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Multiple solutions on a p-biharmonic equation with nonlocal term

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Abstract
By variational methods we consider a p-biharmonic equation with nonlocal term on unbounded domain. We give sufficient conditions for the existence of solutions when some certain assumptions are fulfilled.

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Keywords: variational methods; nonlocal term; biharmonic equation

1 Introduction

In this paper, we consider the p-biharmonic equation

$$\begin{cases} (a + b\int_\Omega (|\Delta u|^p + |u|^p)\,dx)(\Delta_p^2 u + |u|^{p-2}u) \\ \quad = h(x)|u|^{r-2}u + H(x)|u|^{q-2}u + g(x), \quad x \in \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $1 < p < N/2$, $\Delta_p^2 = \Delta(|\Delta u|^{p-2}\Delta u)$, $\Omega \subset \mathbb{R}^N$ is an unbounded domain that is the exterior of some bounded domain $D \subset \mathbb{R}^N$ with smooth boundary $\partial D = \partial\Omega$, and $h(x)$, $H(x)$, and $g(x)$ are nonnegative functions; further assumptions will be listed later.

When $a = 0$ and $p = 2$, problems like (1.1) arise in the study of traveling waves in suspension bridges and the static deflection of an elastic plate in a fluid (see [1–4]). In fact, problems with biharmonic operator have been considered extensively by many authors in recent years [5–11]. In [5], the following biharmonic equation was investigated:

$$\begin{cases} \Delta^2 u + c\Delta u = bg(x,u), \quad x \in \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

By a variational method the authors proved the results on the existence of multiple solutions when $c < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. For the case $c > \lambda_1$, Zhang [11] considered the more general problem

$$\begin{cases} \Delta^2 u + a^2\Delta u = f(x,u) + g(x,u), \quad x \in \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

and some assumptions, approved the results on the existence of infinitely many solutions. The author in [7] studied the following fourth-order problem on bounded domain:

$$\begin{cases} \Delta^2 u + c\Delta u = \lambda h(x)|u|^{p-2}u + f(x,u), & x \in \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

By using variational methods the author proved the existence and multiplicity results, which are closely linked to the parameter λ. The function $f(x,u)$ in (1.4) satisfies the following conditions:

(H_1) $f(x,0) = 0$;
(H_2) $\lim_{s\to 0} \frac{f(x,s)}{s} = \mu$, $\lim_{s\to\infty} \frac{f(x,s)}{s} = l$, uniformly a.e. in $x \in \Omega$, $0 \le \mu < \lambda_1(\lambda_1 - c) < l < \infty$, where λ_1 is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Condition (H_2) implies that $f(x,u)$ and u are of the same order at both $u = 0$ and infinity. In our problem (1.1), however, even for $g(x) \equiv 0$, we allow $p < r < q$ or $p < q < r$; see Theorem 1.2. Therefore, the nonlinear term on the right of (1.1) does not satisfy (H_1) and (H_2).

For p-biharmonic equations, we refer to [12–16] and the references therein. In [13], the authors considered the following p-biharmonic equation with Hardy potential:

$$\begin{cases} \Delta_p^2 u - \frac{\mu}{|x|^{2p}}|u|^{p-2}u = f(x,u), & x \in \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

By the method of invariant set of descending flow, the result on the existence of sign-changing solutions was built. In order to get the result, several assumptions are made on $f(x,u)$. Particularly, $f(x,u)$ is required to satisfy the superlinear condition at $u = 0$; more precisely, $f(x,u)$ satisfies

(f_2) $\lim_{t\to 0} \frac{f(x,t)}{t^{p-1}} = 0$ uniformly in $x \in \overline{\Omega}$.

In problem (1.1), the relation $p < r$ or $p < q$ is also considered; see Theorem 1.3. This, together with $g(x)$, implies that (f_2) does not hold for problem (1.1).

