Huang et al. *Boundary Value Problems* (2017) 2017:41
DOI 10.1186/s13661-017-0775-z

Boundary Value Problems
a SpringerOpen Journal

RESEARCH | Open Access

Multiplicity of solutions for a p-Kirchhoff equation

Jincheng Huang[1*], Zhaomin Jiang[2], Zhiyan Li[1] and Jun Wang[1]

*Correspondence:
slbhuangjc@163.com
[1]Math and Physics Teaching Department, Hohai University, Changzhou, 213022, China
Full list of author information is available at the end of the article

Abstract

In this paper, we consider the following p-Kirchhoff equation:

$$(\mathrm{P}) \qquad -\left[M\left(\|u\|^{p}\right)\right]^{p-1}\Delta_p u = f(x,u) \quad \text{in } \Omega$$

with Dirichlet boundary conditions, where Ω is a bounded domain in $\mathbb{R}^N$. Under proper assumptions on M and f, we obtain three existence theorems of infinitely many solutions for problem (P) by the fountain theorem. Moreover, for a special nonlinearity $f(x,u) = \lambda|u|^{q-2}u + |u|^{r-2}u$ $(1 < q < p < r < p^*)$, we prove that problem (P) has at least two nonnegative solutions via the Nehari manifold method and a sequence of solutions with negative energy by the dual fountain theorem.

MSC: 35B38; 35J20; 35J62

Keywords: Kirchhoff equation; fountain theorem; Nehari manifold; fibering map; dual fountain theorem

1 Introduction

In this paper, we consider the following p-Kirchhoff equation:

$$-\left[M\left(\|u\|^{p}\right)\right]^{p-1}\Delta_p u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where M, f are continuous functions, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\|u\|^p = \int_\Omega |\nabla u|^p\,dx$ $(1 < p < N)$. Let X be the Sobolev space $W_0^{1,p}(\Omega)$ endowed with the norm $\|u\|$.

Problem (1.1) began to attract the attention of researchers mainly after the work of Lions [1], where a functional analysis approach was proposed to attack it. Since then, much attention has been paid to the existence of nontrivial solutions, sign-changing solutions, ground state solutions, multiplicity of solutions and concentration of solutions for the following case:

$$-\left(a + b\int_\Omega |\nabla u|^p\,dx\right)\Delta_p u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1.2}$$

See [2–8] and the references therein.

For example, Wu [2] showed that problem (1.2) has a nontrivial solution and a sequence of high energy solutions by using the mountain pass theorem and symmetric mountain pass theorem. Similar consideration can be found in Nie and Wu [3], where radial potentials were considered. Chen et al. [4] treated equation (1.2) when $f(x,t) = \lambda a(x)|u|^{q-2}u + b(x)|u|^{r-2}u$ $(1 < q < p = 2 < r < 2^*)$. Using the Nehari manifold and fibering maps, they established the existence of multiple positive solutions for (1.2).

However, the study of problem (1.1) becomes more difficult since M is a general function. Alves et al. [9] and Corrêa and Figueiredo [10] showed that the problem has a positive solution by the mountain pass theorem, where M is supposed to satisfy the following conditions:

(M_1) $M(t) \geq m_0$ for all $t \geq 0$.
(M_2') $\hat{M}(t) \geq [M(t)]^{p-1}t$ for all $t \geq 0$, where $\hat{M}(t) = \int_0^t [M(s)]^{p-1}\,ds$.

In [11], Liu established the existence of infinite solutions to a Kirchhoff-type equation like (1.1). By the fountain theorem and dual fountain theorem, they investigated the problem with M satisfying (M_1) and

(M_3') $M(t) \leq m_1$ for all $t > 0$.

Very recently, Figueiredo and Nascimento [12] and Santos Jr. [13] considered solutions of (1.1) by the minimization argument and the minimax method, respectively, where $p = 2$ and M satisfies (M_1) and

(M_4') the function $t \mapsto M(t)$ is increasing, and the function $t \mapsto \frac{M(t)}{t}$ is decreasing.

Note that $M(t) = a + bt$ does not satisfy (M_2') for $p = 2$ and (M_3'). Moreover, $M(t) = a + bt^k$ does not satisfy (M_2'), (M_3') for all $k > 0$ and (M_4') for all $k > 1$.

Motivated mainly by [4, 5, 14], we shall establish conditions on M and f under which problem (1.1) possesses infinitely many solutions in the present paper.

Instead of (M_2')-(M_4'), we make the following assumptions on M:

(M_2) There exists $\sigma > 0$ such that

$$\hat{M}(t) \geq \sigma\big[M(t)\big]^{p-1}t$$

holds for all $t \geq 0$, where $\hat{M}(t) = \int_0^t [M(s)]^{p-1}\,ds$.

(M_3) There exist $\mu > 0, \sigma > 0$ and $s > p^{-1}$ such that for all $t \geq 0$

$$\hat{M}(t) \geq \sigma\big[M(t)\big]^{p-1}t + \mu t^s.$$

We also suppose that f satisfies the following conditions:

(f_1) There are constants $1 < p < q < p^* = \frac{Np}{N-p}$ and $C > 0$ such that

$$\big|f(x,t)\big| \leq C\big(1 + |t|^{q-1}\big)$$

for all $x \in \Omega$, $t \in \mathbb{R}$.

(f_2) $f(x,t) = o(|t|^{p-1})$ as $t \to 0$ uniformly for any $x \in \Omega$.
(f_3) $f(x,-t) = -f(x,t)$ for all $x \in \Omega$, $t \in \mathbb{R}$.
(f_4) There exists $\frac{p}{\sigma} < \alpha < p^*$ such that $0 < \alpha F(x,t) \leq tf(x,t)$ for all $x \in \Omega$, $t \in \mathbb{R}$, where $F(x,t) = \int_0^t f(x,s)\,ds$.

(f_5) There exist $\max\{\frac{p}{\sigma}, p\} < \alpha < p^*$ and $r > 0$ such that

$$\inf_{x\in\Omega,|u|=r} F(x,u) > 0$$

and

$$0 < \alpha F(x,t) \le tf(x,t)$$

for all $x \in \Omega$ and $|t| \ge r$.

(f_6) $0 < \frac{p}{\sigma}F(x,t) \le tf(x,t)$ holds for all $x \in \Omega$, $t \in \mathbb{R}$.

