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Fourth order Hamiltonian system with some singular nonlinear term and multiplicity result

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Abstract

We consider a fourth order Hamiltonian system with some singular nonlinear term and multiplicity result. We get two theorems which show the number of weak solutions of this problem. The first theorem is a result which shows that there exists a weak solution for this problem and the second one is an improved result of the first result, which shows that there exist infinitely many weak solutions for this problem. We get the first result by a variational method and critical point theory, and we get the second result by homology theory.

MSC: 35Q72

Keywords: fourth order Hamiltonian system; singular nonlinear term; variational method; critical point theory; minimax method; homology theory; $(P.S.)_c$ condition

1 Introduction

Let $\bar{N}_\epsilon(\theta)$ be a closure of an ϵ-neighborhood of $\theta = (0,\dots,0)$, $\epsilon > 0$ be a fixed small number, and D be an open subset in R^n with compact complement $\bar{N}_\epsilon(\theta) = R^n \setminus D$, $n \geq 2$. Let $c \in R$ and $|\cdot|$ be a norm in R^n. In this paper we consider the weak solutions $z(t) = (z_1(t),\dots,z_n(t)) \in C^4([0,2\pi],D)$ of a fourth order Hamiltonian system with singular nonlinear term

$$\begin{aligned} &\ddot{\ddot{z}}(t) + c\ddot{z}(t) + \operatorname{grad}_z\left(\frac{1}{|z(t)|^{2p}}\right) = 0, \quad p \geq 1, \\ &z(0) = z(2\pi), \qquad \ddot{z}(0) = \ddot{z}(2\pi). \end{aligned} \tag{1.1}$$

Our problems are characterized as a singular fourth order Hamiltonian system with singularity at $\{z(t) = \theta\}$, $\theta = (0,\dots,0)$. The motivation of this paper is the fourth order elliptic problem with singular potential. We recommend the book [1] for the singular elliptic problems. Many authors considered the fourth order elliptic boundary value problem. In particular, Choi and Jung [2] showed that the problem

$$\begin{aligned} &\Delta^2 u + c\Delta u = bu^+ + s \quad \text{in } \Omega, \\ &u = 0, \qquad \Delta u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

has at least two nontrivial solutions when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$, and $s < 0$ or when $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$, and $s > 0$. We obtained these results by using a variational reduction method. We [3] also proved that when $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b < \lambda_2(\lambda_2 - c)$, and $s < 0$, (1.2) has at least three nontrivial solutions by using degree theory. Tarantello [4] also studied

$$\begin{aligned} &\Delta^2 u + c\Delta u = b\big((u+1)^+ - 1\big), \\ &u = 0, \qquad \Delta u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

She showed that if $c < \lambda_1$ and $b \geq \lambda_1(\lambda_1 - c)$, then (1.3) has a negative solution. She obtained this result by degree theory. Micheletti and Pistoia [5] also proved that if $c < \lambda_1$ and $b \geq \lambda_2(\lambda_2 - c)$ then (1.3) has at least three solutions by the variational linking theorem and Leray-Schauder degree theory.

The eigenvalue problem

$$\begin{aligned} &\ddot{u} + \lambda u = 0 \quad \text{in } (0, 2\pi), \\ &u(0) = u(2\pi) = 0, \end{aligned}$$

has many eigenvalues λ_j, $j \geq 1$, and corresponding eigenfunctions ϕ_j, $j \geq 1$, suitably normalized with respect to $L^2([0, 2\pi])$ inner product and each eigenvalue λ_j is repeated as often as its multiplicity. The eigenvalue problem

$$\begin{aligned} &\ddot{\ddot{u}}(t) + c\ddot{u}(t) = \mu u \quad \text{in } (0, 2\pi), \\ &u(0) = u(2\pi) = 0, \qquad \ddot{u}(0) = \ddot{u}(2\pi) = 0, \end{aligned}$$

has also infinitely many eigenvalues $\mu_j = \lambda_j(\lambda_j - c)$, $j \geq 1$, and corresponding eigenfunctions ϕ_j, $j \geq 1$. We note that $\mu_1 < \mu_2 \leq \mu_3, \ldots, \mu_j \to +\infty$.