In the present paper, inspired by [7, 13, 17–19], we consider the p-biharmonic equation (1.1) with nonlocal term on unbounded domain Ω. By the fountain theorem we prove the existence of infinitely many solutions for problem (1.1). We also obtain results on the existence of positive and negative energy solutions by the mountain pass lemma and Ekeland's variational principle, respectively. To the best of our knowledge, there seems to be little work on the existence of solutions of problems like (1.1).

We recall that $h(x)$, $H(x)$, and $g(x)$ are all nonnegative functions, and we make the following assumptions:

(A_1) $h(x) \in L^\alpha(\Omega) \cap L^\infty(\Omega)$ with $\alpha = \frac{p^*}{p^*-r}$;
(A_2) $H(x) \in L^\beta(\Omega) \cap L^\infty(\Omega)$ with $\beta = \frac{p^*}{p^*-q}$;
(A_3) $g(x) \in L^{p^{*\prime}}(\Omega) \cap L^\infty(\Omega)$ with $p^* = \frac{Np}{N-2p}$ and $p^{*\prime} = \frac{p^*}{p^*-1}$;

We consider problem (1.1) on the space $X = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega \left(|\Delta u|^p + |u|^p \right) dx \right)^{1/p}.$$

Define the functional related to problem (1.1) by

$$\begin{aligned} J(u) &= \frac{a}{p}\int_\Omega \left(|\Delta u|^p + |u|^p\right) dx + \frac{b}{2p}\left(\int_\Omega \left(|\Delta u|^p + |u|^p\right) dx\right)^2 \\ &\quad - \frac{1}{r}\int_\Omega h(x)|u|^r\, dx - \frac{1}{q}\int_\Omega H(x)|u|^q\, dx - \int_\Omega g(x)u\, dx \\ &= \frac{a}{p}\|u\|_X^p + \frac{b}{2p}\|u\|_X^{2p} - \frac{1}{r}\int_\Omega h(x)|u|^r\, dx \\ &\quad - \frac{1}{q}\int_\Omega H(x)|u|^q\, dx - \int_\Omega g(x)u\, dx. \end{aligned} \tag{1.6}$$

In view of (A_1)-(A_3), the functional $J(u)$ is well defined. It is not difficult to check that $J(u) \in C^1(X,R)$ and, for any $\varphi \in X$,

$$\begin{aligned} \left\langle J'(u), \varphi\right\rangle &= \left(a + b\|u\|_X^p\right)\int_\Omega \left(|\Delta u|^{p-2}\Delta u \Delta\varphi + |u|^{p-2}u\varphi\right) dx - \int_\Omega h(x)|u|^{r-2}u\varphi\, dx \\ &\quad - \int_\Omega H(x)|u|^{q-2}u\varphi\, dx - \int_\Omega g(x)\varphi\, dx. \end{aligned} \tag{1.7}$$

Particularly, this results in

$$\begin{aligned} \left\langle J'(u), u\right\rangle &= \left(a + b\|u\|_X^p\right)\int_\Omega \left(|\Delta u|^p + |u|^p\right) dx - \int_\Omega h(x)|u|^r\, dx - \int_\Omega H(x)|u|^q\, dx \\ &\quad - \int_\Omega g(x)u\, dx. \end{aligned} \tag{1.8}$$

Now, we give the definition of a weak solution for problem (1.1).

Definition 1.1 A function $u \in X$ is said to be a weak solution of (1.1) if, for any $\varphi \in X$,

$$\begin{aligned} &\left(a + b\|u\|_X^p\right)\int_\Omega \left(|\Delta u|^{p-2}\Delta u \Delta\varphi + |u|^{p-2}u\varphi\right) dx - \int_\Omega h(x)|u|^{r-2}u\varphi\, dx \\ &\quad - \int_\Omega H(x)|u|^{q-2}u\varphi\, dx - \int_\Omega g(x)\varphi\, dx = 0. \end{aligned} \tag{1.9}$$

It is well known that the solutions of (1.1) are precisely the critical points of the functional $J(u)$.

Our main results are the following.