(f_7) $\frac{F(x,t)}{t^{p/\sigma}} \to +\infty$ as $|t| \to \infty$ uniformly in $x \in \Omega$.

The associated energy functional to equation (1.1) is

$$J(u) = \frac{1}{p}\hat{M}\big(\|u\|^p\big) - \int_\Omega F(x,u)\,dx. \tag{1.3}$$

For any $\phi \in C_0^\infty(\Omega)$, we have

$$\langle J'(u),\phi\rangle = \big[M\big(\|u\|^p\big)\big]^{p-1}\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,dx - \int_\Omega f(x,u)\phi\,dx. \tag{1.4}$$

We have the following results by the fountain theorem.

Theorem 1.1 *Assume* (f_1)-(f_4) *and* (M_1)-(M_2). *Then problem* (1.1) *has a sequence* $\{u_n\}$ *of solutions in* X *with* $J(u_n) \to \infty$ *as* $n \to \infty$.

Theorem 1.2 *Assume* (f_1)-(f_3), (f_5) *and* (M_1)-(M_2). *Then problem* (1.1) *has a sequence* $\{u_n\}$ *of solutions in* X *with* $J(u_n) \to \infty$ *as* $n \to \infty$.

Theorem 1.3 *Assume* (f_1)-(f_3), (f_6)-(f_7) *and* (M_1), (M_3). *Then problem* (1.1) *has a sequence* $\{u_n\}$ *of solutions in* X *with* $J(u_n) \to \infty$ *as* $n \to \infty$.

Furthermore, we also consider a special nonlinearity $f(x,u) = \lambda|u|^{q-2}u + |u|^{r-2}u$ ($1 < q < p < r < p^*$). In this case, the associated energy functional is J_λ defined by

$$J_\lambda(u) = \frac{1}{p}\hat{M}\big(\|u\|^p\big) - \frac{1}{q}\int_\Omega \lambda|u|^q\,dx - \frac{1}{r}\int_\Omega |u|^r\,dx, \tag{1.5}$$

where $\hat{M}(s) = \int_0^s [M(t)]^{p-1}\,dt$.

Note that this nonlinearity does not satisfy conditions (f_2), (f_4)-(f_7). For this case, we will prove that problem (1.1) has at least two nonnegative solutions by extracting a minimizing sequence from the Nehari manifold, and we will obtain a sequence of weak solutions with negative energy by the dual fountain theorem.

Theorem 1.4 *Let* $f(x,u) = \lambda|u|^{q-2}u + |u|^{r-2}u$, *where* $1 < q < \min\{p, \frac{p}{\sigma}\} \le \max\{p, \frac{p}{\sigma}\} < r < p^*$. *Suppose that* M *satisfies* (M_1), (M_2) *and*

(M_4) *M is differentiable for all $t \geq 0$ and there exist some $d > 1$ such that*

$$(r-p)M(t) > dp(p-1)M'(t)t \geq 0.$$

Then there exists $\lambda_0 > 0$ such that problem (1.1) *has at least two nonnegative solutions for all* $0 < \lambda < \lambda_0$.

Theorem 1.5 *Let $f(x,u) = \lambda|u|^{q-2}u + |u|^{r-2}u$, where $1 < q < \min\{p, \frac{p}{\sigma}\} \leq \max\{p, \frac{p}{\sigma}\} < r < p^*$. Suppose that M satisfies* (M_1) *and* (M_2). *Then problem* (1.1) *has a sequence of solutions u_k such that $J_\lambda(u_k) < 0$ and $J_\lambda(u_k) \to 0$ as $k \to \infty$.*

Remark 1.1 Set $M(t) = a + bt^k$ $(a, b, k > 0)$. Then we can easily deduce that
(i) M satisfies (M_2) for all $p > 1$ and $0 < \sigma \leq \frac{1}{(p-1)k+1}$;
(ii) M satisfies (M_3) for one of the following cases:
(1) $s = 1$, $p \geq 2$, $1 - \sigma - \sigma(p-1)k \geq 0$, and $0 < s\mu \leq (1-\sigma)a^{p-1}$;
(2) $s = k+1$, $p \geq 2$, $0 < \sigma < 1$, and $0 < s\mu \leq ((1-\sigma)b - \sigma(p-1)bk)a^{p-2}$;
(iii) M satisfies (M_4) for $r - p > dpk$.

Remark 1.2 Let $M(t) = a + b\ln(1+t)$ $(a, b > 0, t \geq 0)$. By direct calculation, one has

$$\begin{aligned}
\hat{M}(t) &= \int_0^t \big(M(t)\big)^{p-1}\,dt \\
&= t\big(M(t)\big)^{p-1} - \int_0^t b(p-1)\big(M(t)\big)^{p-2}\,dt + \int_0^t \frac{b(p-1)M(t)^{p-2}}{1+t}\,dt \\
&\geq t\big(M(t)\big)^{p-1} - b(p-1)tM(t)^{p-2} \\
&\geq t\big(M(t)\big)^{p-1}\left(1 - \frac{b(p-1)}{a}\right).
\end{aligned}$$

Hence M satisfies (M_2) for $p > 1$, $b(p-1) < a$, $0 < \sigma \leq 1 - \frac{b(p-1)}{a}$.
Moreover, M satisfies (M_3) for $p = 2$, $s = 1$, $0 < \sigma \leq 1$ and $\sigma + \mu \leq a - b$.

The rest of the paper is organized as follows. In Section 2, we present some properties of $(PS)_c$ sequences. The proofs of Theorems 1.1-1.3 are given in Section 3. Then we establish some properties of the Nehari manifold and give the proofs of Theorems 1.4 and 1.5 in the last section.

2 Properties of $(PS)_c$ sequences

We say that $\{u_n\}$ is a $(PS)_c$ sequence for the functional J if

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0 \quad \text{in } X^*,$$

where X^* denotes the dual space of X. If every $(PS)_c$ sequence of J has a strong convergent subsequence, then we say that J satisfies the (PS) condition.
In this section, we derive some results related to the $(PS)_c$ sequence.

Lemma 2.1 *Assume* (f_1) *and* (M_1). *Then any bounded $(PS)_c$ sequence of J has a strong convergent subsequence.*

Proof The proof is almost the same as Lemma 2.1 in [10], though it was supposed $(\tilde{\mathrm{f}}_1)$ $|f(x,t)| \le C|t|^{q-1}$ instead of (f_1) there. □

By Lemma 2.1, in order to get a strong convergent subsequence from any $(\mathrm{PS})_c$ sequence of J, it suffices to verify the boundedness of the $(\mathrm{PS})_c$ sequence. In the following, we present three lemmas about the boundedness of the $(\mathrm{PS})_c$ sequence of J under different assumptions on the functions M and f.