In this paper we are trying to find the weak solutions $z(t) \in C^4([0, 2\pi], D) \cap \Lambda D$ of the system (1.1) satisfying

$$\int_0^{2\pi} \big[\ddot{z}(t) \cdot \ddot{\phi}(t) - c\dot{z}(t) \cdot \dot{\phi}(t)\big]\, dt + \int_\Omega \operatorname{grad}_u \left(\frac{1}{|z(t)|^{2p}}\right) \cdot \phi(t)\, dt = 0$$

for all $\phi(t) \in C^4([0, 2\pi], D) \cap \Lambda D$, where ΛD is introduced in Section 2.

Theorem 1.1 *Assume that $\lambda_j < c < \lambda_{j+1}$, $j \geq 1$. Then the system* (1.1) *has at least one nontrivial weak solution.*

Moreover, we improve Theorem 1.1 as follows.

Theorem 1.2 *Assume that $\lambda_j < c < \lambda_{j+1}$, $j \geq 1$. Then the system* (1.1) *has infinitely many nontrivial weak solutions.*

For the proof of Theorem 1.1 we follow the approach of the variational method and use a minimax method in critical point theory on the loop space ΛD, and for the proof of Theorem 1.2 we follow homology theory. In Section 2, we introduce a loop subspace ΛD

of the Banach space, and we prove that the associated functional J of (1.1) satisfies the $(P.S.)$ condition on the loop subspace ΛD. In Section 3, we use a minimax method and critical point theory for the existence of a nontrivial weak solution of (1.1) and prove Theorem 1.1. We also prove Theorem 1.2 by using critical point theory and homology theory to prove the existence of infinitely many nontrivial weak solutions.

2 Variational approach

Let $L^2([0,2\pi],R)$ be a square integrable function space defined on $[0,2\pi]$. Any element x in $L^2([0,2\pi],R)$ can be written as

$$x = \sum h_k\phi_k \quad \text{with} \sum h_k^2 < \infty.$$

We shall denote the subset of $L^2([0,2\pi],R)$ satisfying the 2π-periodic condition, by $L^2(S^1,R)$. Similar notations will be used for other 2π-periodic function spaces. We define a subspace W of $L^2(S^1,R)$ as follows:

$$W = \left\{x \in L^2(S^1,R) \mid \sum |\mu_k| h_k^2 < \infty\right\}.$$

Then this is a complete normed space with a norm

$$\|x\|_W = \left[\sum |\mu_k| h_k^2\right]^{\frac{1}{2}}.$$

Let

$$W^+ = \{x \in W \mid h_k = 0 \text{ if } \mu_k < 0\},$$
$$W^- = \{x \in W \mid h_k = 0 \text{ if } \mu_k > 0\}.$$

Then $W = W^- \oplus W^+$, for $x \in W$, $x = x^- + x^+ \in W^- \oplus W^+$. Let E be the n Cartesian product space of W, *i.e.*,

$$E = W \times W \times \cdots \times W.$$

Let E^+ and E^- be the subspaces on which the functional

$$z \mapsto A(z) = \int_0^{2\pi} \left[\left|\ddot{z}(t)\right|^2 - c\left|\dot{z}(t)\right|^2\right] dt$$

is positive definite and negative definite, respectively. Then

$$E = E^+ \oplus E^-.$$

Let P^+ be the projection from E onto E^+ and P^- the projection from E onto E^-. The norm in E is given by

$$\|z\|_E^2 = \left\|P^+ z\right\|_E^2 + \left\|P^- z\right\|_E^2,$$

where $\|P^+z\|_E^2 = \sum_{i=1}^n \|P^+z_i\|_W^2$, $\|P^-z\|_E^2 = \sum_{i=1}^n \|P^-z_i\|_W^2$, $z = (z_1,\dots,z_n)$. Let $\nu^1_{\mu_i}, \nu^2_{\mu_i}, \dots, \nu^n_{\mu_i}$ be the eigenvalues of the matrix

$$\det(\mu_i I) = \begin{pmatrix} \mu_i & 0 & 0 & \dots & 0 \\ 0 & \mu_i & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu_i \end{pmatrix} \in M_{n\times n}(R), \quad i = 1,\dots,n,$$

that is,

$$\nu^k_{\mu_i} = \mu_i, \quad i \geq 1, \text{ for all } k = 1,\dots,n.$$

Let $(c^1_{1,\mu_i},\dots,c^1_{n,\mu_i}), (c^2_{1,\mu_i},\dots,c^2_{n,\mu_i}),\dots,(c^n_{1,\mu_i},\dots,c^n_{n,\mu_i})$ be the eigenvectors of the matrix