Theorem 1.2 *Assume* (A_1)-(A_3). *Suppose that* $g(x) \equiv 0$ *and one of the following cases holds*:

(1) $1 < p < r < 2p < q < p^*$ *or* $1 < p < q < 2p < r < p^*$;

(2) $2p < \min\{r, q\} < p^*$.

Then problem (1.1) *has infinitely many solutions.*

Theorem 1.3 *Assume* (A_1)-(A_3). *If* $g(x) \not\equiv 0$ *and* $1 < r < p < q < 2p < p^*$ *or* $1 < q < p < r < 2p < p^*$, *then problem* (1.1) *has at least one solution* u *such that* $J(u) < 0$.

Theorem 1.4 *Assume* (A_1)-(A_3). *Suppose that* $g(x) \not\equiv 0$ *and* $\|g(x)\|_{L^{p^{*\prime}}}$ *is small enough. Then for each of the cases*
(1) $1 < p < r < 2p < q < p^*$ *or* $1 < p < q < 2p < r < p^*$,
(2) $2p < \min\{r, q\} < p^*$,
problem (1.1) *has at least one solution* u *such that* $J(u) > 0$.

This paper is organized as follows. In Section 2, we prove some important lemmas, which will be used in the proof of our main results. In Section 3, we obtain results on the existence of solutions for problem (1.1).

2 Preliminaries

In order to obtain the critical points of the functional $J(u)$, in this section, we build a variational structure and prove some lemmas. We also give a result on compact embedding, which will be used later.

Lemma 2.1 *Assume* (A_1)-(A_2). *Then, the embeddings* $X \hookrightarrow L^q(\Omega, H)$ *and* $X \hookrightarrow L^r(\Omega, h)$ *are compact for* $1 < q, r < p^* = Np/(N-2p)$.

Proof We only prove the compact embedding $X \hookrightarrow L^q(\Omega, H)$, and the other one can be similarly proved. Denote by $B_R(0)$ the ball centered at 0 of radius R large enough such that $D \subset B_R(0)$. It is clear that the embedding $X \hookrightarrow L^q(\Omega_R)$ is compact since the domain $\Omega_R = B_R(0) \setminus D$ is bounded. Let $\{u_n\} \subset X$ be a bounded sequence. Then there exists $u_0 \in X$ such that

$$u_n \rightharpoonup u_0 \quad \text{in } X, \tag{2.1}$$

$$u_n \to u_0 \quad \text{a.e. in } \Omega. \tag{2.2}$$

Then, it follows from (A_2) that, for any $\varepsilon > 0$,

$$\int_{\Omega_R} H(x)|u_n - u_0|^q \,dx \le \left\|H(x)\right\|_\infty \int_{\Omega_R} |u_n - u_0|^q \,dx < \varepsilon. \tag{2.3}$$

On the other hand, by (A_2) and the Hölder inequality we obtain that

$$\|u_n\|_{L^q(\Omega_R^c, H)} = \int_{\Omega_R^c} H(x)|u_n|^q \,dx \le c\|u_n\|_X^q \|H\|_{L^\beta(\Omega_R^c)}. \tag{2.4}$$

Since $H(x) \in L^\beta(\Omega)$,

$$\lim_{R \to +\infty} \left\|H(x)\right\|_{L^\beta(\Omega_R^c)} = 0. \tag{2.5}$$

Furthermore, we get from (2.4) that

$$\|u_n\|_{L^q(\Omega_R^c, H)} \le \varepsilon, \quad n = 1, 2, \ldots. \tag{2.6}$$

On the other hand, since the embedding $X \hookrightarrow L^q(\Omega_R)$ on the bounded domain Ω_R is compact, there exists $N_0 \in \mathbb{N}$ such that

$$\|u_n - u_0\|_{L^q(\Omega_R, H)} \le \varepsilon \tag{2.7}$$

for any $\varepsilon > 0$ and $n > N_0$. Then, we obtain from (2.6) and (2.7) that

$$\|u_n - u_0\|_{L^q(\Omega,H)} \leq \|u_n\|_{L^q(\Omega_R^c,H)} + \|u_0\|_{L^q(\Omega_R^c,H)} + \|u_n - u_0\|_{L^q(\Omega_R,H)} \leq 3\varepsilon, \tag{2.8}$$

which implies that $\{u_n\}$ strongly converges in $L^q(\Omega, H)$, and we complete the proof. □

In order to get our results by variational methods, we need to verify the Palais-Smale conditions at level c ($(PS)_c$ conditions for short).