Lemma 2.2 *Assume that M satisfies* (M_1)-(M_2) *and f satisfies* (f_4). *Then any* $(\mathrm{PS})_c$ *sequence of the functional J is bounded in X.*

Proof Let $\{u_n\}$ be a $(\mathrm{PS})_c$ sequence of the functional J. Then by (M_1)-(M_2) and (f_4), one has

$$\begin{aligned} c+1+\|u_n\| &\ge J(u_n) - \frac{1}{\alpha}\langle J'(u_n), u_n\rangle \\ &= \frac{1}{p}\hat{M}\big(\|u_n\|^p\big) - \int_\Omega F(x,u_n)\,dx - \frac{1}{\alpha}\big[M\big(\|u_n\|^p\big)\big]^{p-1}\|u_n\|^p \\ &\quad + \frac{1}{\alpha}\int_\Omega f(x,u_n)u_n\,dx \\ &\ge \left(\frac{\sigma}{p} - \frac{1}{\alpha}\right)\big[M\big(\|u_n\|^p\big)\big]^{p-1}\|u_n\|^p - \int_\Omega \left(F(x,u_n) - \frac{1}{\alpha}f(x,u_n)u_n\right)dx \\ &\ge \left(\frac{\sigma}{p} - \frac{1}{\alpha}\right)m_0^{p-1}\|u_n\|^p. \end{aligned}$$

Therefore, $\{u_n\}$ is bounded in X. □

Lemma 2.3 *If assumptions* (M_1), (M_2), (f_1), (f_2) *and* (f_5) *are satisfied, then any* $(\mathrm{PS})_c$ *sequence of the functional J is bounded in X.*

Proof Set $h(t) = F(x,t^{-1}z)t^\alpha$, $t \in [1,\infty)$. For $|z| \ge r$ and $1 \le t \le r^{-1}|z|$, we deduce from (f_5) that

$$\begin{aligned} h'(t) &= f\big(x,t^{-1}z\big)\big(-zt^{-2}\big)t^\alpha + F\big(x,t^{-1}z\big)\alpha t^{\alpha-1} \\ &= t^{\alpha-1}\big[\alpha F\big(x,t^{-1}z\big) - t^{-1}zf\big(x,t^{-1}z\big)\big] \le 0. \end{aligned}$$

Hence $h(1) \ge h(r^{-1}|z|)$. Therefore,

$$F(x,z) \ge r^{-\alpha}F\big(x,r|z|^{-1}z\big)|z|^\alpha \ge C_1|z|^\alpha,$$

where $C_1 = r^{-\alpha}\inf_{x\in\Omega,|u|=r}F(x,u) > 0$. Then there exists β such that $\max\{\frac{p}{\sigma},p\} < \beta < \alpha$ and

$$\lim_{|u|\to\infty}\frac{F(x,u)}{|u|^\beta} = +\infty.$$

Let $\{u_n\}$ be a $(\mathrm{PS})_c$ sequence of the functional J. In the following, we prove that $\{u_n\}$ is bounded in X. Suppose, on the contrary, that $\{u_n\}$ is unbounded. Then we can assume, without loss of generality, that $\|u_n\| \to \infty$ as $n \to \infty$.

By integrating (M_2), we obtain

$$\hat{M}(t) \leq \hat{M}(t_0)\left(\frac{t}{t_0}\right)^{1/\sigma}, \tag{2.1}$$

and so

$$M(t) \leq \left(\frac{\hat{M}(t_0)}{\sigma t_0^{1/\sigma}}\right)^{\frac{1}{p-1}} t^{\frac{1-\sigma}{\sigma(p-1)}} \tag{2.2}$$

holds for all $t \geq t_0 > 0$. Consequently,

$$\begin{aligned}\frac{[M(\|u_n\|^p)]^{p-1}\|u_n\|^p}{\|u_n\|^\beta} &\leq \frac{\frac{\hat{M}(t_0)}{\sigma t_0^{1/\sigma}}\|u_n\|^{p\frac{1-\sigma}{\sigma}}\|u_n\|^p}{\|u_n\|^\beta} \\ &= \frac{\hat{M}(t_0)}{\sigma t_0^{1/\sigma}}\|u_n\|^{\frac{p}{\sigma}-\beta} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Note that

$$\frac{\langle J'(u_n), u_n\rangle}{\|u_n\|^\beta} = \frac{[M(\|u_n\|^p)]^{p-1}\|u_n\|^p}{\|u_n\|^\beta} - \int_\Omega \frac{f(x,u_n)u_n}{\|u_n\|^\beta}\,dx,$$

we deduce that

$$\lim_{n\to\infty}\int_\Omega \frac{f(x,u_n)u_n}{\|u_n\|^\beta}\,dx = 0.$$

Set $v_n = \frac{u_n}{\|u_n\|}$. Since X is a Banach space and $\|v_n\| = 1$, passing to a subsequence if necessary, there is a point $v \in X$ such that

$$v_n \rightharpoonup v \quad \text{weakly in } X, \qquad v_n \to v \quad \text{strongly in } L^\beta(\Omega), \quad \text{and} \quad v_n \to v \quad \text{a.e. in } \Omega.$$

Denote $\Omega_0 := \{x \in \Omega | v(x) \neq 0\}$. Then $|u_n(x)| \to \infty$ for a.e. $x \in \Omega_0$. By assumptions (f_1), (f_2) and (f_5), we know that there exist constants $C_2, C_3 > 0$ such that

$$f(x,u)u \geq C_2|u|^\beta - C_3|u|^p \quad \text{for all } (x,u) \in \Omega \times \mathbb{R}.$$

Therefore

$$\int_\Omega \frac{f(x,u_n)u_n}{\|u_n\|^\beta}\,dx \geq C_2\int_\Omega |v_n|^\beta\,dx - C_3\int_\Omega \frac{|v_n|^p}{\|u_n\|^{\beta-p}}\,dx.$$

Consequently,

$$\lim_{n\to\infty}\int_\Omega \frac{f(x,u_n)u_n}{\|u_n\|^\beta}\,dx \geq C_2\int_\Omega |v|^\beta\,dx = C_2\int_{\Omega_0} |v|^\beta\,dx.$$

