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# Positive solutions of a system for nonlinear singular higher-order fractional differential equations with fractional multi-point boundary conditions

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**Abstract**
This paper deals with the existence and multiplicity of positive solutions for a system of nonlinear singular higher-order fractional differential equations with fractional multi-point boundary conditions. The main tool used in the proof is fixed point index theory. Some limit type conditions for ensuring the existence of positive solutions are given, and our conditions are suitable for more general functions.

**MSC:** 34B10; 34B16; 34B18; 45G10

**Keywords:** singular fractional differential equations; positive solution; cone; fixed point index

## 1 Introduction

We discuss the following multi-point boundary problem of the system for nonlinear singular higher-order fractional differential equations:

$$\begin{cases} D^{\alpha}_{0+}u(x) + h_1(x)f_1(x,u(x),v(x)) = 0, & x \in (0,1), \\ D^{\beta}_{0+}v(x) + h_2(x)f_2(x,u(x),v(x)) = 0, & x \in (0,1), \\ u^{(i)}(0) = 0, \quad 0 \le i \le n-2, & D^{\mu}_{0+}u(1) = \sum_{k=1}^{p} a_k D^{\mu}_{0+}u(\xi_k), \\ v^{(j)}(0) = 0, \quad 0 \le j \le m-2, & D^{\nu}_{0+}v(1) = \sum_{k=1}^{q} b_k D^{\nu}_{0+}v(\eta_k), \end{cases} \tag{1}$$

where $D^{\alpha}_{0+}$, $D^{\beta}_{0+}$ are the standard Riemann-Liouville fractional derivative of order $\alpha \in (n-1,n]$, $\beta \in (m-1,m]$, $\mu \in [1,n-2]$, $\nu \in [1,m-2]$ for $n,m \in \mathbb{N}^+$ and $n,m \ge 3$, $a_i, b_j \in \mathbb{R}^+$, $i = 1,2,\dots,p$, $j = 1,2,\dots,q$ for $p,q \in \mathbb{N}^+$, $f_k \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $h_k \in C((0,1),\mathbb{R}^+)$, $\mathbb{R}^+ = [0,+\infty)$, $h_k(x)$ $(k=1,2)$ is allowed to be singular at $x=0$ and/or $x=1$ and

$$0 < \xi_1 < \xi_2 < \cdots < \xi_p < 1, \qquad 0 < \eta_1 < \eta_2 < \cdots < \eta_q < 1.$$

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electro-analytical chemistry, biology, control theory, fitting of experimental data, and so forth. Hence fractional differential equations have attracted great research interest in recent years, and for more details we refer the reader to [1–7] and the references cited therein. Recently, the existence and multiplicity of positive solutions for the nonlinear fractional differential equations have been researched, see [8–16] and the references therein. For instance, Zhang *et al.* [17] studied the existence of two positive of following singular fractional boundary value problems:

$$\begin{cases} D_{0+}^{\alpha}u(t)+f(t,u(t))=0, & 0<t<1, \\ u(0)=0, \quad D_{0+}^{\beta}u(0)=0, & D_{0+}^{\beta}u(1)=\sum_{i=0}^{\infty}\xi_i D_{0+}^{\beta}u(\eta_i), \end{cases}$$

where $D_{0+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order $\alpha \in (2,3]$, $\beta \in [1,2], \xi_i, \eta_i \in (0,1), \alpha-\beta \geq 1$ with $\sum_{i=0}^{\infty}\xi_i\eta_i^{\alpha-\beta-1}<1$.

In [18–21], the authors studied the existence of a positive solutions of two types of systems for nonlinear fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t)+\lambda f(t,u(t),v(t))=0, & t\in(0,1), \\ D_{0+}^{\beta}v(t)+\mu g(t,u(t),v(t))=0, & t\in(0,1), \end{cases} \tag{2}$$

with boundary conditions:

$$\begin{cases} u(0)=u'(0)=\cdots=u^{(n-2)}(0)=0, & u(1)=\int_0^1 v(t)\,dH(t), \\ v(0)=v'(0)=\cdots=v^{(n-2)}(0)=0, & v(1)=\int_0^1 u(t)\,dK(t), \end{cases} \tag{3}$$

$$\begin{cases} u^{(i)}(0)=0, \quad 0\leq i\leq n-2, & u(1)=\sum_{k=1}^{p}a_k u(\xi_k), \\ v^{(j)}(0)=0, \quad 0\leq j\leq m-2, & v(1)=\sum_{k=1}^{q}b_k v(\eta_k), \end{cases}$$

and

$$\begin{cases} u^{(i)}(0)=v^{(i)}(0)=0, & 0\leq i\leq n-2, \\ D_{0+}^{\gamma}u(1)=\phi_1(u), & D_{0+}^{\gamma}v(1)=\phi_2(v), \end{cases}$$

where $D_{0+}^{\alpha}, D_{0+}^{\beta}$, and $D_{0+}^{\gamma}$ are the standard Riemann-Liouville fractional derivative, $\alpha,\beta \in (n-1,n], \gamma\in[1,n-2]$ for $n\geq 3, \lambda,\mu>0$. Equations (2) with $\lambda f(t,u,v)$ and $\mu g(t,u,v)$ replaced by $\widetilde{f}(t,v)$ and $\widetilde{g}(t,u)$, respectively, the existence and multiplicity of positive solutions of the system (2), (3) was investigated in [22]. The extreme limits

$$f_{\delta}^{s} =: \limsup_{u+v\to\delta}\max_{t\in[0,1]}\frac{f(t,u,v)}{u+v}, \qquad g_{\delta}^{s} =: \limsup_{u+v\to\delta}\max_{t\in[0,1]}\frac{g(t,u,v)}{u+v},$$
$$f_{\delta}^{i} =: \liminf_{u+v\to\delta}\min_{t\in[\theta,1-\theta]}\frac{f(t,u,v)}{u+v}, \qquad g_{\delta}^{i} =: \liminf_{u+v\to\delta}\min_{t\in[\theta,1-\theta]}\frac{g(t,u,v)}{u+v},$$

are used in [19, 20], where $\theta\in(0,\frac{1}{2}), \delta=0^{+}$ or $+\infty$. Some similar extreme limits are used in [18, 21, 23–25]. However, for equation systems [18–21, 23–25] and a single equation using the extreme limits, there is no essential difference.

Motivated by the above mentioned work, in this paper, we present some limit type conditions and discuss the existence and multiplicity of positive solutions of the singular fractional multi-point boundary problems (1) by using fixed point index theory in a cone. The results obtained here are different from those in [18–21, 23–25], and some examples explain our conditions are applicable for more general functions.

## 2 Preliminaries

**Definition 2.1** [26] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0,+\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,ds, \quad t > 0,$$

provided the right side is pointwise defined on $(0,+\infty)$, where $\Gamma(\alpha)$ is the Euler gamma function. The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $u : (0,+\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)\,ds, \quad t > 0,$$

where $n = [\alpha] + 1, [\alpha]$ denotes the integer part of number $\alpha$, provided the right side is pointwise defined on $(0,+\infty)$.

**Lemma 2.2** [27] *Let $x \in L^p(0,1)$ $(1 \le p \le +\infty)$, $\rho > \sigma > 0$.*

(i) $D_{0+}^{\sigma} I_{0+}^{\rho} x(t) = I_{0+}^{\rho-\sigma} x(t), D_{0+}^{\sigma} I_{0+}^{\sigma} x(t) = x(t), I_{0+}^{\rho} I_{0+}^{\sigma} x(t) = I_{0+}^{\rho+\sigma} x(t)$ *hold at almost every point $t \in (0,1)$. If $\rho + \sigma > 1$, then the above third equation holds at any point of* $[0,1]$;

(ii) $D_{0+}^{\sigma} t^{\rho-1} = \Gamma(\rho) t^{\rho-\sigma-1} / \Gamma(\rho-\sigma), t > 0$.