Lemma 2.2 *For each of the cases listed below, $J(u)$ satisfies the $(PS)_c$ conditions for all $c \in R^1$:*

(1) $1 < r < p < q < 2p < p^*$ *or* $1 < q < p < r < 2p < p^*$;
(2) $1 < r < p < 2p < q < p^*$ *or* $1 < q < p < 2p < r < p^*$;
(3) $1 < p < r < 2p < q < p^*$ *or* $1 < p < q < 2p < r < p^*$;
(4) $2p < \min\{r, q\} < p^*$.

Proof Let $\{u_n\} \subset X$ be a $(PS)_c$ sequence, that is,

$$J(u_n) \to c, \qquad J'(u_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty. \tag{2.9}$$

Now, we divide the proof into two steps.

Step 1. $\{u_n\}$ is bounded in X.

In this step, we need to consider the relations of the exponents p, q, r, and $2p$. We only give the proof for the case of $r < p < q < 2p$, and the other cases can be proved similarly.

Let $\theta > 2p$. Then it follows from (2.9) that

$$\begin{aligned} c + \|u_n\|_X &\geq J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{\theta}\right) a\|u_n\|_X^p + \left(\frac{1}{2p} - \frac{1}{\theta}\right) b\|u_n\|_X^{2p} \\ &\quad - \left(\frac{1}{r} - \frac{1}{\theta}\right)\int_\Omega h(x)|u_n|^r\,dx - \left(\frac{1}{q} - \frac{1}{\theta}\right)\int_\Omega H(x)|u_n|^q\,dx \\ &\quad - \left(1 - \frac{1}{\theta}\right)\int_\Omega g(x)u_n\,dx. \end{aligned} \tag{2.10}$$

By the Sobolev inequalities and the continuous embedding $X \hookrightarrow L^{p^*}(\Omega)$, we get from (A_1)-(A_3) that

$$\begin{aligned} &\int_\Omega h(x)|u_n|^r\,dx \leq c\|h\|_{L^\alpha}\|u_n\|_X^r, \\ &\int_\Omega H(x)|u_n|^q\,dx \leq c\|H\|_{L^\beta}\|u_n\|_X^q \end{aligned} \tag{2.11}$$

and

$$\int_\Omega g(x)u_n\,dx \leq c\|g\|_{L^{p^{*\prime}}}\|u_n\|_X, \tag{2.12}$$

where c denotes a positive constant that may change from line to line. Then, it follows from (2.10)-(2.12) that

$$\begin{aligned} c + \|u_n\|_X \geq & \left(\frac{1}{p} - \frac{1}{\theta}\right) a\|u_n\|_X^p + \left(\frac{1}{2p} - \frac{1}{\theta}\right) b\|u_n\|_X^{2p} - c\left(\frac{1}{r} - \frac{1}{\theta}\right) \|h\|_{L^\alpha} \|u_n\|_X^r \\ & - c\left(\frac{1}{q} - \frac{1}{\theta}\right) \|H\|_{L^\beta} \|u_n\|_X^q - c\left(1 - \frac{1}{\theta}\right) \|g\|_{L^{p^{*\prime}}} \|u_n\|_X. \end{aligned} \tag{2.13}$$

Note that since $r < p < q < 2p$, $\{u_n\}$ is bounded in X.

For the other cases, for example, $1 < r < p < 2p < q$, we can choose $\theta = q$, for the case $2p < \min\{r, q\}$, we can choose θ such that $2p < \theta < \min\{r, q\}$, and the results on the boundedness can be similarly obtained, so we omit the proof.

Step 2. $\{u_n\}$ has a convergent subsequence in X.