If $\mathrm{meas}(\Omega_0) > 0$, then

$$0 = \lim_{n\to\infty}\int_\Omega \frac{f(x,u_n)u_n}{\|u_n\|^\beta}\,dx \geq C_2\int_{\Omega_0} |v|^\beta\,dx > 0.$$

This is a contradiction. Hence $\operatorname{meas}(\Omega_0) = 0$. So, $v(x) = 0$ a.e. in Ω. Moreover, by (f_1), (f_2) and (f_5) we know that there is a constant $C_4 > 0$ such that

$$\frac{1}{\alpha} uf(x,u) - F(x,u) \geq -C_4|u|^p \quad \text{for all } (x,u) \in \Omega \times \mathbb{R}.$$

Consequently,

$$\begin{aligned}
&\frac{1}{\|u_n\|^p}\left[J(u_n) - \frac{1}{\alpha}\langle J'(u_n), u_n\rangle\right] \\
&\quad\geq \left(\frac{\sigma}{p} - \frac{1}{\alpha}\right)\left[M\left(\|u_n\|^p\right)\right]^{p-1} \\
&\qquad - \int_\Omega \left(F(x,u_n) - \frac{1}{\alpha} f(x,u_n)u_n\right)\frac{1}{\|u_n\|^p}\,dx \\
&\quad\geq \left(\frac{\sigma}{p} - \frac{1}{\alpha}\right) m_0^{p-1} - C_4 \int_\Omega |v_n|^p\,dx.
\end{aligned}$$

This implies $0 \geq (\frac{\sigma}{p} - \frac{1}{\alpha}) m_0^{p-1}$. But this is again impossible. Therefore $\{u_n\}$ is bounded in X. □

Note that $\alpha > \frac{p}{\sigma}$ in assumptions (f_4) and (f_5). Now, we consider the case $\alpha = \frac{p}{\sigma}$. In this case, we should strengthen our assumption on M. Then, we have the following result.

Lemma 2.4 *Assume that conditions* (M_1), (M_3) *and* (f_6) *are satisfied. Then any* $(PS)_c$ *sequence of the functional* J *is bounded.*

Proof It follows from the assumptions that

$$\begin{aligned}
c + 1 + \|u_n\| &\geq J(u_n) - \frac{\sigma}{p}\langle J'(u_n), u_n\rangle \\
&\geq \frac{\mu}{p}\|u_n\|^{ps} - \int_\Omega \left(F(x,u_n) - \frac{\sigma}{p} f(x,u_n)u_n\right)dx \\
&\geq \frac{\mu}{p}\|u_n\|^{ps}.
\end{aligned}$$

Since $ps > 1$, $\|u_n\|$ is bounded in X. □

3 Proofs of Theorems 1.1-1.3

In this section, we use the following fountain theorem to prove Theorems 1.1-1.3.

Lemma 3.1 (Fountain theorem [15]) *Let* X *be a Banach space with the norm* $\|\cdot\|$, *and let* X_i *be a sequence of subspace of* X *with* $\dim X_i < \infty$ *for each* $i \in \mathbb{N}$. *Further, set*

$$X = \overline{\bigoplus_{i=1}^{\infty} X_i}, \qquad Y_k = \bigoplus_{i=1}^{k} X_i, \qquad Z_k = \overline{\bigoplus_{i=k}^{\infty} X_i}.$$

Consider an even functional $\Phi \in C^1(X,\mathbb{R})$. *Assume that for each* $k \in \mathbb{N}$, *there exist* $\rho_k > \gamma_k > 0$ *such that*

(Φ_1) $a_k := \max_{u\in Y_k,\|u\|=\rho_k} \Phi(u) \le 0$,
(Φ_2) $b_k := \inf_{u\in Z_k,\|u\|=\gamma_k} \Phi(u) \to +\infty$, $k \to +\infty$,
(Φ_3) Φ *satisfies the* $(\mathrm{PS})_c$ *condition for every* $c > 0$.

Then Φ *has an unbounded sequence of critical values.*

Proof of Theorem 1.1 Since $X = W_0^{1,p}(\Omega)$ is a reflexive and separable Banach space, it is well known that there exist $e_j \in X$ and $e_j^* \in X^*$ $(j = 1, 2, \ldots)$ such that

(1) $\langle e_i, e_j^*\rangle = \delta_{ij}$, where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \ne j$.

(2) $X = \overline{\mathrm{span}\{e_1, e_2, \ldots\}}$, $X^* = \overline{\mathrm{span}\{e_1^*, e_2^*, \ldots\}}$.

Set $X_i = \mathrm{span}\{e_i\}$, $Y_k = \bigoplus_{i=1}^k X_i$, $Z_k = \overline{\bigoplus_{i=k}^\infty X_i}$.

In the following, we verify that J satisfies all the conditions of the fountain theorem.

1. By (f_3), the energy functional J is even.

2. In view of (f_2) and (f_4), there exist positive constants C_5 and C_6 such that

$$F(x,u) \ge C_5|u|^\alpha - C_6 \quad \text{for all } (x,u) \in \Omega \times \mathbb{R}.$$

Moreover, inequality (2.1) implies that there exist constants $C_7, C_8 > 0$ such that

$$\hat{M}(t) \le C_7 t^{1/\sigma} + C_8 \tag{3.1}$$

for all $t \ge 0$. Hence

$$J(u) \le \frac{1}{p}\bigl(C_7\|u\|^{\frac{p}{\sigma}} + C_8\bigr) - \int_\Omega \bigl(C_5|u|^\alpha - C_6\bigr)\,dx.$$

Since all norms are equivalent on the finite dimensional space Y_k and $\alpha > \frac{p}{\sigma}$, we have

$$a_k := \max_{u\in Y_k,\|u\|=\rho_k} J(u) < 0$$

for $\|u\| = \rho_k$ sufficiently large.

3. Set $\beta_k = \sup_{u\in Z_k,\|u\|=1}(\int_\Omega |u|^q\,dx)^{1/q}$. From the fact $Z_{k+1} \subset Z_k$, it is clear that $0 \le \beta_{k+1} \le \beta_k$. Hence $\beta_k \to \beta_0 \ge 0$ as $k \to +\infty$. By the definition of β_k, there exists $u_k \in Z_k$ with $\|u_k\| = 1$ such that

$$-1/k \le \beta_k - \left(\int_\Omega |u_k|^q\,dx\right)^{1/q} \le 0$$

for all $k \ge 1$. Then there exists a subsequence of $\{u_k\}$ (not relabeled) such that $u_k \rightharpoonup u$ in X and $\langle u, e_j^*\rangle = \lim_{k\to\infty}\langle u_k, e_j^*\rangle = 0$ for all $j \ge 1$. Thus $u = 0$. This shows $u_k \rightharpoonup 0$ in X and so $u_k \to 0$ in $L^q(\Omega)$. Thus $\beta_0 = 0$.