**Lemma 2.3** [27] *Let $\alpha > 0, n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$, $n$ is the smallest integer greater than or equal to $\alpha$. Then, for any $y_1 \in L^1(0,1)$, the solution of the fractional differential equation $D_{0+}^{\alpha} u(t) + y_1(t) = 0$ $(0 < t < 1)$ is*

$$u(t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y_1(s)\,ds = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad 0 < t < 1,$$

*where $c_1, c_2, \ldots, c_n$ are arbitrary real constants.*

**Lemma 2.4** *Let $\sum_{j=1}^{p} a_j \xi_j^{\alpha-\mu-1} \in [0,1), \alpha \in (n-1,n], \mu \in [1,n-2]$ $(n \ge 3)$ and $y_1 \in C[0,1]$. Then the solution of the fractional boundary value problem*

$$\begin{cases} D_{0+}^{\alpha} u(t) + y_1(t) = 0, & 0 < t < 1, \\ u^{(i)}(0) = 0, & 0 \le i \le n-2, \\ D_{0+}^{\mu} u(1) = \sum_{j=1}^{p} a_j D_{0+}^{\mu} u(\xi_j), \end{cases} \tag{4}$$

*is given by*

$$u(t) = \int_0^1 G_1(t,s) y_1(s)\,ds, \tag{5}$$

$$G_1(t,s) = g_1(t,s) + \frac{t^{\alpha-1}}{d_1}\sum_{j=1}^{p} a_j h_1(\xi_j, s), \tag{6}$$

*where* $d_1 = 1 - \sum_{j=1}^{p} a_j \xi_j^{\alpha-\mu-1}$,

$$g_1(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-\mu-1} - (t-s)^{\alpha-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-1}(1-s)^{\alpha-\mu-1}, & 0 \le t \le s \le 1, \end{cases} \tag{7}$$

$$h_1(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-\mu-1}(1-s)^{\alpha-\mu-1} - (t-s)^{\alpha-\mu-1}, & 0 \le s \le t \le 1, \\ t^{\alpha-\mu-1}(1-s)^{\alpha-\mu-1}, & 0 \le t \le s \le 1. \end{cases} \tag{8}$$

*Proof* By using Lemma 2.3, the solution for the above equation is

$$u(t) = \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y_1(s)\,ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_1, c_2, \ldots, c_n$ are arbitrary real constants. By $u(0) = 0$, we have $c_n = 0$. Then

$$u(t) = \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y_1(s)\,ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_{n-1} t^{\alpha-n+1}. \tag{9}$$

Differentiating (9), we have

$$u'(t) = \frac{1-\alpha}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-2} y_1(s)\,ds + c_1(\alpha-1)t^{\alpha-2} + \cdots + c_{n-1}(\alpha-n+1)t^{\alpha-n}.$$

By $u'(0) = 0$, we have $c_{n-1} = 0$. Similarly, we get $c_2 = c_3 = \cdots = c_{n-2} = 0$. Hence

$$u(t) = \frac{-1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y_1(s)\,ds + c_1 t^{\alpha-1}. \tag{10}$$

By $D_{0+}^{\mu} u(1) = \sum_{j=1}^{p} a_j D_{0+}^{\mu} u(\xi_j)$ and Lemma 2.2, we get

$$D_{0+}^{\mu} u(t) = \frac{1}{\Gamma(\alpha-\mu)}\left[c_1 \Gamma(\alpha) t^{\alpha-\mu-1} - \int_0^t (t-s)^{\alpha-\mu-1} y_1(s)\,ds\right],$$
$$c_1 = \frac{1}{d_1 \Gamma(\alpha)}\left[\int_0^1 (1-s)^{\alpha-\mu-1} y_1(s)\,ds - \sum_{j=1}^{p} a_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-\mu-1} y_1(s)\,ds\right].$$

Substituting $c_1$ into (10), we see that the unique solution of the problem (4) is

$$\begin{aligned} u(t) &= \frac{t^{\alpha-1}}{d_1 \Gamma(\alpha)}\left[\int_0^1 (1-s)^{\alpha-\mu-1} y_1(s)\,ds - \sum_{j=1}^{p} a_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-\mu-1} y_1(s)\,ds\right] \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} y_1(s)\,ds \\ &= \frac{1}{\Gamma(\alpha)}\left[\int_0^t \left[t^{\alpha-1}(1-s)^{\alpha-\mu-1} - (t-s)^{\alpha-1}\right] y_1(s)\,ds\right. \\ &\quad + \int_t^1 t^{\alpha-1}(1-s)^{\alpha-\mu-1} y_1(s)\,ds - \frac{1-d_1}{d_1}\int_0^1 t^{\alpha-1}(1-s)^{\alpha-\mu-1} y_1(s)\,ds \end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{j=1}^{p} a_j \int_0^{\xi_j} (\xi_j - s)^{\alpha-\mu-1} y_1(s)\,ds \Bigg] \\
&= \int_0^1 g_1(t,s) y_1(s)\,ds + \frac{t^{\alpha-1}}{d_1} \sum_{j=1}^{p} a_j \Bigg[ \int_{\xi_j}^1 \xi_j^{\alpha-\mu-1} (1-s)^{\alpha-\mu-1} y_1(s)\,ds \\
&\quad + \int_0^{\xi_j} \big[ \xi_j^{\alpha-\mu-1}(1-s)^{\alpha-\mu-1} - (\xi_j - s)^{\alpha-\mu-1} \big] y_1(s)\,ds \Bigg] \\
&= \int_0^1 g_1(t,s) y_1(s)\,ds + \frac{t^{\alpha-1}}{d_1} \sum_{j=1}^{p} a_j \int_0^1 h_1(\xi_j,s) y_1(s)\,ds \\
&= \int_0^1 G_1(t,s) y_1(s)\,ds,
\end{aligned}
$$

*i.e.* (5) holds.

Conversely, if $u \in C[0,1]$ is a solution of the integral equation (5), from Lemma 2.2 we easily see that $u$ satisfies the equation and boundary conditions of (4). □

**Lemma 2.5** *Under the assumptions of Lemma* 2.4, *the functions* $g_1(t,s)$ *and* $h_1(\xi_j,s)$ *defined by* (7) *and* (8) *have the following properties*:

(i) $g_1(t,s) \geq 0$ *is continuous on* $[0,1] \times [0,1]$ *and* $g_1(t,s) > 0$ *for all* $t,s \in (0,1)$.

(ii) $\max_{t\in[0,1]} g_1(t,s) = g_1(1,s)$ *for all* $s \in [0,1]$, *where*

$$g_1(1,s) = \frac{1}{\Gamma(\alpha)} \big[ (1-s)^{\alpha-\mu-1} - (1-s)^{\alpha-1} \big].$$

(iii) $g_1(t,s) \geq t^{\alpha-1} g_1(1,s)$ *for all* $t,s \in [0,1]$, *and there are* $\theta \in (0,\frac{1}{2})$, $\gamma_\alpha \in (0,1)$ *such that* $\min_{t\in J_\theta} g_1(t,s) \geq \gamma_\alpha g_1(1,s)$ *for each* $s \in [0,1]$, *where* $J_\theta = [\theta, 1-\theta]$, $\gamma_\alpha = \theta^{\alpha-1}$.

(iv) $h_1(t,s) \geq 0$ *is continuous on* $[0,1] \times [0,1]$ *and* $h_1(t,s) > 0$ *for all* $t,s \in (0,1)$.

*Proof* For the proof of (i), (ii), and (iv), respectively, see Theorem 3.2 in [28] and Lemma 2.6 in [19]. It remains to prove (iii). We have by (7)

$$
\begin{aligned}
&g_1(t,s) \geq \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \left[ (1-s)^{\alpha-\mu-1} - \left(1 - \frac{s}{t}\right)^{\alpha-1} \right] \geq t^{\alpha-1} g_1(1,s), \quad 0 \leq s < t \leq 1, \\
&g_1(t,s) \geq \frac{1}{\Gamma(\alpha)} t^{\alpha-1} (1-s)^{\alpha-\mu-1} \geq t^{\alpha-1} g_1(1,s), \quad 0 \leq t \leq s \leq 1.
\end{aligned}
$$

Hence $g_1(t,s) \geq t^{\alpha-1} g_1(1,s)$ for all $t,s \in [0,1]$, and so $\min_{t\in J_\theta} g_1(t,s) \geq \gamma_\alpha g_1(1,s)$ for all $s \in [0,1]$. □

From Lemma 2.5 it is easy to get the following result.

**Lemma 2.6** *Under the assumptions of Lemma* 2.4, *the Green's function* $G_1(t,s)$ *defined by* (6) *has the following properties*:

(i) $G_1(t,s) \geq 0$ *is continuous on* $[0,1] \times [0,1]$ *and* $G_1(t,s) > 0$ *for all* $t,s \in (0,1)$.

