain

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega}\frac{\partial U_i^1}{\partial z}\left(\frac{\partial U_i^2}{\partial z}-\frac{\partial U_i^1}{\partial z}\right)dx'\,dz \\
&\qquad+\sum_{i=1}^{2}\widehat{l}\int_{\omega}U_i^1\big(x',h(x')\big)\big(U_i^2\big(x',h(x')\big)-U_i^1\big(x',h(x')\big)\big)\,dx' \\
&\qquad+\int_{\omega}\hat{k}\big(\big|U^2\big|-\big|U^1\big|\big)\,dx'+\widehat{\alpha}\int_{\Omega}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial U_i^2}{\partial z}\right)^2\right)^{\frac{1}{2}}dx'\,dz \\
&\qquad-\widehat{\alpha}\int_{\Omega}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial U_i^1}{\partial z}\right)^2\right)^{\frac{1}{2}}dx'\,dz \\
&\quad\geq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\big(U_i^2-U_i^1\big)\,dx'\,dz,
\end{aligned}\tag{4.24}$$

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega}\frac{\partial U_i^2}{\partial z}\left(\frac{\partial U_i^1}{\partial z}-\frac{\partial U_i^2}{\partial z}\right)dx'\,dz \\
&\qquad+\sum_{i=1}^{2}\widehat{l}\int_{\omega}U_i^2\big(x',h(x')\big)\big(U_i^1\big(x',h(x')\big)-U_i^2\big(x',h(x')\big)\big)\,dx' \\
&\qquad+\int_{\omega}\hat{k}\big(\big|U^1\big|-\big|U^2\big|\big)\,dx'+\widehat{\alpha}\int_{\Omega}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial U_i^1}{\partial z}\right)^2\right)^{\frac{1}{2}}dx'\,dz \\
&\qquad-\widehat{\alpha}\int_{\Omega}\left(\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\partial U_i^2}{\partial z}\right)^2\right)^{\frac{1}{2}}dx'\,dz \\
&\quad\geq\sum_{i=1}^{2}\int_{\Omega}\hat{f}_i\big(U_i^1-U_i^2\big)\,dx'\,dz.
\end{aligned}\tag{4.25}$$

By adding the two inequalities (4.24), (4.25), it follows that

$$\begin{aligned}
&\sum_{i=1}^{2}\int_{\Omega}\frac{\partial U_i^1}{\partial z}\left(\frac{\partial U_i^2}{\partial z}-\frac{\partial U_i^1}{\partial z}\right)dx'\,dz+\sum_{i=1}^{2}\int_{\Omega}\frac{\partial U_i^2}{\partial z}\left(\frac{\partial U_i^1}{\partial z}-\frac{\partial U_i^2}{\partial z}\right)dx'\,dz \\
&\quad\geq\sum_{i=1}^{2}\widehat{l}\int_{\omega}\big|U_i^2\big(x',h(x')\big)-U_i^1\big(x',h(x')\big)\big|^2\,dx',
\end{aligned}$$

then $\sum_{i=1}^{2}\int_{\Omega}|\frac{\partial}{\partial z}(U_i^2-U_i^1)|^2\,dx'\,dz\leq 0$.

Using the Poincaré inequality, we deduce that

$$\big\|U_i^2-U_i^1\big\|_{V_z}=0,\tag{4.26}$$

so $u^\star$ is unique.

The uniqueness of $p^\star$ in $L_0^2(\omega) \cap H^1(\omega)$ follows then from the specific weak Reynolds equation (4.17), indeed we have

$$\int_\omega \left(\frac{h^3}{12}\nabla p^1 - \frac{h}{2}U^1(x',0) - \frac{h}{2}U^1(x',h(x')) + \widetilde{F}\right)\nabla\hat{\phi}\,dx' + \int_{\partial\omega} h\widetilde{U^1}\hat{\phi}\cdot n\,d\sigma = 0 \quad \forall\hat{\phi} \in H^1(\omega), \tag{4.27}$$

$$\int_\omega \left(\frac{h^3}{12}\nabla p^2 - \frac{h}{2}U^2(x',0) - \frac{h}{2}U^2(x',h(x')) + \widetilde{F}\right)\nabla\hat{\phi}\,dx' + \int_{\partial\omega} h\widetilde{U^2}\hat{\phi}\cdot n\,d\sigma = 0 \quad \forall\hat{\phi} \in H^1(\omega). \tag{4.28}$$

Subtracting (4.28) and (4.27) and using (4.26), we obtain

$$\int_\omega \frac{h^3}{12}\left(\nabla p^1 - \nabla p^2\right)\nabla\hat{\phi}\,dx' = 0,$$

taking $\hat{\phi} = p^1 - p^2$ and by Poincaré's inequality, we get $\|p^1 - p^2\|_{L^2(\omega)} = 0$.

This ends the proof of the uniqueness. □

5 Conclusions

In this research, we studied the asymptotic analysis of a non-linear problem in a three-dimensional thin domain Ω^ε with Fourier and Tresca boundary conditions. Firstly, the problem statement and variational formulation were formulated. We then have studied the asymptotic behavior in the one-dimensional case when the domain parameter tends to zero. In the latter case, the main results concerning the limit of weak problem and its uniqueness have been obtained.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Bordj Bou Arreridj University, El Anasser, 34000, Algeria. [2]Applied Math Lab, Department of Mathematics, Setif 1 University, Setif, 19000, Algeria.