$$\det(\mu_i I) = \begin{pmatrix} \mu_i & 0 & 0 & \dots & 0 \\ 0 & \mu_i & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu_i \end{pmatrix} \in M_{n\times n}(R)$$

corresponding to the eigenvalues $\nu^1_{\mu_i}, \nu^2_{\mu_i},\dots,\nu^n_{\mu_i}$, respectively. Since $\nu^k_{\mu_i} = \mu_i$ for all $k = 1,2,\dots,n$, $(c^1_{1,\mu_i},\dots,c^1_{n,\mu_i}) = \cdots = (c^n_{1,\mu_i},\dots,c^n_{n,\mu_i})$. Let us set

$$(c_{1,\mu_i},\dots,c_{n,\mu_i}) = \left(c^1_{1,\mu_i},\dots,c^1_{n,\mu_i}\right) = \cdots = \left(c^n_{1,\mu_i},\dots,c^n_{n,\mu_i}\right).$$

Let us set

$$\begin{aligned} W_{\mu_i} &= \operatorname{span}\{\phi_i \mid \mu_j = \mu_i\}, \\ E_{\mu_i} &= \left\{(c_{1,\mu_i}\phi,\dots,c_{n,\mu_i}\phi) \in E \mid (c_1,\dots,c_n) \in R^n, \phi \in W_{\mu_i}\right\}, \\ E^1_{\mu_i} &= \left\{\left(c^1_{1,\mu_i}\phi,\dots,c^1_{n,\mu_i}\phi\right) \in E \mid \phi \in W_{\mu_i}\right\}, \\ &\vdots \\ E^n_{\mu_i} &= \left\{\left(c^n_{1,\mu_i}\phi,\dots,c^n_{n,\mu_i}\phi\right) \in E \mid \phi \in W_{\mu_i}\right\}. \end{aligned}$$

We note that

$$E_{\mu_i} \equiv E^1_{\mu_i} \equiv \cdots \equiv E^n_{\mu_i}$$

and

$$E = \bigoplus_{i\geq 1} E_{\mu_i}.$$

Let us introduce an open set of the Hilbert space E as follows:

$$\Lambda D = \left\{z \in E \mid z(t) \in D = R^n \setminus \bar{N}_\epsilon(\theta), \epsilon > 0 \text{ is a small number}, \forall t \in S^1\right\}.$$

Let us consider the functional on ΛD

$$J(z) = \frac{1}{2}\int_0^{2\pi} \left[\left|\ddot{z}(t)\right|^2 - c\left|\dot{z}(t)\right|^2\right] dt + \int_0^{2\pi} \frac{1}{|z(t)|^{2p}}\, dt, \quad p \geq 1. \tag{2.1}$$

The Euler equation for J is (1.1).

By Lemma 2.1, $J \in C^1(\Lambda D, R)$, and so the weak solutions of system (1.1) coincide with the critical points of the associated functional $J(z)$.

Lemma 2.1 *$J(z)$ is continuous and Fréchet differentiable in ΛD with Fréchet derivative*

$$DJ(z)v = \int_0^{2\pi} \left[\ddot{z}(t) \cdot \ddot{w}(t) - c\dot{z}(t) \cdot \dot{w}(t) + \operatorname{grad}_z \frac{1}{|z(t)|^{2p}} \cdot w(t)\right] dx \quad \forall w \in \Lambda D. \tag{2.2}$$

Moreover, $DJ \in C$. That is, $J \in C^1$.

Proof First we prove that $J(z)$ is continuous. For $z, w \in \Lambda D$,

$$\begin{aligned}
&\left|J(z+w) - J(z)\right| \\
&\quad = \left|\frac{1}{2}\int_0^{2\pi} (z \ddddot{+} w + cz \ddot{+} w) \cdot (z+w)\, dt \right. \\
&\qquad + \int_0^{2\pi} \frac{1}{|z(t)+w(t)|^{2p}}\, dt \\
&\qquad \left. - \frac{1}{2}\int_0^{2\pi} (\ddddot{z} + c\ddot{z}) \cdot z\, dt - \int_0^{2\pi} \frac{1}{|z(t)|^{2p}}\, dt\right| \\
&\quad = \left|\frac{1}{2}\int_0^{2\pi} \left[(\ddddot{z} + c\ddot{z}) \cdot w + (\ddddot{w} + c\ddot{w}) \cdot z + (\ddddot{w} + c\ddot{w}) \cdot w\right] dt \right. \\
&\qquad \left. + \int_0^{2\pi} \left(\frac{1}{|z(t)+w(t)|^{2p}} - \frac{1}{|z(t)|^{2p}}\right) dt\right|.
\end{aligned}$$