(ii) $\max_{t\in[0,1]} G_1(t,s) = G_1(1,s)$ *for each* $s \in [0,1]$, *where*

$$G_1(1,s) = g_1(1,s) + \frac{1}{d_1} \sum_{j=1}^{p} a_j h_1(\xi_j,s) \leq \frac{(1-s)^{\alpha-\mu-1}}{d_1 \Gamma(\alpha)}. \tag{11}$$

(iii) $G_1(t,s) \ge t^{\alpha-1}G_1(1,s)$ *for all* $t,s \in [0,1]$, *there are* $\theta \in (0,\frac{1}{2}), \gamma_\alpha \in (0,1)$ *such that* $\min_{t\in J_\theta} G_1(t,s) \ge \gamma_\alpha G_1(1,s)$ *for each* $s \in [0,1]$, *where* $J_\theta = [\theta, 1-\theta], \gamma_\alpha = \theta^{\alpha-1}$.

We can also formulate similar results as Lemmas 2.4-2.6 above, for the fractional differential equation with fractional multi-point boundary conditions

$$\begin{cases} D_{0+}^{\beta} v(t) + y_2(t) = 0, \quad 0 < t < 1, \\ v^{(i)}(0) = 0, \quad 0 \le i \le m-2, \\ D_{0+}^{\nu} v(1) = \sum_{j=1}^{q} b_j D_{0+}^{\nu} v(\eta_j), \end{cases}$$

where $m, q \in \mathbb{N}^+, m \ge 3, 0 < \eta_1 < \cdots < \eta_q < 1, b_j \ge 0$ for all $j = 1,2,\dots,q$ and $y_2 \in C[0,1]$. We denote by $d_2 = 1 - \sum_{j=1}^{q} b_j \eta_j^{\beta-\nu-1}, \gamma_\beta$ and $g_2(t,s), h_2(\eta_j,s), G_2(t,s), G_2(1,s)$ the corresponding constants and functions for the problem (2) defined in a similar manner to $d_1, \gamma_\alpha$ and $g_1(t,s), h_1(\xi_j,s), G_1(t,s), G_1(1,s)$, respectively. From Lemma 2.6 we know that $G_1(t,s)$ and $G_2(t,s)$ have the same properties, and there exists $\gamma_\beta = \theta^{\beta-1}$ such that $\min_{t\in J_\theta} G_2(t,s) \ge \gamma_\beta G_2(1,s)$. Let $\gamma = \min\{\gamma_\alpha, \gamma_\beta\}$,

$$\delta_k = \int_\theta^{1-\theta} G_k(1,y)h_k(y)\,dy, \qquad \mu_k = \int_0^1 G_k(1,y)h_k(y)\,dy \quad (k = 1,2).$$

For convenience we list the following assumptions:

($H_1$) $h_k \in C((0,1), \mathbb{R}^+), h_k(x) \not\equiv 0$ on any subinterval of $(0,1)$ and

$$0 < \int_0^1 (1-y)^{\alpha-\mu-1} h_1(y)\,dy =: l_1 < +\infty, \qquad 0 < \int_0^1 (1-y)^{\beta-\nu-1} h_2(y)\,dy =: l_2 < +\infty.$$

($H_2$) There exist $a, b \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

(1) $a(\cdot)$ is concave and strictly increasing on $\mathbb{R}^+$ with $a(0) = 0$;

(2) $f_{10} = \liminf_{v\to 0+} \frac{f_1(x,u,v)}{a(v)} > 0, f_{20} = \liminf_{u\to 0+} \frac{f_2(x,u,v)}{b(u)} > 0$ uniformly with respect to $(x,u) \in J_\theta \times \mathbb{R}^+$ and $(x,v) \in J_\theta \times \mathbb{R}^+$, respectively (specifically, $f_{10} = f_{20} = +\infty$);

(3) $\lim_{u\to 0+} \frac{a(Cb(u))}{u} = +\infty$ for any constant $C > 0$.

($H_3$) There exists $\tau \in (0, +\infty)$ such that

$$f_1^\infty = \limsup_{v\to+\infty} \frac{f_1(x,u,v)}{v^\tau} < +\infty, \qquad f_2^\infty = \limsup_{u\to+\infty} \frac{f_2(x,u,v)}{u^{\frac{1}{\tau}}} = 0$$

uniformly with respect to $(x,u) \in [0,1] \times \mathbb{R}^+$ and $(x,v) \in [0,1] \times \mathbb{R}^+$, respectively (specifically, $f_1^\infty = f_2^\infty = 0$).

($H_4$) There exist $p, q \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

(1) $p$ is concave and strictly increasing on $\mathbb{R}^+$;

(2) $f_{1\infty} = \liminf_{v\to+\infty} \frac{f_1(x,u,v)}{p(v)} > 0, f_{2\infty} = \liminf_{u\to+\infty} \frac{f_2(x,u,v)}{q(u)} > 0$ uniformly with respect to $(x,u) \in J_\theta \times \mathbb{R}^+$ and $(x,v) \in J_\theta \times \mathbb{R}^+$, respectively (specifically, $f_{1\infty} = f_{2\infty} = +\infty$);

(3) $\lim_{u\to+\infty} \frac{p(Cq(u))}{u} = +\infty$ for any constant $C > 0$.

($H_5$) There exists $\varsigma \in (0, +\infty)$ such that

$$f_1^0 = \limsup_{v\to 0+} \frac{f_1(x,u,v)}{v^\varsigma} < +\infty, \qquad f_2^0 = \limsup_{u\to 0+} \frac{f_2(x,u,v)}{u^{\frac{1}{\varsigma}}} = 0$$

uniformly with respect to $(x,u) \in [0,1] \times \mathbb{R}^+$ and $(x,v) \in [0,1] \times \mathbb{R}^+$, respectively (specifically, $f_1^0 = f_2^0 = 0$).

($H_6$) There exists $r > 0$ such that $f_1(x,u,v)$ and $f_2(x,u,v)$ are nondecreasing in the second variable and the third variable $u,v \in [0,r]$ for all $x \in [0,1]$, and

$$f_1(x,\gamma r,\gamma r) \geq (\gamma\delta_1)^{-1} r, \qquad f_2(x,\gamma r,\gamma r) \geq (\gamma\delta_2)^{-1} r, \quad \forall x \in [\theta, 1-\theta].$$

($H_7$) There exists $R > r > 0$ such that $f_1(x,u,v)$ and $f_2(x,u,v)$ are nondecreasing in the second variable and the third variable $u,v \in [0,R]$ for all $x \in [0,1]$, and

$$f_1(x,R,R) \leq (2\mu_1)^{-1} R, \qquad f_2(x,R,R) \leq (2\mu_2)^{-1} R, \quad \forall x \in [0,1].$$

Let $E = C[0,1]$, $\|u\| = \max_{t\in[0,1]} |u(t)|$, the product space $E \times E$ be equipped with norm $\|(u,v)\| = \|u\| + \|v\|$ for $(u,v) \in E \times E$, and

$$P = \Bigl\{ u \in E : u(t) \geq 0, t \in [0,1], \min_{t\in J_\theta} u(t) \geq \gamma \|u\| \Bigr\}.$$

Then $E$ is a real Banach space and $P$ is a cone of $E$. By ($H_1$), we can define operators $A_k : P \times P \to E$ as follows:

$$A_k(u,v)(x) = \int_0^1 G_k(x,y) h_1(y) f_k\bigl(y, u(y), v(y)\bigr)\, dy \quad (k = 1,2), \tag{12}$$

$A(u,v) = (A_1(u,v), A_2(u,v))$. Clearly $(u,v)$ is a positive solution of the system (1) if and only if $(u,v) \in P \times P \setminus \{(0,0)\}$ is a fixed point of $A$. Let $B_r = \{u \in E : \|u\| < r\}$ for $r > 0$.

**Lemma 2.7** *Assume that the condition* ($H_1$) *is satisfied, then* $A : P \times P \to P \times P$ *is a completely continuous operator.*

*Proof* First of all, we show that $A_1 : P \times P \to P$ is uniformly bounded continuous operator. For any $(u,v) \in P \times P$, it follows from (12) that $A_1(u,v)(x) \geq 0, x \in [0,1]$,

$$\bigl\|A_1(u,v)\bigr\| \leq \int_0^1 G_1(1,y) h_1(y) f_1\bigl(y, u(y), v(y)\bigr)\, dy$$

and

$$\min_{x\in J_\theta} A_1(u,v)(x) \geq \gamma \int_0^1 G_1(1,y) h_1(y) f_1\bigl(y, u(y), v(y)\bigr)\, dy \geq \gamma \bigl\|A_1(u,v)\bigr\|.$$

Hence $A_1(P \times P) \subset P$.