Since $\{u_n\}$ is bounded, extracting a subsequence if necessary, we assume that $u_n \rightharpoonup u$ in X. By lemma 2.1 we get that

$$\int_\Omega h(x)|u_n - u|^r \, dx \to 0, \qquad \int_\Omega H(x)|u_n - u|^q \, dx \to 0 \quad \text{as } n \to \infty. \tag{2.14}$$

Furthermore, the Hölder inequality yields that

$$\int_\Omega h(x)|u_n|^{r-2}u_n|u_n - u| \, dx \to 0, \qquad \int_\Omega H(x)|u_n|^{q-2}u_n|u_n - u| \, dx \to 0 \tag{2.15}$$

as $n \to +\infty$. Let $\varphi = u_n - u$ in (1.7). Then

$$\begin{aligned} o(1) &= \langle J'(u_n), u_n - u \rangle \\ &= \left(a + b\|u_n\|_X^p\right) \int_\Omega \left(|\Delta u_n|^{p-2}\Delta u_n \Delta(u_n - u) + |u_n|^{p-2}u_n(u_n - u)\right) dx \\ &\quad - \int_\Omega h(x)|u_n|^{r-2}u_n|u_n - u| \, dx - \int_\Omega H(x)|u_n|^{q-2}u_n|u_n - u| \, dx \\ &\quad - \int_\Omega g(x)(u_n - u) \, dx. \end{aligned} \tag{2.16}$$

Thus, it follows from (2.15)-(2.16) and (A_3) that

$$\int_\Omega \left(|\Delta u_n|^{p-2}\Delta u_n \Delta(u_n - u) + |u_n|^{p-2}u_n(u_n - u)\right) dx = o(1). \tag{2.17}$$

On the other hand, by the weak convergence $u_n \rightharpoonup u$ we get that

$$\int_\Omega \left(|\Delta u|^{p-2}\Delta u \Delta(u_n - u) + |u|^{p-2}u(u_n - u)\right) dx = o(1). \tag{2.18}$$

Therefore, we obtain from (2.17) and (2.18) that

$$\begin{aligned} &\int_\Omega \left(|\Delta u_n|^{p-2}\Delta u_n - |\Delta u|^{p-2}\Delta u\right)(\Delta u_n - \Delta u) \, dx \\ &\quad + \int_\Omega \left(|u_n|^{p-2}u_n - |u|^{p-2}u\right)(u_n - u) \, dx \to 0 \end{aligned} \tag{2.19}$$

as $n \to +\infty$.

Consequently, using the standard inequalities

$$\langle |\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta, \xi - \zeta \rangle \geq \begin{cases} c|\xi - \zeta|^p & \text{for } p \geq 2, \\ c|\xi - \zeta|^2(|\xi| + |\zeta|)^{p-2} & \text{for } 1 < p < 2, \end{cases} \tag{2.20}$$

we get that

$$\int_\Omega \left(|\Delta(u_n - u)|^p + |u_n - u|^p\right) dx \to 0 \quad \text{as } n \to +\infty, \tag{2.21}$$

that is, $u_n \to u$ in X, and we complete the proof. □

In the following, we introduce an important lemma, which will be used to prove the existence of multiple solutions. Note that X is a reflexive and separable Banach space and, therefore, there exist $e_j \in X$ and $e_j^* \in X^*$ $(j = 1, 2, \ldots)$ such that

(1) $\langle e_i, e_j^* \rangle = \delta_{ij}$, where $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$;

(2) $X = \overline{\operatorname{span}\{e_1, e_2, \ldots\}}$, $X^* = \overline{\operatorname{span}\{e_1^*, e_2^*, \ldots\}}$.