For any $\epsilon > 0$, (f_1) and (f_2) imply

$$\bigl|F(x,u)\bigr| \le \epsilon|u|^p + C(\epsilon)|u|^q$$

for some $C(\epsilon) > 0$. Therefore, for any $u \in Z_k$, there holds

$$\begin{aligned} J(u) &\geq \frac{1}{p}\sigma\left[M\left(\|u\|^p\right)\right]^{p-1}\|u\|^p - \int_\Omega F(x,u)\,dx \\ &\geq \frac{\sigma}{p}m_0^{p-1}\|u\|^p - \epsilon\int_\Omega |u|^p\,dx - C(\epsilon)\int_\Omega |u|^q\,dx \\ &\geq \left(\frac{\sigma}{p}m_0^{p-1} - \epsilon S_p^{-1}\right)\|u\|^p - C(\epsilon)\beta_k^q\|u\|^q, \end{aligned}$$

where S_p is the best Sobolev constant for the embedding of X into $L^p(\Omega)$, i.e.,

$$\|u\|_{L^p(\Omega)} \leq S_p^{-1/p}\|u\|.$$

Select ϵ so small that $\frac{\sigma}{p}m_0^{p-1} - \epsilon S_p^{-1} > 0$ and let

$$\gamma_k = \left(\frac{\frac{\sigma}{p}m_0^{p-1} - \epsilon S_p^{-1}}{2C(\epsilon)\beta_k^q}\right)^{\frac{1}{q-p}},$$

we obtain

$$b_k := \inf_{u\in Z_k,\|u\|=\gamma_k} J(u) \geq \frac{1}{2}\left(\frac{\sigma}{p}m_0^{p-1} - \epsilon S_p^{-1}\right)\gamma_k^p.$$

Since $\beta_k \to 0$, we have $b_k \to +\infty$ as $k \to +\infty$.

4. By Lemmas 2.1 and 2.2, J satisfies the $(\mathrm{PS})_c$ condition. Consequently, the conclusion follows from the fountain theorem. □

Proof of Theorem 1.2 It follows from Lemmas 2.1 and 2.3 that J satisfies the $(\mathrm{PS})_c$ condition. Similar to the proof of Theorem 1.1, we have that all the conditions of Lemma 3.1 are fulfilled. □

Proof of Theorem 1.3 By Lemmas 2.1 and 2.4, J satisfies the $(\mathrm{PS})_c$ condition. From the proof of Theorem 1.1, it is sufficient to show that condition (Φ_1) in Lemma 3.1 is satisfied.

By (f_1), (f_2) and (f_7), we deduce that for any $M > 0$, there exists a constant $C(M) > 0$ such that

$$F(x,u) \geq M|u|^{\frac{p}{\sigma}} - C(M).$$

Since (M_3) implies (M_2), it follows that (3.1) still holds. Therefore

$$J(u) \leq \frac{1}{p}\left(C_7\|u\|^{\frac{p}{\sigma}} + C_8\right) - \int_\Omega \left(M|u|^{\frac{p}{\sigma}} - C(M)\right)dx.$$

Note that all norms are equivalent on the finite dimensional space Y_k, there exists a constant $\mu_1 > 0$ such that

$$\begin{aligned} J(u) &\leq \frac{1}{p}\left(C_7\|u\|^{\frac{p}{\sigma}} + C_8\right) - \mu_1 M\|u\|^{\frac{p}{\sigma}} + C(M)|\Omega| \\ &= \left(\frac{C_7}{p} - \mu_1 M\right)\|u\|^{\frac{p}{\sigma}} + \frac{C_8}{p} + C(M)|\Omega|. \end{aligned}$$

Fix $M > \frac{C_7}{p\mu_1}$, then there exists large $\rho_k > 0$ such that

$$a_k := \max_{u \in Y_k, \|u\| = \rho_k} J(u) < 0.$$

This completes the proof. □

4 Proofs of Theorems 1.4 and 1.5

In this section, we consider a special case $f(x,u) = \lambda|u|^{q-2}u + |u|^{r-2}u$ $(1 < q < p < r < p^*)$. In this case, the associated energy functional is

$$J_\lambda(u) = \frac{1}{p}\hat{M}(\|u\|^p) - \frac{1}{q}\int_\Omega \lambda|u|^q\,dx - \frac{1}{r}\int_\Omega |u|^r\,dx, \tag{4.1}$$

where $\hat{M}(s) = \int_0^s [M(t)]^{p-1}\,dt$. It is well known that the energy functional $J_\lambda(u)$ is of class C^1 in $X = H_0^1(\Omega)$ and the solutions of problem (1.1) are the critical points of the energy functional. Since J_λ is not bounded below on X, it is useful to consider the problem on the Nehari manifold

$$\mathcal{N} = \{u \in X\backslash\{0\} | \langle J'_\lambda(u), u\rangle = 0\},$$

where $\langle\cdot,\cdot\rangle$ denotes the usual duality. Clearly, $u \in \mathcal{N}$ if and only if

$$[M(\|u\|^p)]^{p-1}\|u\|^p = \int_\Omega \lambda|u|^q\,dx + \int_\Omega |u|^r\,dx.$$

Since $\mathcal{N}$ is a much smaller set than X, it is easier to study $J_\lambda(u)$ on the Nehari manifold. Moreover, we have the following result.