We have

$$\begin{aligned}
&\left|\int_0^{2\pi} \left[\frac{1}{|z(t)+w(t)|^{2p}} - \frac{1}{|z(t)|^{2p}}\right] dt\right| \\
&\quad \leq \left|\int_0^{2\pi} \left[\operatorname{grad}_z \frac{1}{|z(t)|^{2p}} \cdot w + O\big(\|w\|_E\big)\right] dt\right| = O\big(\|w\|_E\big).
\end{aligned} \tag{2.3}$$

Thus we have

$$\left|J(z+w) - J(z)\right| = O\big(\|w\|_E\big).$$

Next we shall prove that $J(z)$ is Fréchet differentiable in ΛD. For $z, w \in \Lambda D$,

$$\begin{aligned}
&\left|J(z+w) - J(z) - DJ(z)w\right| \\
&\quad = \left|\frac{1}{2}\int_0^{2\pi} (z \ddddot{+} w + cz \ddot{+} w) \cdot (z+w)\, dt + \int_0^{2\pi} \frac{1}{|z(t)+w(t)|^{2p}}\, dt \right.
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{2}\int_0^{2\pi} (\dddot{z} + c\ddot{z}) \cdot z\,dt - \int_0^{2\pi} \frac{1}{|z(t)|^{2p}}\,dt \\
&- \int_0^{2\pi} \left(\dddot{z} + c\ddot{z} + \operatorname{grad}_z \frac{1}{|z(t)|^{2p}}\right) \cdot w\,dt \Bigg| \\
= &\left| \frac{1}{2}\int_0^{2\pi} \left[(\dddot{w} + c\ddot{w}) \cdot z + (\dddot{w} + c\ddot{w}) \cdot w\right] dt \right. \\
&\left. + \int_0^{2\pi} \left(\frac{1}{|z(t) + w(t)|^{2p}} - \frac{1}{|z(t)|^{2p}}\right) dt - \int_0^{2\pi} \operatorname{grad}_z \frac{1}{|z(t)|^{2p}} \cdot w\,dt \right|.
\end{aligned}
$$

Thus by (2.3), we have

$$
\left|J(z + w) - J(z) - DJ(z)w\right| = O\left(\|w\|_E\right). \tag{2.4}
$$

Similarly, it is easily checked that $J \in C^1$. □

Lemma 2.2 *Assume that $\lambda_j < c < \lambda_{j+1}$, $j \geq 1$. Let $\{z_k\} \subset \Lambda D$, $z_k(t) \in Z$, and $z_k \rightharpoonup z$ weakly in ΛD with $z \in \partial \Lambda D$. Then $J(z_k) \to \infty$, where Z is a neighborhood of $\theta = (0, \dots, 0)$.*

Proof Since $\frac{1}{z(t)^{2p}}$ has a singular point $\theta = (0, \dots, 0)$ in R^n, the conclusion follows. □

Now, we shall prove that $J(z)$ satisfies $(P.S.)_\gamma$ condition for any $\gamma \in R$.

Lemma 2.3 *Assume that $\lambda_j < c < \lambda_{j+1}$, $j \geq 1$. Then if $\|z_k\|_E \to \infty$, then there exist $(z_{h_k})_k$ and z in ΛD such that*

$$
\operatorname{grad}_z \frac{1}{|z_{h_k}(t)|^{2p}} \to z \in \Lambda D, \qquad \frac{z_{h_k}}{\|z_{h_k}\|_E} \to \theta, \quad \theta = (0, \dots, 0).
$$

Proof Let $\|z_k\|_E \to \infty$. Then $\frac{1}{|z_k(t)|^{2p}}$ is bounded, it follows that

$$
\int_0^{2\pi} \frac{\frac{1}{|z_k(t)|^{2p}}}{\|z_k\|_E}\,dt \to 0. \tag{2.5}
$$

Since

$$
\int_0^{2\pi} \frac{[\operatorname{grad}_z \frac{1}{|z_k(t)|^{2p}} \cdot z_k(t) - 2\frac{1}{|z_k(t)|^{2p}}]}{\|z_k\|_E}\,dt \longrightarrow 0,
$$

by (2.5), we have

$$
\int_0^{2\pi} \frac{\operatorname{grad}_z \frac{1}{|z_k(t)|^{2p}} \cdot z_k(t)}{\|z_k\|_E}\,dt \longrightarrow 0. \tag{2.6}
$$