Let $\Omega \subset P \times P$ be a bounded set, we assume that $\|(u,v)\| \leq d$ for any $(u,v) \in \Omega$. Let $M = \max_{x\in[0,1],(u,v)\in\Omega} f_1(x,u,v) + 1$. Equation (11) and ($H_1$) imply that

$$\bigl\|A_1(u,v)\bigr\| \leq \int_0^1 G_1(1,y) h_1(y) f_1\bigl(y, u(y), v(y)\bigr)\, dy \leq M \int_0^1 G_1(1,y) h_1(y)\, dy < +\infty,$$

from this we know that $A_1(\Omega)$ is a bounded set.

We show that $A_1 : P \times P \to P$ is continuous. Let $(u_n,v_n),(u_0,v_0) \in P \times P$, $\|(u_n,v_n) - (u_0,v_0)\| = \|(u_n - u_0, v_n - v_0)\| \to 0$ $(n \to \infty)$. Then $\{(u_n,v_n)\}$ is a bounded set, we assume that $\|(u_n, v_n)\| \leq d$ $(n = 0, 1, 2, \ldots)$. From $(\mathrm{H}_1)$, $f_1 \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$,

$$\left\|A_1(u_n, v_n) - A_1(u_0, v_0)\right\| \leq \int_0^1 G_1(1, y) h_1(y) \left|f_1\big(y, u_n(y), v_n(y)\big) - f_1\big(y, u_0(y), v_0(y)\big)\right| dy$$

and the Lebesgue control convergent theorem, we know that $A_1$ is a continuous operator.

Now we show that $A_1$ is equicontinuous on $\Omega$. For any given $\varepsilon > 0$, taking $\delta \in (0, \min\{\frac{d_1\Gamma(\alpha)\varepsilon}{Ml_1(\alpha-1)}, 1\})$, for each $(u, v) \in \Omega, x_1, x_2 \in [0,1], x_1 < x_2$, and $x_2 - x_1 < \delta$, we have by (7) and (8)

$$\begin{aligned}
&\left|A_1(u,v)(x_2) - A_1(u,v)(x_1)\right| \\
&\quad = \left|\int_0^1 \big[G_1(x_2, y) - G_1(x_1, y)\big] h_1(y) f_1\big(y, u(y), v(y)\big)\, dy\right| \\
&\quad = \left(\int_0^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^1\right) \big[G_1(x_2, y) - G_1(x_1, y)\big] h_1(y) f_1\big(y, u(y), v(y)\big)\, dy \\
&\quad \leq \frac{M}{d_1\Gamma(\alpha)} \big(x_2^{\alpha-1} - x_1^{\alpha-1}\big) \left(\int_0^{x_1} + \int_{x_1}^{x_2} + \int_{x_2}^1\right) (1-y)^{\alpha-\mu-1} h_1(y)\, dy \\
&\quad = \frac{Ml_1}{d_1\Gamma(\alpha)} \big(x_2^{\alpha-1} - x_1^{\alpha-1}\big) < \frac{Ml_1(\alpha-1)}{d_1\Gamma(\alpha)} (x_2 - x_1) < \frac{Ml_1(\alpha-1)}{d_1\Gamma(\alpha)} \delta < \varepsilon.
\end{aligned}$$

By means of the Arzela-Ascoli theorem, $A_1 : P \times P \to P$ is completely continuous. Similarly, we can prove that $A_2 : P \times P \to P$ is completely continuous. Hence $A : P \times P \to P \times P$ is a completely continuous operator. □

**Lemma 2.8** [29] *Assume that* $A : \overline{B}_r \cap P \to P$ *is a completely continuous operator. If there exists* $u_0 \in P \setminus \{0\}$ *such that*

$$u \neq Au + \lambda u_0, \quad \forall \lambda \geq 0, u \in \partial B_r \cap P,$$

*then the fixed point index* $i(A, B_r \cap P, P) = 0$.

**Lemma 2.9** [29, 30] *Assume that* $A : \overline{B}_r \cap P \to P$ *is a completely continuous operator.*

(1) *If* $u \nleq Au$ *or* $\|Au\| \leq \|u\|$ *for all* $u \in \partial B_r \cap P$, *then the fixed point index* $i(A, B_r \cap P, P) = 1$.

(2) *If* $u \ngeq Au$ *or* $\|Au\| \geq \|u\|$ *for all* $u \in \partial B_r \cap P$, *then the fixed point index* $i(A, B_r \cap P, P) = 0$.

In the following, we adopt the convention that $C_1, C_2, C_3, \ldots$ stand for different positive constants. Let $\Omega_r = \{(u, v) \in E \times E : \|(u, v)\| < r\}$ for $r > 0$.

## 3 Existence of a positive solution

**Theorem 3.1** *Assume that the conditions* $(\mathrm{H}_1)$-$(\mathrm{H}_3)$ *are satisfied, then the system* (1) *has at least one positive solution.*

*Proof* By $(\mathrm{H}_2)$, there are $\xi_1 > 0, \eta_1 > 0$ and a sufficiently small $\rho > 0$ such that

$$\begin{aligned}
&f_1(x, u, v) \geq \xi_1 a(v), \quad \forall (x, u) \in J_\theta \times \mathbb{R}^+, 0 \leq v \leq \rho, \\
&f_2(x, u, v) \geq \eta_1 b(u), \quad \forall (x, v) \in J_\theta \times \mathbb{R}^+, 0 \leq u \leq \rho,
\end{aligned} \tag{13}$$

and

$$a\big(K_1 b(u)\big) \geq \frac{2K_1}{\xi_1\eta_1\delta_1\delta_2\gamma^3}u, \quad \forall u \in [0,\rho], \tag{14}$$

where $K_1 = \max\{\eta_1\gamma G_2(1,y)h_2(y) : y \in J_\theta\}$. We claim that

$$(u,v) \neq A(u,v) + \lambda(\varphi,\varphi), \quad \forall \lambda \geq 0, (u,v) \in \partial\Omega_\rho \cap (P \times P),$$

where $\varphi \in P \setminus \{0\}$. If not, there are $\lambda \geq 0$ and $(u,v) \in \partial\Omega_\rho \cap (P \times P)$ such that $(u,v) = A(u,v) + \lambda(\varphi,\varphi)$, then $u \geq A_1(u,v), v \geq A_2(u,v)$. By using the monotonicity and concavity of $a(\cdot)$, Jensen's inequality and Lemma 2.6, we have by (13) and (14)

$$\begin{aligned}
u(x) &\geq \int_0^1 G_1(x,y)h_1(y)f_1\big(y,u(y),v(y)\big)\,dy \\
&\geq \xi_1\gamma_\alpha \int_0^1 G_1(1,y)h_1(y)a\big(v(y)\big)\,dy \\
&\geq \xi_1\gamma_\alpha \int_0^1 G_1(1,y)h_1(y)a\bigg(\int_0^1 \eta_1 G_2(y,z)h_2(z)b\big(u(z)\big)\,dz\bigg)\,dy \\
&\geq \xi_1\gamma \int_\theta^{1-\theta} G_1(1,y)h_1(y)\int_0^1 a\big(\eta_1\gamma G_2(1,z)h_2(z)b\big(u(z)\big)\big)\,dz\,dy \\
&\geq \xi_1\gamma \int_\theta^{1-\theta} G_1(1,y)h_1(y)\int_0^1 a\big(K_1^{-1}\eta_1\gamma G_2(1,z)h_2(z)K_1 b\big(u(z)\big)\big)\,dz\,dy \\
&\geq \xi_1\eta_1\gamma^2 K_1^{-1}\int_\theta^{1-\theta}\int_\theta^{1-\theta} G_1(1,y)h_1(y)G_2(1,z)h_2(z)a\big(K_1 b\big(u(z)\big)\big)\,dz\,dy \\
&\geq \xi_1\eta_1\gamma^2\delta_1 K_1^{-1}\int_\theta^{1-\theta} G_2(1,z)h_2(z)a\big(K_1 b\big(u(z)\big)\big)\,dz \\
&\geq \frac{2}{\delta_2\gamma}\int_\theta^{1-\theta} G_2(1,z)h_2(z)u(z)\,dz \geq 2\|u\|, \quad x \in J_\theta.
\end{aligned} \tag{15}$$