Acknowledgements
The authors would like to thank very much the referees for their expert and constructive comments, which have further enriched our article.
This work is supported by the MESRS of Algeria (CNEPRU Project C00L03UN190120150002).

Publisher's Note

Received: 20 December 2016 Accepted: 9 April 2017 Published online: 20 April 2017

References

1. Duvaut, G, Lions, J-L: Les Inéquations en Mécanique et en Physique. Dunod, Paris (1972)
2. Fuchs, M, Grotowski, J-F, Reuling, J: On variational model for quasi-static Bingham fluids. Math. Models Methods Appl. Sci. **19**, 991-1015 (1996)
3. Fuchs, M, Seregin, G: Regularity results for the quasi-static Bingham variational inequality in dimensions two and three. Math. Z. **227**, 525-541 (1998)
4. El Mir, R: Comportement asymptotique d'un fluide de Bingham dans un filmi mince avec des conditions non-linéaires sur le bord. Thèse de Doctorat, France (2006)

5. Bunoiu, R, Kesavan, S: Asymptotic behavior of a Bingham fluid in thin layers. J. Math. Anal. Appl. **293**(2), 405-418 (2004)
6. Dilmi, M, Benseridi, H, Saadallah, A: Asymptotic analysis of a Bingham fluid in a thin domain with Fourier and Tresca boundary conditions. Adv. Appl. Math. Mech. **6**, 797-810 (2014)
7. Benseridi, H, Saadallah, A, Dilmi, M: On the asymptotic analysis of a non-isothermal Bingham fluid with mixed boundary conditions and friction law. Acta Math. Sin. Engl. Ser. (to appear)
8. Dean, E-J, Glowinski, R, Guidoboni, G: On the numerical simulation of Bingham visco-plastic flow: old and new results. J. Non-Newton. Fluid Mech. **142**, 36-62 (2007)
9. De Los Reyes, J-C, Gonzalez, S: Path following methods for steady laminar Bingham flow in cylindrical pipes. ESAIM: M2AN **43**(1), 81-117 (2009)
10. Glowinski, R, Lions, J-L, Trémolières, R: Analyse Numerique des Inéquations Variationnelles, Bd. 1: Théorie Générale - Premières Applications, 268 S., 180 FF; Bd. 2: Applications aux Phénomènes Stationnaires et D'évolution, 290 S., 210 FF. Dunod, Paris (1976)
11. Aissaoui, A, Hemici, N: A frictional contact problem with damage and adhesion for an electro elastic-viscoplastic body. Electron. J. Differ. Equ. **2014**, 11 (2014)
12. Bhattarai, S: Stability of solitary-wave solutions of coupled NLS equations with power-type nonlinearities. Adv. Nonlinear Anal. **4**(2), 73-90 (2015)
13. Ionescu, I-R, Radulescu, V: Nonlinear eigenvalue problems arising in earthquake initiation. Adv. Differ. Equ. **8**(7), 769-786 (2003)
14. Vallée, C, Radulescu, V, Atchonouglo, K: New variational principles for solving extended Dirichlet-Neumann problems. J. Elast. **123**(1), 1-18 (2016)
15. Ionescu, I-R, Nguyen, Q-L, Wolf, S: Slip-dependent friction in dynamic elasticity. Nonlinear Anal. **53**, 375-390 (2003)
16. Colli, P, Gilardi, G, Sprekels, J: A boundary control problem for the pure Cahn-Hilliard equation with dynamic boundary conditions. Adv. Nonlinear Anal. **4**(4), 311-325 (2015)
17. Di Pietro, D-A, Ern, A: A hybrid high-order locking-free method for linear elasticity on general meshes. Comput. Methods Appl. Mech. Eng. **283**, 1-21 (2015)
18. Ghergu, M, Radulescu, V: Nonlinear PDEs. Mathematical Models in Biology, Chemistry and Population Genetics. Springer Monographs in Mathematics. Springer, Heidelberg (2012)
19. Kabaria, H, Lew, A-J, Cockburn, B: A hybridizable discontinuous Galerkin formulation for non-linear elasticity. Comput. Methods Appl. Mech. Eng. **283**, 303-329 (2015)
20. Lamichhane, B-P: A new stabilization technique for the non-conforming Crouzeix-Raviart element applied to linear elasticity. Appl. Math. Lett. **39**, 35-41 (2015)
21. Lerguet, Z, Zellagui, Z, Benseridi, H, Drabla, S: Variational analysis of an electro viscoelastic contact problem with friction. J. Assoc. Arab Univ. Basic Appl. Sci. **14**(1), 93-100 (2013)
22. Lions, J-L: Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires. Dunod, Paris (1969)
23. Boukrouche, M, Lukaszewicz, G: On a lubrication problem with Fourier and Tresca boundary conditions. Math. Models Methods Appl. Sci. **14**(6), 913-941 (2004)