We write

$$X_j = \operatorname{span}\{e_j\}, \qquad Y_k = \bigoplus_{j=1}^{k} X_j, \qquad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}, \quad j, k = 1, 2, \ldots. \tag{2.22}$$

Lemma 2.3 (Fountain theorem [20]) *Assume that* $J_1(u) \in C^1(X, \mathbb{R}^1)$ *and* $J_1(u) = J_1(-u)$. *Suppose that, for every* $k \in \mathbb{N}$, *there exist* $\rho_k > \gamma_k > 0$ *such that*

(B$_1$) $a_k = \inf_{u \in Z_k, \|u\|_X = \gamma_k} J_1(u) \to +\infty$ *as* $k \to \infty$,

(B$_2$) $b_k = \sup_{u \in Y_k, \|u\|_X = \rho_k} J_1(u) \leq 0$,

(B$_3$) $J_1(u)$ *satisfies the* $(PS)_c$ *conditions for every* $c > 0$.

Then $J_1(u)$ *has a sequence of critical points* $\{u_k\}$ *such that* $J_1(u_k) \to \infty$ *as* $k \to \infty$.

The following lemma will be used in the proof of Theorem 1.3.

Lemma 2.4 *Assume* (A$_1$)-(A$_3$) *and let* $1 < r < p < q < 2p < p^*$ or $1 < q < p < r < 2p < p^*$. *Then there exist* $\rho, \alpha > 0$ *such that* $J(u) \geq \alpha$ *with* $\|u\|_X = \rho$.

Proof By assumptions (A$_1$)-(A$_3$) and the Hölder inequality we obtain that

$$\begin{aligned} J(u) \geq{}& \frac{1}{p} a\|u\|_X^p + \frac{b}{2p}\|u\|_X^{2p} - \frac{1}{r} c\|u\|_X^r \|h\|_{L^\alpha} - \frac{1}{q} c\|u\|_X^q \|H\|_{L^\beta} \\ & - c\|u\|_X \|g\|_{L^{p^{*\prime}}}. \end{aligned} \tag{2.23}$$

Since $r < 2p$ and $q < 2p$, there exist $\alpha > 0$ and large $\rho > 0$ such that $J(u) \geq \alpha$ for $\|u\|_X = \rho$. Thus, we complete the proof. □

3 Main results and proofs

In this section, we give the proofs of Theorems 1.2-1.4.

Proof of Theorem 1.2 We only give the proof for the case of $p < r < 2p < q$, and the other cases can be similarly proved. In order to prove Theorem 1.2, we only need to verify conditions (B_1)-(B_3) in Lemma 2.3. For this purpose, we define

$$\beta_k = \sup_{u\in Z_k, u\neq 0} \frac{\|u\|_{L^r(\Omega,h)}}{\|u\|_X} = \sup_{u\in Z_k, \|u\|_X=1} \|u\|_{L^r(\Omega,h)}, \tag{3.1}$$

$$\sigma_k = \sup_{u\in Z_k, u\neq 0} \frac{\|u\|_{L^q(\Omega,H)}}{\|u\|_X} = \sup_{u\in Z_k, \|u\|_X=1} \|u\|_{L^q(\Omega,H)}. \tag{3.2}$$

Then

$$\|u\|_{L^r(\Omega,h)} \le \beta_k \|u\|_X, \qquad \|u\|_{L^q(\Omega,H)} \le \sigma_k \|u\|_X, \quad \forall u \in Z_k. \tag{3.3}$$

Moreover, proceeding in a similar manner to that of [21], we get that

$$\beta_k \to 0, \qquad \sigma_k \to 0 \quad \text{as } k \to \infty.$$

In view of (3.3), we obtain from (1.6) and the Hölder inequality that

$$J(u) \ge \frac{1}{p} a\|u\|_X^p + \frac{b}{2p}\|u\|_X^{2p} - c\beta_k^r \|u\|_X^r - c\sigma_k^q \|u\|_X^q. \tag{3.4}$$

Choosing

$$\gamma_k = \min\left\{ \left(\frac{qa}{2p\sigma_k^q}\right)^{\frac{1}{q-p}}, \left(\frac{ra}{2p\beta_k^r}\right)^{\frac{1}{r-p}} \right\},$$

it follows from $\sigma_k \to 0$ and $\beta_k \to 0$ that $\gamma_k \to +\infty$ as $k \to \infty$. Furthermore,

$$J(u) \ge \frac{b}{2p}\|u\|_X^{2p} \to +\infty \tag{3.5}$$

as $k \to \infty$ with $\|u\|_X = \gamma_k$, which verifies (B_1).