Consequently, $\|u\| = 0$. Next, (13) and (14) yield

$$\begin{aligned}
a\big(v(x)\big) &\geq a\bigg(\int_0^1 G_2(x,y)h_2(y)f_2\big(y,u(y),v(y)\big)\,dy\bigg) \\
&\geq \int_0^1 a\big(\eta_1\gamma G_2(1,y)h_2(y)b\big(u(y)\big)\big)\,dy \\
&\geq \eta_1\gamma K_1^{-1}\int_\theta^{1-\theta} G_2(1,y)h_2(y)a\big(K_1 b\big(u(y)\big)\big)\,dy \\
&\geq \frac{2}{\xi_1\delta_1\delta_2\gamma^2}\int_\theta^{1-\theta} G_2(1,y)h_2(y)u(y)\,dy \\
&\geq \frac{2}{\delta_1\delta_2\gamma}\int_\theta^{1-\theta} G_2(1,y)h_2(y)\,dy\int_0^1 G_1(1,z)h_1(z)a\big(v(z)\big)\,dz \\
&\geq \frac{2}{\delta_1\gamma}\int_\theta^{1-\theta} G_1(1,z)h_1(z)a\big(v(z)\big)\,dz \geq 2a\big(\|v\|\big), \quad x \in J_\theta,
\end{aligned} \tag{16}$$

this means that $a(\|v\|) = 0$. It follows from the strict monotonicity of $a(v)$ and $a(0) = 0$ that $\|v\| = 0$. Hence $\|(u,v)\| = 0$, which is a contradiction. Lemma 2.8 implies that

$$i\big(A, \Omega_\rho \cap (P \times P), P \times P\big) = 0. \tag{17}$$

On the other hand, by ($H_3$), there exist $\zeta > 0$ and $C_1 > 0, C_2 > 0$ such that

$$\begin{aligned} &f_1(x,u,v) \le \zeta v^\tau + C_1, \quad \forall (x,u,v) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \\ &f_2(x,u,v) \le \varepsilon_2 u^{\frac{1}{\tau}} + C_2, \quad \forall (x,u,v) \in [0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \end{aligned} \tag{18}$$

where

$$\varepsilon_2 = \min\left\{ \frac{1}{\mu_2(8\zeta\mu_1)^{\frac{1}{\tau}}}, \frac{1}{8\mu_2(\zeta\mu_1)^{\frac{1}{\tau}}} \right\}.$$

Let

$$W = \big\{(u,v) \in P \times P : (u,v) = \lambda A(u,v), 0 \le \lambda \le 1\big\}.$$

We prove that $W$ is bounded. Indeed, for any $(u,v) \in W$, there exists $\lambda \in [0,1]$ such that $u = \lambda A_1(u,v), v = \lambda A_2(u,v)$. Then (18) implies that

$$\begin{aligned} &u(x) \le A_1(u,v)(x) \le \zeta \int_0^1 G_1(1,y)h_1(y)v^\tau(y)\,dy + C_3, \\ &v(x) \le A_2(u,v)(x) \le \varepsilon_2 \int_0^1 G_2(1,y)h_2(y)u^{\frac{1}{\tau}}(y)\,dy + C_4. \end{aligned}$$

Consequently,

$$\begin{aligned} u(x) &\le \zeta \int_0^1 G_1(1,y)h_1(y)\,dy\left(\varepsilon_2 \int_0^1 G_2(1,z)h_2(z)u^{\frac{1}{\tau}}(z)\,dz + C_4\right)^\tau + C_3 \\ &\le \zeta\mu_1\left(\varepsilon_2 \int_0^1 G_2(1,z)h_2(z)\|u\|^{\frac{1}{\tau}}\,dz + C_4\right)^\tau + C_3 \\ &\le \zeta\mu_1\left[\left(\frac{\|(u,v)\|}{8\zeta\mu_1}\right)^{\frac{1}{\tau}} + C_4\right]^\tau + C_3, \end{aligned} \tag{19}$$

$$\begin{aligned} v(x) &\le \varepsilon_2 \int_0^1 G_2(1,y)h_2(y)\,dy\left(\zeta \int_0^1 G_1(1,z)h_1(z)v^\tau(z)\,dz + C_3\right)^{\frac{1}{\tau}} + C_4 \\ &\le \varepsilon_2\mu_2\left(\zeta \int_0^1 G_1(1,z)h_1(z)\|v\|^\tau\,dz + C_3\right)^{\frac{1}{\tau}} + C_4 \\ &\le \frac{1}{8(\zeta\mu_1)^{\frac{1}{\tau}}}\left(\zeta\mu_1\big\|(u,v)\big\|^\tau + C_3\right)^{\frac{1}{\tau}} + C_4. \end{aligned} \tag{20}$$

Since

$$\lim_{w\to+\infty} \frac{\zeta\mu_1[(\frac{w}{8\zeta\mu_1})^{\frac{1}{\tau}} + C_4]^\tau}{w} = \frac{1}{8}, \qquad \lim_{w\to+\infty} \frac{(\zeta\mu_1 w^\tau + C_3)^{\frac{1}{\tau}}}{8(\zeta\mu_1)^{\frac{1}{\tau}} w} = \frac{1}{8},$$

there exists $r_1 > r$, when $\|(u,v)\| > r_1$, (19) and (20) yield

$$u(x) \leq \frac{1}{4}\|(u,v)\| + C_3, v(x) \leq \frac{1}{4}\|(u,v)\| + C_4.$$

Hence $\|(u,v)\| \leq 2(C_3 + C_4)$ and $W$ is bounded.

Select $G > 2(C_3 + C_4)$. We obtain from the homotopic invariant property of fixed point index that

$$i\bigl(A, \Omega_G \cap (P \times P), P \times P\bigr) = i\bigl(\theta, \Omega_G \cap (P \times P), P \times P\bigr) = 1. \tag{21}$$

Equations (17) and (21) yield

$$\begin{aligned} &i\bigl(A, (\Omega_G \setminus \overline{\Omega}_\rho) \cap (P \times P), P \times P\bigr) \\ &\quad = i\bigl(A, \Omega_G \cap (P \times P), P \times P\bigr) - i\bigl(A, \Omega_\rho \cap (P \times P), P \times P\bigr) = 1. \end{aligned}$$

So $A$ has at least one fixed point on $(\Omega_G \setminus \overline{\Omega}_\rho) \cap (P \times P)$. This means that the system (1) has at least one positive solution. □

**Theorem 3.2** *Assume that the conditions* ($H_1$), ($H_4$), *and* ($H_5$) *are satisfied. Then the system* (1) *has at least one positive solution.*

*Proof* By ($H_4$), there are $\xi_2 > 0, \eta_2 > 0, C_5 > 0, C_6 > 0$, and $C_7 > 0$ such that

$$f_1(x,u,v) \geq \xi_2 p(v) - C_5, \qquad f_2(x,u,v) \geq \eta_2 q(u) - C_6, \quad (x,u,v) \in J_\theta \times \mathbb{R}^+ \times \mathbb{R}^+,$$

and

$$p\bigl(K_2 q(u)\bigr) \geq \frac{2K_2}{\xi_2 \eta_2 \delta_1 \delta_2 \gamma^3} u - C_7, \quad u \in \mathbb{R}^+, \tag{22}$$

where $K_2 = \max\{\eta_2 \gamma G_2(1,y)h_2(y) : y \in J_\theta\}$. Then we have

$$\begin{aligned} A_1(u,v)(x) &\geq \xi_2 \int_0^1 G_1(x,y)h_1(y)p\bigl(v(y)\bigr)\,dy - C_8, \quad x \in J_\theta, \\ A_2(u,v)(x) &\geq \eta_2 \int_0^1 G_2(x,y)h_2(y)q\bigl(u(y)\bigr)\,dy - C_9, \quad x \in J_\theta. \end{aligned} \tag{23}$$