Lemma 4.1 *Assume $\sigma r > p$ and M satisfies* (M_1), (M_2). *Then the energy functional J_λ is coercive and bounded below on $\mathcal{N}$.*

Proof We denote by C_s the best Sobolev constant for the embedding of X in $L^s(\Omega)$ with $1 < s < p^*$. In particular,

$$\|u\|_{L^s(\Omega)} \le C_s^{-1/p}\|u\| \quad \text{for all } u \in X\backslash\{0\}.$$

Let $u \in \mathcal{N}$. Then we have

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{p}\hat{M}(\|u\|^p) - \frac{1}{q}\int_\Omega \lambda|u|^q\,dx - \frac{1}{r}\int_\Omega |u|^r\,dx \\
&\ge \frac{1}{p}\sigma[M(\|u\|^p)]^{p-1}\|u\|^p - \frac{1}{q}\int_\Omega \lambda|u|^q\,dx - \frac{1}{r}\left\{[M(\|u\|^p)]^{p-1}\|u\|^p - \int_\Omega \lambda|u|^q\,dx\right\} \\
&= \left(\frac{\sigma}{p} - \frac{1}{r}\right)[M(\|u\|^p)]^{p-1}\|u\|^p - \lambda\left(\frac{1}{q} - \frac{1}{r}\right)\int_\Omega |u|^q\,dx \\
&\ge \left(\frac{\sigma}{p} - \frac{1}{r}\right)m_0^{p-1}\|u\|^p - \lambda\left(\frac{1}{q} - \frac{1}{r}\right)C_q^{-\frac{q}{p}}\|u\|^q.
\end{aligned}$$

Since $\frac{\sigma}{p} > \frac{1}{r}$ and $q < p < r$, J_λ is coercive and bounded below on $\mathcal{N}$. □

The Nehari manifold $\mathcal{N}$ is closely linked to the behavior of the fibering map $K_u : t \to J_\lambda(tu)$. For $u \in X$, we have

$$\begin{aligned}
K_u(t) &= \frac{1}{p}\hat{M}\big(t^p\|u\|^p\big) - \frac{1}{q}t^q\int_\Omega \lambda|u|^q\,dx - \frac{1}{r}t^r\int_\Omega |u|^r\,dx;\\
K_u'(t) &= \big[M\big(t^p\|u\|^p\big)\big]^{p-1}t^{p-1}\|u\|^p - \lambda t^{q-1}\int_\Omega |u|^q\,dx - t^{r-1}\int_\Omega |u|^r\,dx;\\
K_u''(t) &= \big[M\big(t^p\|u\|^p\big)\big]^{p-1}(p-1)t^{p-2}\|u\|^p\\
&\quad + p(p-1)t^{2p-2}\|u\|^{2p}\big[M\big(t^p\|u\|^p\big)\big]^{p-2}M'\big(t^p\|u\|^p\big)\\
&\quad - \lambda(q-1)t^{q-2}\int_\Omega |u|^q\,dx - (r-1)t^{r-2}\int_\Omega |u|^r\,dx.
\end{aligned}$$

Clearly, $tu \in \mathcal{N}$ if and only if $K_u'(t) = 0$. It is natural to split $\mathcal{N}$ into three parts corresponding to local minima, local maxima and points of inflection, i.e.,

$$\begin{aligned}
\mathcal{N}^+ &= \big\{u \in \mathcal{N} | K_u''(1) > 0\big\},\\
\mathcal{N}^0 &= \big\{u \in \mathcal{N} | K_u''(1) = 0\big\},\\
\mathcal{N}^- &= \big\{u \in \mathcal{N} | K_u''(1) < 0\big\}.
\end{aligned}$$

Then we have the following lemmas.

Lemma 4.2 *Suppose that u_0 is a local minimizer of J_λ on $\mathcal{N}$ and $u_0 \notin \mathcal{N}^0$. Then u_0 is a critical point of J_λ.*

Proof Our proof is almost the same as that of Binding et al. [16] and Brown and Zhang [17]. □

Lemma 4.3 *Suppose that M satisfies* (M_1) *and* (M_4). *Then there exists $\lambda_0 > 0$ such that $\mathcal{N}^0 = \emptyset$ for all $0 < \lambda < \lambda_0$.*

Proof For each $u \in \mathcal{N}$, we have

$$\begin{aligned}
K_u''(1) &= (p-q)\big[M\big(\|u\|^p\big)\big]^{p-1}\|u\|^p + p(p-1)\|u\|^{2p}\big[M\big(\|u\|^p\big)\big]^{p-2}M'\big(\|u\|^p\big)\\
&\quad - (r-q)\int_\Omega |u|^r\,dx \qquad (4.2)\\
&= -(r-p)\big[M\big(\|u\|^p\big)\big]^{p-1}\|u\|^p + p(p-1)\|u\|^{2p}\big[M\big(\|u\|^p\big)\big]^{p-2}M'\big(\|u\|^p\big)\\
&\quad + \lambda(r-q)\int_\Omega |u|^q\,dx. \qquad (4.3)
\end{aligned}$$

Furthermore, if $u \in \mathcal{N}^0$, then

$$\begin{aligned}
(p-q)m_0^{p-1}\|u\|^p &\le (p-q)\big[M\big(\|u\|^p\big)\big]^{p-1}\|u\|^p + p(p-1)\|u\|^{2p}\big[M\big(\|u\|^p\big)\big]^{p-2}M'\big(\|u\|^p\big)\\
&= (r-q)\int_\Omega |u|^r\,dx \le (r-q)C_r^{-\frac{r}{p}}\|u\|^r
\end{aligned}$$

and

$$\begin{aligned}\frac{(r-p)(d-1)}{d}m_0^{p-1}\|u\|^p &\le \frac{(r-p)(d-1)}{d}\big[M\big(\|u\|^p\big)\big]^{p-1}\|u\|^p\\ &\le (r-p)\big[M\big(\|u\|^p\big)\big]^{p-1}\|u\|^p\\ &\quad - p(p-1)\|u\|^{2p}\big[M\big(\|u\|^p\big)\big]^{p-2}M'\big(\|u\|^p\big)\\ &\le \lambda(r-q)C_q^{-\frac{q}{p}}\|u\|^q.\end{aligned}$$

Consequently,

$$\left(\frac{(p-q)m_0^{p-1}}{(r-q)C_r^{-r/p}}\right)^{1/(r-p)} \le \|u\| \le \left(\frac{\lambda d(r-q)C_q^{-q/p}}{(r-p)(d-1)m_0^{p-1}}\right)^{1/(p-q)}.$$

Therefore,

$$\lambda \ge \lambda_0 := \left(\frac{(p-q)m_0^{p-1}}{(r-q)C_r^{-r/p}}\right)^{(p-q)/(r-p)} \frac{(r-p)(d-1)m_0^{p-1}}{d(r-q)C_q^{-q/p}}.$$

Hence $\mathcal{N}^0 = \emptyset$ for all $0 < \lambda < \lambda_0$. □

Lemma 4.4 *Suppose that conditions* (M_1), (M_2) *hold. Assume also* $0 < \lambda < \lambda_0\frac{d}{d-1}$ *and* $q < \frac{p}{\sigma} < r$. *Then, for each* $u \in X\backslash\{0\}$, *there exist* t^+ *and* t^- *such that* $t^+u \in \mathcal{N}^+$ *and* $t^-u \in \mathcal{N}^-$.