Note that since Y_k is a finite-dimensional space, all the norms on Y_k are equivalent. Therefore, there exists large $\rho_k > \sigma_k$ such that $J(u) < 0$ with $u \in Y_k$ and $\|u_k\| = \rho_k$. Thus, condition (B_2) is satisfied. It is obvious that (B_3) holds by Lemma 2.2. As a result, we get from Lemma 2.3 that $J(u)$ admits a sequence of critical points, that is, problem (1.1) has infinitely many weak solutions. □

In the following, we give the proof of Theorem 1.3, and our proof based on the Ekeland's variational principle [22].

Proof of Theorem 1.3 Since $g(x) \not\equiv 0$, we can choose $\varphi \in C_0^\infty(\Omega)$ such that $\int_\Omega g(x)\varphi\,dx > 0$. Then,

$$\begin{aligned} J(t\varphi) = {} & \frac{t^p}{p} a\|\varphi\|_X^p + \frac{t^{2p}}{2p} b\|\varphi\|_X^{2p} - \frac{t^r}{r}\int_\Omega h(x)|\varphi|^r\,dx - \frac{t^q}{q}\int_\Omega H(x)|\varphi|^q\,dx \\ & - t\int_\Omega g(x)\varphi\,dx < 0 \end{aligned} \tag{3.6}$$

for small $t > 0$. It follows from (3.6) and Lemma 2.4 that

$$C_\rho = \inf_{u\in\overline{B_\rho}(0)} J(u) < 0 \quad \text{and} \quad \inf_{u\in\partial B_\rho(0)} J(u) > 0, \tag{3.7}$$

where $\rho > 0$ is given in Lemma 2.4, and $B_\rho(0) \subset X$ is an open ball with radius ρ. For $\varepsilon_n \to 0$ small enough, we have

$$0 < \varepsilon_n < \inf_{u\in\partial B_\rho(0)} J(u) - \inf_{u\in\overline{B_\rho}(0)} J(u). \tag{3.8}$$

Then, by Ekeland's variational principle, there exists $\{u_n\} \subset \overline{B_\rho}(0)$ such that

$$C_\rho \le J(u_n) < C_\rho + \varepsilon_n \tag{3.9}$$

and

$$J(u_n) < J(u) + \varepsilon_n \|u_n - u\|_X, \quad \forall u \in \overline{B_\rho}(0), u \neq u_n. \tag{3.10}$$

Therefore, relations (3.8) and (3.9) imply that

$$J(u_n) < C_\rho + \varepsilon_n \le \inf_{u\in\overline{B_\rho}(0)} J(u_n) + \varepsilon_n < \inf_{u\in\partial B_\rho(0)} J(u_n), \tag{3.11}$$

which implies that $u_n \in B_\rho(0)$.

We define the function $G(u) : \overline{B_\rho}(0) \to R$ by

$$G(u) = J(u) + \varepsilon_n \|u_n - u\|_X, \quad u \in \overline{B_\rho}(0). \tag{3.12}$$

Then, it follows from (3.10) that $G(u_n) < G(u)$ for $u \neq u_n$ and u_n is a strict local minimum of $G(u)$. Furthermore, we get that

$$t^{-1}\big(G(u_n + tv) - G(u_n)\big) \ge 0 \quad \text{for small } t > 0 \text{ and } \forall v \in B_1(0), \tag{3.13}$$

which implies that

$$t^{-1}\big(J(u_n + tv) - J(u_n)\big) + \varepsilon_n \|v\|_X \ge 0. \tag{3.14}$$

Passing to the limit as $t \to 0^+$ in (3.14), it follows that

$$\big\langle J'(u_n), v\big\rangle + \varepsilon_n \|v\|_X \ge 0, \quad \forall v \in B_1(0). \tag{3.15}$$