We affirm that the set

$$W = \bigl\{(u,v) \in P \times P : (u,v) = A(u,v) + \lambda(\varphi,\varphi), \lambda \geq 0\bigr\}$$

is bounded, where $\varphi \in P \setminus \{0\}$. Indeed, $(u,v) \in W$ implies that $u \geq A_1(u,v), v \geq A_2(u,v)$ for some $\lambda \geq 0$. We have by (23)

$$u(x) \geq \xi_2 \int_0^1 G_1(x,y)h_1(y)p\bigl(v(y)\bigr)\,dy - C_8, \quad x \in J_\theta, \tag{24}$$

$$v(x) \geq \eta_2 \int_0^1 G_2(x,y)h_2(y)q\bigl(u(y)\bigr)\,dy - C_9, \quad x \in J_\theta. \tag{25}$$

By the monotonicity and concavity of $p(\cdot)$ as well as Jensen's inequality, (25) implies that

$$\begin{aligned}p\big(v(x)+C_9\big) &\geq p\left(\int_0^1 \eta_2 G_2(x,y)h_2(y)q\big(u(y)\big)\,dy\right)\\ &\geq \int_0^1 p\big(\eta_2\gamma G_2(1,y)h_2(y)q\big(u(y)\big)\big)\,dy\\ &\geq \eta_2\gamma K_2^{-1}\int_\theta^{1-\theta} G_2(1,y)h_2(y)p\big(K_2 q\big(u(y)\big)\big)\,dy,\quad x\in J_\theta. \end{aligned}\tag{26}$$

Since $p(v(x))\geq p(v(x)+C_9)-p(C_9)$, we have by (22), (24), and (26)

$$\begin{aligned}u(x) &\geq \xi_2\gamma\int_0^1 G_1(1,y)h_1(y)\big[p\big(v(y)+C_9\big)-p(C_9)\big]\,dy - C_8\\ &\geq \xi_2\gamma\int_\theta^{1-\theta} G_1(1,y)h_1(y)p\big(v(y)+C_9\big)\,dy - C_{10}\\ &\geq \xi_2\eta_2\gamma^2 K_2^{-1}\int_\theta^{1-\theta} G_1(1,y)h_1(y)\int_\theta^{1-\theta} G_2(1,z)h_2(z)p\big(K_2 q\big(u(z)\big)\big)\,dz\,dy - C_{10}\\ &\geq \xi_2\eta_2\gamma^2\delta_1 K_2^{-1}\int_\theta^{1-\theta} G_2(1,z)h_2(z)p\big(K_2 q\big(u(z)\big)\big)\,dz - C_{10}\\ &\geq 2(\delta_2\gamma)^{-1}\int_\theta^{1-\theta} G_2(1,z)h_2(z)u(z)\,dz - C_{11} \geq 2\|u\| - C_{11},\quad x\in J_\theta. \end{aligned}\tag{27}$$

Hence $\|u\|\leq C_{11}$.

Since $p(v(x))\geq \gamma p(\|v\|)$ for $x\in J_\theta, v\in P$, it follows from (26), (22), and (24) that

$$\begin{aligned}p\big(v(x)\big) &\geq p\big(v(x)+C_9\big)-p(C_9)\\ &\geq \eta_2\gamma K_2^{-1}\int_\theta^{1-\theta} G_2(1,y)h_2(y)p\big(K_2 q\big(u(y)\big)\big)\,dy - p(C_9)\\ &\geq \frac{2}{\xi_2\delta_1\delta_2\gamma^2}\int_\theta^{1-\theta} G_2(1,y)h_2(y)u(y)\,dy - C_{12}\\ &\geq \frac{2}{\delta_1\delta_2\gamma}\int_\theta^{1-\theta} G_2(1,y)h_2(y)\,dy\int_0^1 G_1(1,z)h_1(z)p\big(v(z)\big)\,dz - C_{13}\\ &\geq 2\delta_1^{-1}\int_\theta^{1-\theta} G_1(1,z)h_1(z)p\big(\|v\|\big)\,dz - C_{13}\\ &= 2p\big(\|v\|\big) - C_{13},\quad x\in J_\theta. \end{aligned}$$

Hence $p(\|v\|)\leq C_{13}$. By (1) and (3) of the condition $(\mathrm{H}_5)$, we know that $\lim_{v\to+\infty} p(v)=+\infty$, thus there exists $C_{14}>0$ such that $\|v\|\leq C_{14}$. This shows $W$ is bounded. Then there exists a sufficiently large $K>0$ such that

$$(u,v)\neq A(u,v)+\lambda(\varphi,\varphi),\quad \forall(u,v)\in\partial\Omega_K\cap(P\times P),\lambda\geq 0.$$

Lemma 2.8 yields

$$i\big(A,\Omega_K\cap(P\times P),P\times P\big)=0.\tag{28}$$

On the other hand, by ($H_5$), there is a $\sigma > 0$ and sufficiently small $\rho > 0$ such that

$$\begin{aligned} &f_1(x,u,v) \le \sigma v^{\varsigma}, \quad \forall (x,u) \in [0,1] \times \mathbb{R}^+, v \in [0,\rho], \\ &f_2(x,u,v) \le \varepsilon_1 u^{\frac{1}{\varsigma}}, \quad \forall (x,v) \in [0,1] \times \mathbb{R}^+, u \in [0,\rho], \end{aligned} \tag{29}$$

where

$$\varepsilon_1 = \min\bigl\{\bigl(2\sigma\mu_1\mu_2^{\varsigma}\bigr)^{-\frac{1}{\varsigma}}, \mu_2^{-1}\bigr\}.$$

We claim that

$$(u,v) \nleq A(u,v), \quad \forall (u,v) \in \partial\Omega_\rho \cap (P \times P). \tag{30}$$

If not, there exists a $(u,v) \in \partial\Omega_\rho \cap (P \times P)$ such that $(u,v) \le A(u,v)$, that is, $u \le A_1(u,v), v \le A_2(u,v)$. Then (29) implies that

$$\begin{aligned} u(x) &\le \int_0^1 G_1(x,y)h_1(y)f_1\bigl(y,u(y),v(y)\bigr)\,dy \\ &\le \sigma \int_0^1 G_1(1,y)h_1(y)v^{\varsigma}(y)\,dy \\ &\le \sigma \int_0^1 G_1(1,y)h_1(y)\biggl(\int_0^1 G_2(y,z)h_2(z)f_2\bigl(z,u(z),v(z)\bigr)\,dz\biggr)^{\varsigma}\,dy \\ &\le \sigma \int_0^1 G_1(1,y)h_1(y)\,dy\biggl(\int_0^1 G_2(1,z)h_2(z)f_2\bigl(z,u(z),v(z)\bigr)\,dz\biggr)^{\varsigma} \\ &= \sigma\mu_1\biggl(\int_0^1 G_2(1,z)h_2(z)f_2\bigl(z,u(z),v(z)\bigr)\,dz\biggr)^{\varsigma} \\ &\le \sigma\mu_1\varepsilon_1^{\varsigma}\biggl(\int_0^1 G_2(1,z)h_2(z)u^{\frac{1}{\varsigma}}(z)\,dz\biggr)^{\varsigma} \\ &\le \sigma\mu_1\varepsilon_1^{\varsigma}\mu_2^{\varsigma}\|u\| \le \frac{1}{2}\|u\|, \quad x \in [0,1], \end{aligned} \tag{31}$$

and

$$\begin{aligned} v(x) &\le \int_0^1 G_2(x,y)h_2(y)f_2\bigl(y,u(y),v(y)\bigr)\,dy \\ &\le \varepsilon_1 \int_0^1 G_2(1,y)h_2(y)u^{\frac{1}{\tau}}(y)\,dy \le \varepsilon_1\mu_2\|u\|^{\frac{1}{\varsigma}} \le \|u\|^{\frac{1}{\varsigma}}, \quad x \in [0,1]. \end{aligned} \tag{32}$$

Equations (31) and (32) imply that $\|(u,v)\| = 0$, which contradicts $\|(u,v)\| = \rho$, and the inequality (30) holds. Lemma 2.9 yields

$$i\bigl(A, \Omega_\rho \cap (P \times P), P \times P\bigr) = 1. \tag{33}$$

We have by (28) and (33)

$$\begin{aligned} &i\bigl(A, (\Omega_K \setminus \overline{\Omega}_\rho) \cap (P \times P), P \times P\bigr) \\ &\quad = i\bigl(A, \Omega_K \cap (P \times P), P \times P\bigr) - i\bigl(A, \Omega_\rho \cap (P \times P), P \times P\bigr) = -1. \end{aligned}$$

Hence $A$ has a fixed point on $(\Omega_K \setminus \overline{\Omega}_\rho) \cap (P \times P)$. This means that the system (1) has at least one positive solution. □