Thus the sequence $(\int_0^{2\pi} \frac{\operatorname{grad}_z \frac{1}{|z_k(t)|^{2p}} \cdot z_k(t)}{\|z_k\|_E}\,dt)_k$ is bounded. It follows from (2.6) that there exists a subsequence $(z_{h_k})_k$ such that

$$
\lim_{k\to\infty} \frac{\int_0^{2\pi} [\operatorname{grad}_z \frac{1}{|z_{h_k}(t)|^{2p}} \cdot z_{h_k}(t)]\,dt}{\|z_{h_k}\|_E} = \lim_{k\to\infty} \int_0^{2\pi} \operatorname{grad}_z \frac{1}{|z_{h_k}(t)|^{2p}} \cdot \frac{z_{h_k}(t)}{\|z_{h_k}\|_E}\,dt = 0. \tag{2.7}
$$

Since $\text{grad}_z \frac{1}{|z_k(t)|^{2p}}$ is bounded when $\|z_k\|_E \to \infty$, it follows from (2.7) that there exists z in ΛD such that

$$\text{grad}_z \frac{1}{|z_{h_k}(t)|^{2p}} \to z \in \Lambda D, \qquad \frac{z_{h_k}}{\|z_{h_k}\|_E} \to \theta.$$

Thus the lemma is proved. □

Lemma 2.4 *Assume that* $\lambda_j < c < \lambda_{j+1}$, $j \geq 1$. *Then* $J(z)$ *satisfies the* $(P.S.)_\gamma$ *condition for any* $\gamma \in R$.

Proof Let $\gamma \in R$ and $(z_k)_k \subset \Lambda D$ be a sequence such that $J(z_k) \to \gamma$ and

$$DJ(z_k) = \ddddot{z}_k(t) + c\ddot{z}_k(t) + \text{grad}_z\left(\frac{1}{|z_k(t)|^{2p}}\right) \longrightarrow \theta, \quad \theta = (0, \ldots, 0) \text{ in } \Lambda D$$

or equivalently

$$\|P_+z_k(t)\| - \|P_-z_k(t)\| + (D_{tttt} + cD_{tt})^{-1}\left(\text{grad}_z\left(\frac{1}{|z_k(t)|^{2p}}\right)\right) \longrightarrow \theta, \tag{2.8}$$

where $D_{tttt}z_k(t) = \ddddot{z}_k(t)$, $(D_{tttt} + cD_{tt})^{-1}$ is a compact operator. We shall show that $(z_k)_k$ has a convergent subsequence. We claim that $\{z_k\}$ is bounded in ΛD. By contradiction, we suppose that $\|z_k\|_E \to \infty$ and set $w_k = \frac{z_k}{\|z_k\|_E}$. Since $(w_k)_k$ is bounded, up to a subsequence, $(w_k)_k$ converges weakly to some w_0 in ΛD. Since $J(z_k) \to \gamma$ and $DJ(z_k) \to 0$, we have

$$\frac{DJ(z_k) \cdot (z_k)}{\|z_k\|_E} = \frac{2J(z_k)}{\|z_k\|_E} + \frac{\int_0^{2\pi} [\text{grad}_z \frac{1}{|z_k(t)|^{2p}} \cdot z_k(t) - 2\frac{1}{|z_k(t)|^{2p}}]\, dt}{\|z_k\|_E} \longrightarrow 0.$$

Thus we have

$$\frac{\int_0^{2\pi} [\text{grad}_z \frac{1}{|z_k(t)|^{2p}} \cdot z_k(t) - 2\frac{1}{|z_k(t)|^{2p}}]\, dt}{\|z_k\|_E} \longrightarrow 0.$$

By Lemma 2.3 and (2.8), there exist $(z_{h_k})_k$ and z in ΛD such that

$$\text{grad}_u \frac{1}{|z_{h_k}(t)|^{2p}} \to z \in \Lambda D, \quad \text{and} \quad \frac{z_{h_k}}{\|z_{h_k}\|_E} \to \theta.$$