Proof Fix $u \in X\backslash\{0\}$. Then it follows from condition (M_1) that

$$\begin{aligned}K_u'(t) &= \big[M\big(t^p\|u\|^p\big)\big]^{p-1}t^{p-1}\|u\|^p - \lambda t^{q-1}\int_\Omega |u|^q\,dx - t^{r-1}\int_\Omega |u|^r\,dx\\ &\ge m_0^{p-1}t^{p-1}\|u\|^p - \lambda t^{q-1}\int_\Omega |u|^q\,dx - t^{r-1}\int_\Omega |u|^r\,dx\\ &= t^{p-1}\big(m_0^{p-1}\|u\|^p - h(t)\big),\end{aligned}$$

where $h(t) = \lambda t^{q-p}\int_\Omega |u|^q\,dx + t^{r-p}\int_\Omega |u|^r\,dx$. Since

$$h'(t) = \lambda(q-p)t^{q-p-1}\int_\Omega |u|^q\,dx + (r-p)t^{r-p-1}\int_\Omega |u|^r\,dx,$$

we obtain $h'(t_M) = 0$ for

$$t_M = \left(\frac{\lambda(p-q)\int_\Omega |u|^q\,dx}{(r-p)\int_\Omega |u|^r\,dx}\right)^{\frac{1}{r-q}}.$$

Moreover,

$$\begin{aligned}h(t_M) &= \left(\frac{r-p}{p-q}+1\right)t_M^{r-p}\int_\Omega |u|^r\,dx\\ &= \frac{r-q}{p-q}\left(\frac{\lambda(p-q)\int_\Omega |u|^q\,dx}{(r-p)\int_\Omega |u|^r\,dx}\right)^{\frac{r-p}{r-q}}\int_\Omega |u|^r\,dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{r-q}{p-q}\left(\frac{\lambda(p-q)}{r-p}\right)^{\frac{r-p}{r-q}}\left(\int_\Omega |u|^q\,dx\right)^{\frac{r-p}{r-q}}\left(\int_\Omega |u|^r\,dx\right)^{\frac{p-q}{r-q}} \\
&\le \frac{r-q}{p-q}\left(\frac{\lambda(p-q)}{r-p}\right)^{\frac{r-p}{r-q}} C_q^{-\frac{q(r-p)}{p(r-q)}} C_r^{-\frac{r(p-q)}{p(r-q)}}\|u\|^p.
\end{aligned}
$$

Hence $m_0^{p-1}\|u\|^p > h(t_M)$ and so $K_u'(t_M) > 0$ for all

$$
0 < \lambda < m_0^{(p-1)\frac{r-q}{r-p}} C_q^{q/p} C_r^{\frac{r(p-q)}{p(r-p)}} \frac{r-p}{p-q}\left(\frac{p-q}{r-q}\right)^{\frac{r-q}{r-p}} = \lambda_0 \frac{d}{d-1}.
$$

On the other hand, it follows from (2.2) that

$$
\begin{aligned}
K_u'(t) &= \left[M\left(t^p\|u\|^p\right)\right]^{p-1} t^{p-1}\|u\|^p - \lambda t^{q-1}\int_\Omega |u|^q\,dx - t^{r-1}\int_\Omega |u|^r\,dx \\
&\le \frac{\hat{M}(t_0)}{\sigma t_0^{1/\sigma}}\|u\|^{\frac{p}{\sigma}} t^{\frac{p}{\sigma}-1} - \lambda t^{q-1}\int_\Omega |u|^q\,dx - t^{r-1}\int_\Omega |u|^r\,dx.
\end{aligned}
$$

Since $q < \frac{p}{\sigma} < r$, there exist $0 < t_1 < t_M < t_2$ such that $K_u'(t_1) < 0$, $K_u'(t_2) < 0$. Note that $\mathcal{N}^0 = \emptyset$, we deduce that there exist t^+, t^- such that $K_u'(t^+) = K_u'(t^-) = 0$ and $K_u''(t^+) > 0 > K_u''(t^-)$. Hence $t^+u \in \mathcal{N}^+$ and $t^-u \in \mathcal{N}^-$. □

Proof of Theorem 1.4 By Lemma 4.3, we write $\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^-$ and define

$$
\alpha_\lambda^+ = \inf_{u\in\mathcal{N}^+} J_\lambda(u), \qquad \alpha_\lambda^- = \inf_{u\in\mathcal{N}^-} J_\lambda(u).
$$

In view of Lemma 4.1 and the Ekeland variational principle [18], there exist minimizing sequences $\{u_n^+\}$ and $\{u_n^-\}$ for J_λ on $\mathcal{N}^+$ and $\mathcal{N}^-$ such that

$$
J_\lambda\left(u_n^+\right) = \alpha_\lambda^+ + o(1), \qquad J_\lambda\left(u_n^-\right) = \alpha_\lambda^- + o(1)
$$

and

$$
J_\lambda'\left(u_n^+\right) = o(1), \qquad J_\lambda'\left(u_n^-\right) = o(1).
$$

Furthermore, Lemma 2.1 implies that there exist u_0^+ and u_0^- such that $u_n^+ \to u_0^+$ and $u_n^- \to u_0^-$ strongly in X. Note that $u_n^+ \in \mathcal{N}^+$ implies $K_{u_n^+}'(1) = 0$ and $K_{u_n^+}''(1) > 0$. Letting $n \to \infty$, we deduce that $K_{u^+}'(1) = 0$ and $K_{u^+}''(1) \ge 0$, and so $u^+ \in \mathcal{N}^+ \cup \mathcal{N}^0$. By Lemma 4.3, we obtain $u^+ \in \mathcal{N}^+$. Similarly, $u^- \in \mathcal{N}^-$. Since $J_\lambda(u) = J_\lambda(|u|)$, we may assume u_0^+ and u_0^- are nonnegative. Moreover, it can be deduced from Lemma 4.2 that u_0^+ and u_0^- are nonnegative solutions of equation (1.1). Finally, since $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$, we infer that u_0^+ and u_0^- are two distinct solutions. □

Finally, we prove Theorem 1.5 by the following dual fountain theorem.