**Theorem 3.3** *Assume that the conditions* $(\mathrm{H}_1)$, $(\mathrm{H}_6)$, *and* $(\mathrm{H}_7)$ *are satisfied. Then the system* (1) *has at least one positive solution.*

*Proof* Since $\gamma r \leq u(x), v(x) \leq r$ for $(u,v) \in \partial\Omega_r \cap (P \times P), x \in [\theta, 1-\theta]$, we know from $(\mathrm{H}_6)$ that

$$\begin{aligned} A_1(u,v)(x) &\geq \int_\theta^{1-\theta} G_1(x,y)h_1(y)f_1(y,\gamma r,\gamma r)\,dy \\ &\geq \delta_1^{-1} r \int_\theta^{1-\theta} G_1(1,y)h_1(y)\,dy = r, \quad x \in [\theta, 1-\theta], \\ A_2(u,v)(x) &\geq {}_\theta^{1-\theta} G_2(x,y)h_2(y)f_2(y,\gamma r,\gamma r)\,dy \\ &\geq \delta_2^{-1} r \int_c^{1-c} G_2(1,y)h_2(y)\,dy = r, \quad x \in [\theta, 1-\theta]. \end{aligned}$$

Hence $\|A(u,v)\| > r = \|(u,v)\|$ for any $(u,v) \in \partial\Omega_r \cap (P \times P)$. Lemma 2.9 yields

$$i\big(A, \Omega_r \cap (P \times P), P \times P\big) = 0. \tag{34}$$

On the other hand, for any $x \in [0,1], 0 \leq u, v \leq R$, $(\mathrm{H}_7)$ implies that

$$\begin{aligned} A_1(u,v)(x) &\leq \int_0^1 G_1(1,y)h_1(y)f_1(y,R,R)\,dy \leq \frac{R}{2}, \\ A_2(u,v)(x) &\leq \int_0^1 G_2(1,y)h_2(y)f_2(y,R,R)\,dy \leq \frac{R}{2}. \end{aligned}$$

Hence $\|A(u,v)\| \leq R = \|(u,v)\|$ for $(u,v) \in \partial\Omega_R \cap (P \times P)$. Lemma 2.9 yields

$$i\big(A, \Omega_R \cap (P \times P), P \times P\big) = 1. \tag{35}$$

We have by (34) and (35)

$$\begin{aligned} & i\big(A, (\Omega_R \setminus \overline{\Omega}_r) \cap (P \times P), P \times P\big) \\ &\quad = i\big(A, \Omega_R \cap (P \times P), P \times P\big) - i\big(A, \Omega_r \cap (P \times P), P \times P\big) = 1. \end{aligned} \tag{36}$$

So $A$ has a fixed point on $(\Omega_R \setminus \overline{\Omega}_r) \cap (P \times P)$. This means that the system (1) has at least one positive solution. □

## 4 Existence of multiple positive solutions

**Theorem 4.1** *Assume that the conditions* $(\mathrm{H}_1)$, $(\mathrm{H}_3)$, $(\mathrm{H}_5)$, *and* $(\mathrm{H}_6)$ *hold. Then the system* (1) *has at least two positive solutions.*

*Proof* We may take $G > r > \sigma$ such that (21), (33), and (34) hold. Then we have

$$\begin{aligned} & i\big(A, (\Omega_G \setminus \overline{\Omega}_r) \cap (P \times P), P \times P\big) \\ &\quad = i\big(A, \Omega_G \cap (P \times P), P \times P\big) - i\big(A, \Omega_r \cap (P \times P), P \times P\big) = 1, \end{aligned}$$

$$\begin{aligned} & i\big(A,(\Omega_r \setminus \overline{\Omega}_\sigma) \cap (P \times P), P \times P\big) \\ & \quad = i\big(A, \Omega_r \cap (P \times P), P \times P\big) - i\big(A, \Omega_\sigma \cap (P \times P), P \times P\big) = -1. \end{aligned} \tag{37}$$

Hence $A$ has a fixed point on $(\Omega_G \setminus \overline{\Omega}_r) \cap (P \times P)$ and $(\Omega_r \setminus \overline{\Omega}_\sigma) \cap (P \times P)$, respectively. This means the system (1) has at least two positive solutions. □

**Theorem 4.2** *Assume that the conditions* $(\mathrm{H}_1)$, $(\mathrm{H}_2)$, $(\mathrm{H}_4)$, *and* $(\mathrm{H}_7)$ *hold. Then the system* (1) *has at least two positive solutions.*

*Proof* We may take $K > R > \rho$ such that (17), (28), and (35) hold. Then we have

$$\begin{aligned} & i\big(A,(\Omega_K \setminus \overline{\Omega}_R) \cap (P \times P), P \times P\big) \\ & \quad = i\big(A, \Omega_K \cap (P \times P), P \times P\big) - i\big(A, \Omega_R \cap (P \times P), P \times P\big) = -1, \\ & i\big(A,(\Omega_R \setminus \overline{\Omega}_\rho) \cap (P \times P), P \times P\big) \\ & \quad = i\big(A, \Omega_R \cap (P \times P), P \times P\big) - i\big(A, \Omega_\rho \cap (P \times P), P \times P\big) = 1. \end{aligned} \tag{38}$$

Hence $A$ has a fixed point on $(\Omega_K \setminus \overline{\Omega}_R) \cap (P \times P)$ and $(\Omega_R \setminus \overline{\Omega}_\rho) \cap (P \times P)$, respectively. This means the system (1) has at least two positive solutions. □

**Theorem 4.3** *Assume that the conditions* $(\mathrm{H}_1)$, $(\mathrm{H}_4)$, $(\mathrm{H}_5)$, $(\mathrm{H}_6)$, *and* $(\mathrm{H}_7)$ *hold. Then the system* (1) *has at least three positive solutions.*

*Proof* We may take $K > R > r > \sigma$ such that (28), (33), (34), and (35) hold. From the proof of Theorem 3.3, Theorem 4.1, and Theorem 4.2 we know that (36), (37), and (38) hold. Hence $A$ has a fixed point on $(\Omega_K \setminus \overline{\Omega}_R) \cap (P \times P)$, $(\Omega_R \setminus \overline{\Omega}_r) \cap (P \times P)$ and $(\Omega_r \setminus \overline{\Omega}_\sigma) \cap (P \times P)$, respectively. Hence the system (1) has at least three positive solutions. □

Similar to the proof of Theorem 3.3, we can get the following result.

**Theorem 4.4** *Assume that* $(\mathrm{H}_1)$ *holds. If there are* $2l$ *positive numbers* $d_k, D_k$ $(k = 1, 2, \ldots, l)$ *with*

$$d_1 < D_1 < d_2 < D_2 < \cdots < d_l < D_l,$$

*such that* $f_1(x, u, v)$ *and* $f_2(x, u, v)$ *are nondecreasing in the second variable and the third variable* $u, v \in [0, D_l]$ *for all* $x \in [0, 1]$, *and*

$(\mathrm{H}_8)$ $f_1(x, \gamma d_k, \gamma d_k) \geq (\gamma \delta_1)^{-1} d_k, f_2(x, \gamma d_k, \gamma d_k) \geq (\gamma \delta_2)^{-1} d_k$ *for all* $x \in J_\theta$, $k = 1, 2, \ldots, l$,

$(\mathrm{H}_9)$ $f_1(x, D_k, D_k) \leq \mu_1^{-1} \frac{D_k}{2}, f_2(x, D_k, D_k) \leq \mu_2^{-1} \frac{D_k}{2}$ *for all* $x \in [0, 1], k = 1, 2, \ldots, l$.

*Then the system* (1) *has at least* $l$ *positive solutions* $(u_k, v_k)$ *satisfying*

$$d_k \leq \big\|(u_k, v_k)\big\| \leq D_k, \quad k = 1, 2, \ldots, l.$$

## 5 Some examples

In the following, we give some examples to illustrate our main results. In Examples 5.1-5.4, the meaning of $\alpha, \beta, \mu, \nu$ is the same as in the system (1).