On the other hand, when v is replaced by $-v$, we get from (3.15) that

$$-\big\langle J'(u_n), v\big\rangle + \varepsilon_n \|v\|_X \ge 0, \quad \forall v \in B_1(0). \tag{3.16}$$

Consequently, it follows from (3.15)-(3.16) that $\|J'(u_n)\| \le \varepsilon_n$. Therefore, there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that $J(u_n) \to C_\rho$ and $J'(u_n) \to 0$ in X^* as $n \to \infty$. According to

Lemma 2.2, there is a convergent subsequence of $\{u_n\}$ in X, still denoted by $\{u_n\}$, such that $u_n \to u$ in X and $J(u) = C_\rho < 0$. Thus, u is a solution of problem (1.1). This completes the proof. □

In the last part of this section, we give the proof of Theorem 1.4. To complete the proof by a variational method, we need the following mountain pass lemma.

Proposition 3.1 (Mountain pass lemma [23]) *Let X be a Banach space, and let $J \in C^1(X,\mathbb{R})$ satisfy the $(PS)_c$ conditions. Suppose that $J(0) = 0$ and*

(M_1) *there are constants $\rho, \alpha > 0$ such that $J_{\partial B_\rho} \geq \alpha$;*
(M_2) *there is $e \in X \setminus B_\rho$ such that $J(e) \leq 0$.*

Then $J(u)$ possesses a critical value $c \geq \alpha$. Moreover, c can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma[0,1]} J(u) \geq \alpha > 0,$$

where

$$\Gamma = \bigl\{\gamma \in C\bigl([0,1],X\bigr) : \gamma(0) = 0, \gamma(1) = e\bigr\}.$$

Proof of Theorem 1.4 In order to make the proof more concise, we only consider the case of $p < r < 2p < q$. Lemma 2.2 shows that $J(u)$ satisfies the $(PS)_c$ conditions. Then, in the following, we will verify conditions (M_1) and (M_2) in Proposition 3.1. By the Young inequality with $0 < \varepsilon < a/2$ it follows from (2.23) that

$$\begin{aligned} J(u) &\geq \frac{1}{p}(a-\varepsilon)\|u\|_X^p + \frac{b}{2p}\|u\|_X^{2p} - \frac{1}{r}c\|u\|_X^r\|h\|_{L^\alpha} - \frac{1}{q}c\|u\|_X^q\|H\|_{L^\beta} - c(\varepsilon)\|g\|_{L^{p^{*\prime}}} \\ &\geq \frac{a}{2p}\|u\|_X^p + \frac{b}{2p}\|u\|_X^{2p} - \frac{1}{r}c\|u\|_X^r\|h\|_{L^\alpha} - \frac{1}{q}c\|u\|_X^q\|H\|_{L^\beta} \\ &\quad - c(\varepsilon)\|g\|_{L^{p^{*\prime}}}. \end{aligned} \tag{3.17}$$

Since $p < r < 2p < q$ and $0 < \|g\|_{L^{p^{*\prime}}}$ is small enough, there exist $\alpha > 0$ and small $\rho > 0$ such that $J(u) \geq \alpha$ with $\|u\|_X = \rho$, and this verifies (M_1).

For fixed $\varphi \in X$, note that

$$J(t\varphi) = \frac{t^p}{p}a\|\varphi\|_X^p + \frac{t^{2p}}{2p}b\|\varphi\|_X^{2p} - \frac{t^r}{r}\int_\Omega h|\varphi|^r\,dx - \frac{t^q}{q}\int_\Omega H|\varphi|^q\,dx - t\int_\Omega g\varphi\,dx \to -\infty$$

as $t \to +\infty$. Then, there exists large $t_0 > 0$ such that $\|t_0\varphi\|_X > \rho$ and $J(t_0\varphi) < 0$. Thus, (M_2) holds for $e = t_0\varphi$. As a result, by Proposition 3.1 there exists $u \in X$, a solution of problem (1.1), such that $J(u) \geq \alpha > 0$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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