Theorem 4.1 (Dual fountain theorem [19]) *Assume that* $J \in C^1(X,\mathbb{R})$ *satisfies* $J(-u) = J(u)$. *If for every* $k \in \mathbb{N}$ *there exist* $\rho_k > r_k > 0$ *such that*

(B_1) $a_k := \inf_{u\in Z_k,\|u\|=\rho_k} J(u) \ge 0$ *as* $k \to \infty$,

(B_2) $b_k := \max_{u\in Y_k,\|u\|=r_k} J(u) < 0$,
(B_3) $d_k := \inf_{u\in Z_k,\|u\|\le\rho_k} J(u) \to 0$ *as* $k\to\infty$,
(B_4) *J satisfies the* $(\mathrm{PS})_c^*$ *condition for every* $c \in [d_{k_0}, 0)$, *that is, any sequence* $\{u_{n_j}\} \subset X$ *such that*

$$u_{n_j} \in Y_{n_j},\quad J(u_{n_j}) \to c, \qquad J|'_{Y_{n_j}} \to 0,\quad \text{as } n_j \to \infty$$

has a convergent subsequence.

Then J has a sequence of negative critical points $\{u_k\}$ *with* $J(u_k) \to 0$.

Proof of Theorem 1.5 1. Let

$$\beta_k := \sup_{u\in Z_k,\|u\|=1} \left(\int_\Omega |u|^q\,dx\right)^{1/q}.$$

Then by (M_1)-(M_2), for all $u \in Z_k$, there holds

$$\begin{aligned} J_\lambda(u) &= \frac{1}{p}\hat{M}\big(\|u\|^p\big) - \frac{1}{q}\int_\Omega \lambda|u|^q\,dx - \frac{1}{r}\int_\Omega |u|^r\,dx \\ &\ge \frac{1}{p}\sigma m_0^{p-1}\|u\|^p - \frac{\lambda}{q}\beta_k^q\|u\|^q - \frac{1}{r}C_r^{-\frac{r}{p}}\|u\|^r. \end{aligned}$$

Since $p < r$, we have

$$\frac{1}{2p}\sigma m_0^{p-1}\|u\|^p \ge \frac{1}{r}C_r^{-\frac{r}{p}}\|u\|^r \quad \text{for all } \|u\| \le R = \left(\frac{\sigma r C_r^{r/p} m_0^{p-1}}{2p}\right)^{1/(r-p)}.$$

Therefore,

$$J_\lambda(u) \ge \frac{1}{2p}\sigma m_0^{p-1}\|u\|^p - \frac{\lambda}{q}\beta_k^q\|u\|^q \quad \text{for all } u \in Z_k \text{ with } \|u\| \le R. \tag{4.4}$$

Choose

$$\rho_k = \left(\frac{2p\lambda\beta_k^q}{q\sigma m_0^{p-1}}\right)^{1/(p-q)}.$$

It follows from $\beta_k \to 0$ that $\rho_k \to 0$. Hence there exists $k_0 > 0$ such that $\rho_k \le R$ for all $k > k_0$. Consequently, $J_\lambda(u) \ge 0$ for all $k > k_0$, $u \in Z_k$ and $\|u\| = \rho_k$. This gives (B_1).

2. Since in the finite dimensional space Y_k all norms are equivalent, there exist positive constants C_9, C_{10} such that

$$\int_\Omega |u|^q\,dx \ge C_9\|u\|^q \quad \text{and} \quad \int_\Omega |u|^r\,dx \ge C_{10}\|u\|^r.$$

Then, by (2.1), we obtain for all $u \in Y_k$

$$J_\lambda(u) \le \frac{\hat{M}(t_0)}{pt_0^{1/\sigma}}\|u\|^{\frac{p}{\sigma}} - \frac{\lambda}{q}C_9\|u\|^q - \frac{C_{10}}{r}\|u\|^r.$$

Notice that $\frac{p}{\sigma} > q$ and $r > q$, we deduce that $J_\lambda(u) < 0$ for $\|u\| = r_k$ sufficiently small and (B_2) is proved.

3. It follows from (4.4) that, for all $u \in Z_k$ with $\|u\| \le \rho_k$ and $k > k_0$,

$$J_\lambda(u) \ge -\frac{\lambda}{q}\beta_k^q \rho_k^q.$$

Since $\beta_k \to 0$ and $\rho_k \to 0$ as $k \to \infty$, relation (B_3) is satisfied.

4. Finally, we prove that J_λ satisfies the $(PS)_c^*$ condition. Let $\{u_{n_j}\}$ be a sequence such that $\{u_{n_j}\} \subset Y_{n_j}$, $J_\lambda(u_{n_j}) \to c$ and $J|'_{Y_{n_j}} \to 0$ as $n_j \to \infty$. Then by (M_1)-(M_2) we have

$$\begin{aligned} c + 1 + \|u_{n_j}\| &\ge J_\lambda(u_{n_j}) - \frac{1}{r}\langle J'_\lambda(u_{n_j}), u_{n_j}\rangle \\ &= \frac{1}{p}\hat{M}(\|u_{n_j}\|^p) - \frac{1}{r}\big[M(\|u_{n_j}\|^p)\big]^{p-1}\|u_{n_j}\|^p - \lambda\left(\frac{1}{q} - \frac{1}{r}\right)\int_\Omega |u_{n_j}|^q\,dx \\ &\ge \left(\frac{\sigma}{p} - \frac{1}{r}\right)m_0^{p-1}\|u_{n_j}\|^p - \lambda\left(\frac{1}{q} - \frac{1}{r}\right)C_q^{-q/p}\|u_{n_j}\|^q. \end{aligned}$$

This implies $\|u_{n_j}\|$ is bounded. Obviously, f satisfies (f_1). Hence, by Lemma 2.1, J_λ satisfies the $(PS)_c^*$ condition.

We complete the proof by applying the dual fountain theorem. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Math and Physics Teaching Department, Hohai University, Changzhou, 213022, China. [2]School of Mathematics and Physics, Jiangsu University of Technology, Changzhou, 213015, China.

Acknowledgements
The authors are grateful for the referee's helpful suggestions and comments. This work is supported by the Fundamental Research Funds for the Central Universities (2016B07514).

Publisher's Note

Received: 21 January 2017 Accepted: 20 March 2017 Published online: 24 March 2017

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