**Example 5.1** Let $h_1(x) = 1/(1-x)^{\alpha-\mu-1}, h_2(x) = 1/(1-x)^{\beta-\nu-1}, x \in (0,1), f_1(x,u,v) = e^x(1+e^{-(u+v)}), f_2(x,u,v) = 1-e^{-(u+v)}, x \in [0,1], u,v \in \mathbb{R}^+$, $a(v) = v^{\frac{1}{2}}, b(u) = u^{\frac{1}{2}}, \tau = 1/2$. Clearly,

$$\int_0^1 (1-y)^{\alpha-\mu-1}h_1(y)\,dy = \int_0^1 (1-y)^{\beta-\nu-1}h_2(y)\,dy = 1,$$

but $\int_0^1 h_k(y)\,dy = +\infty$ $(k=1,2)$ for $\alpha-\mu-1 \geq 1, \beta-\nu-1 \geq 1$. The results of [18–25, 28, 31] are not suitable for the problem. It is easy to verify that the conditions $(\mathrm{H}_1)$-$(\mathrm{H}_3)$ hold, hence Theorem 3.1 implies that the system (1) has at least one positive solution. Here $f_1(x,u,v)$ and $f_2(x,u,v)$ are sublinear on $u$ and $v$ at 0 and $+\infty$.

**Example 5.2** Let $h_k(x)$ be as in Example 5.1, $f_1(x,u,v) = e^x(1+e^{-(u+v)}), f_2(x,u,v) = u^{\frac{3}{2}}, a(v) = v^{\frac{1}{3}}, b(u) = u^2, \tau = 1/2$. It is easy to verify that the conditions $(\mathrm{H}_1)$-$(\mathrm{H}_3)$ hold, Theorem 3.1 implies that the system (1) has at least one positive solution. Here $f_1(x,u,v)$ is sublinear on $u$ and $v$ at 0 and $+\infty$, whereas $f_2(x,u,v)$ is superlinear on $u$ at 0 and $+\infty$.

**Example 5.3** Let $h_k(x)$ be as in Example 5.1, $f_1(x,u,v) = (1+e^{-u})v^3, f_2(x,u,v) = u^3$, $p(v) = v^{\frac{1}{2}}, q(u) = u^3, \varsigma = 3$. It is easy to verify that the conditions $(\mathrm{H}_1)$, $(\mathrm{H}_4)$, and $(\mathrm{H}_5)$ hold. Theorem 3.2 shows that the system (1) has at least one positive solution. Here $f_1(x,u,v)$ is superlinear on $v$ at 0 and $+\infty$, $f_2(x,u,v)$ is superlinear on $u$ at 0 and $+\infty$.

**Example 5.4** Let $h_k(x)$ be as in Example 5.1, $f_1(x,u,v) = (1+e^{-u})v^{\frac{2}{3}}, f_2(x,u,v) = (1+e^{-v})u^5$, $p(v) = v^{\frac{1}{3}}, q(u) = u^4, \varsigma = 1/3$. It is easy to see that the conditions $(\mathrm{H}_1)$, $(\mathrm{H}_4)$, and $(\mathrm{H}_5)$ hold. Theorem 3.2 shows that the system (1) has at least one positive solution. Here $f_1(x,u,v)$ is sublinear on $v$ at 0 and $+\infty$, whereas $f_2(x,u,v)$ is superlinear on $u$ at 0 and $+\infty$.

**Example 5.5** Consider the system of nonlinear singular fractional differential equations with fractional three-point boundary conditions:

$$\begin{cases} D_{0+}^{\frac{5}{2}}u(x) + h_1(x)f_1(x,u(x),v(x)) = 0, & x \in (0,1), \\ D_{0+}^{\frac{5}{2}}v(x) + h_1(x)f_2(x,u(x),v(x)) = 0, & x \in (0,1), \\ u(0) = u'(0) = 0, \qquad u'(1) = \frac{\sqrt{2}}{2}u'(\frac{1}{2}), \\ v(0) = v'(0) = 0, \qquad v'(1) = \frac{\sqrt{2}}{2}v'(\frac{1}{2}), \end{cases} \tag{39}$$

where $\alpha = \beta = \frac{5}{2}, \mu = \nu = 1, a_1 = b_1 = \frac{\sqrt{2}}{2}, \xi_1 = \eta_1 = \frac{1}{2}, h_1(x) = (1-x)^{-\frac{1}{2}}$,

$$f_1(x,u,v) = 3\sqrt{2\pi}v^{\frac{1}{2}}, \quad x \in [0,1], v \geq 0, \qquad f_2(x,u,v) = 3\sqrt{2\pi}\begin{cases} u^3, & x,u \in [0,1], \\ u^{\frac{3}{2}}, & x \in [0,1], u \geq 1. \end{cases}$$

By a simple calculation, we have $d_1 = \frac{1}{2}, \gamma = \frac{1}{8}, \delta_1 = \int_{\frac{1}{4}}^{\frac{3}{4}} G_1(1,s)h_1(s)\,ds = \frac{3\sqrt{2}-1}{3\sqrt{2\pi}}$. Take $r = 1, \tau = \varsigma = \frac{1}{2}$ in $(\mathrm{H}_3)$ and $(\mathrm{H}_5)$, it is easy to verify that the conditions $(\mathrm{H}_1)$, $(\mathrm{H}_3)$, $(\mathrm{H}_5)$, and $(\mathrm{H}_6)$ hold. From Theorem 4.1 one concludes that the system (39) has two positive solutions.

**Example 5.6** Consider the singular system (39), where

$$f_1(x,u,v) = 3\sqrt{2\pi}v^3, \quad x \in [0,1], v \geq 0, \qquad f_2(x,u,v) = 3\sqrt{2\pi}\begin{cases} u^{\frac{2}{3}}, & x,u \in [0,1], \\ u^{\frac{1}{4}}, & x \in [0,1], u \geq 1. \end{cases}$$

Take $r = 1, \tau = \varsigma = 3$ in ($H_3$) and ($H_5$), it is easy to verify that the conditions ($H_1$), ($H_3$), ($H_5$), and ($H_6$) hold. From Theorem 4.1 one concludes that the system (39) has two positive solutions.

**Example 5.7** Consider the singular system (39), where

$$f_1(x,u,v) = \frac{\sqrt{\pi}}{4}\begin{cases} \frac{1}{4}v^{\frac{2}{3}}, & x, v \in [0,1], \\ \frac{v^2}{4}, & x \in [0,1], v \geq 1, \end{cases}$$

$$f_2(x,u,v) = \frac{\sqrt{\pi}}{4}\begin{cases} \frac{u^2+2u^{\frac{1}{2}}}{12}, & x, u \in [0,1], \\ \frac{u^2}{4}, & x \in [0,1], u \geq 1. \end{cases}$$

By a simple calculation, we get $\mu_1 = \mu_2 = \int_0^1 G_1(1,s)h_1(s)\,ds = \frac{2(3\sqrt{2}-2)}{3\sqrt{2\pi}}$. Take $R = 4, a(v) = p(v) = v^{\frac{2}{3}}, b(u) = u^{\frac{1}{2}}, q(u) = u^2$, it is easy to see that the conditions ($H_1$), ($H_2$), ($H_4$), and ($H_7$) hold. From Theorem 4.2 one concludes that the system (39) has two positive solutions.

**Example 5.8** Consider the singular system (39), where

$$f_1(x,u,v) = \frac{\sqrt{\pi}}{4}v^{\frac{1}{2}}, \quad x \geq 0, v \geq 0, \qquad f_2(x,u,v) = \frac{\sqrt{\pi}}{4}u^{\frac{1}{2}}, \quad x \geq 0, u \geq 0.$$

Take $R = 4, a(v) = p(v) = v^{\frac{1}{2}}$, $b(u) = q(u) = u^{\frac{1}{2}}$, it is easy to see that the conditions ($H_1$), ($H_2$), ($H_4$), and ($H_7$) hold. From Theorem 4.2 one concludes that the system (39) has two positive solutions.

**Remark 5.9** From Examples 5.1-5.8 we know that the conditions ($H_2$)-($H_5$) are applicable to more general functions and our results are different from those in [18–21, 23–25].

**Remark 5.10** If $n, m \geq 2$, $n-1 < \alpha \leq n, m-1 < \beta \leq m, \mu = \nu = 0$ in the system (1), all our conclusion is true because the corresponding Green's function $g_k(t,s)$ ($k = 1,2$) satisfies a Harnack-like inequality (see [19]). Hence our results improve and generalize some corresponding results in [19, 28, 31] and [32].

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The authors, S Xie and Y Xie, contributed to each part of this work equally and read and approved the final version of the manuscript.

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