Thus we have $w_0 = 0$, which is absurd because $\|w_0\|_E = 1$. Thus $\{z_k\}$ is bounded in ΛD. Thus $(z_k)_k$ has a convergent subsequence converging weakly to some z in ΛD. We claim that this subsequence of $(z_k)_k$ converges strongly to z. By $DJ(z_k) \to \theta$, we have

$$DJ(z_k) = \ddddot{z}_k + c\ddot{z}_k + \text{grad}_z \frac{1}{|z_k(t)|^{2p}} \longrightarrow \theta.$$

We claim that the mapping $z_k \to\mapsto (\text{grad}_z \frac{1}{|z_k(t)|^{2p}})_k$ is compact. Since the embedding $\Lambda D \hookrightarrow C^2([0, 2\pi] \times \Lambda D, R^n)$ is compact, the sequence $(\int_0^{2\pi} [\text{grad}_z \frac{1}{|z_k(t)|^{2p}} \cdot z_k(t)\, dt)_n$ has a convergent subsequence which converges to $\int_0^{2\pi} [\text{grad}_z \frac{1}{|z(t)|^{2p}} \cdot z(t)\, dt$. Because $\{z_k\}$ is bounded and the subsequence of $(z_k)_k$ converges weakly to some z in ΛD, $(\text{grad}_z \frac{1}{|z_k(t)|^{2p}})_k$ is bounded. Since $(D_{tttt} + cD_{tt})^{-1}$ is compact, by (2.8), $(P_+z_k)_k$ and $(P_-z_k)_k$ have subsequences converging strongly. Thus $(z_k)_k$ has a subsequence converging strongly. Thus the lemma is proved. □

3 Proofs of Theorems 1.1 and 1.2

Lemma 3.1 *There exists a sequence of integers*

$$b_1 < b_2 < \cdots < b_i < \cdots, \quad b_i \to \infty,$$

such that $H_{b_i}(\Lambda D) \neq 0$.

Proof Let $\epsilon > 0$ be a fixed small number such that $\bar{N}_\epsilon(\theta)$ contains θ, and choose $R > 0$ such that $\bar{N}_\epsilon(\theta) \subset \operatorname{int}(B_R)$. Then we have

$$R^n \setminus B_R \subset D \subset R^n \setminus \{\theta\}.$$

Since $R^n \setminus B_R$ is a deformation retract of $R^n \setminus \{\theta\}$, $\Lambda(R^n \setminus B_R)$ is a deformation retract of $\Lambda(R^n \setminus \{\theta\})$, so $\Lambda(R^n \setminus B_R)$ is a deformation retract of λD. Then we have

$$\begin{aligned} H_*(\Lambda D) &\cong H_*\bigl(\Lambda\bigl(R^n \setminus B_R\bigr)\bigr) \oplus H_*\bigl(\Lambda D, \Lambda\bigl(R^n \setminus B_R\bigr)\bigr) \\ &\cong H_*\bigl(\Lambda\bigl(S^{n-1}\bigr)\bigr) \oplus H_*\bigl(\Lambda D, \Lambda\bigl(S^{n-1}\bigr)\bigr). \end{aligned}$$

By [6], the Poincaré series of $\Lambda(S^{n-1})$ is written as

$$P_t\bigl(\Lambda\bigl(S^{n-1}\bigr)\bigr) = \bigl(1 + t^n\bigr) + \frac{t^{n-1}}{1 - t^{2(n-1)}}\bigl(1 + t^n\bigr)\bigl(1 + t^{n-1}\bigr)$$

with Z_2 coefficients. Thus the lemma is proved. □

Let us set a level set

$$J_\gamma = \bigl\{z \in \Lambda D \mid J(z) \leq \gamma\bigr\}$$

and

$$\beta = \bigl\{[z] \subset \Lambda D \mid z \in \Lambda D, z(t) \text{ is a loop on } D, \forall t \in S^1\bigr\}.$$

Lemma 3.2 *Assume that* $\lambda_j < c < \lambda_{j+1}$, $j \geq 1$. *For each* $\gamma > 0$, *there exists a finite dimensional singular complex* $\Omega = \Omega_\gamma$ *such that the level set* J_γ *is deformed into* Ω.

Proof Let us choose $z \in J_\gamma$. Then $z \in \Lambda D$ and we have

$$J(z) = \frac{1}{2}\int_0^{2\pi} \bigl[\bigl|\ddot{z}(t)\bigr|^2 - c\bigl|\dot{z}(t)\bigr|^2\bigr]\,dt + \int_0^{2\pi} \frac{1}{|z(t)|^{2p}}\,dt \leq \gamma.$$

We note that there exists a constant $R_0 > 0$ such that

$$\begin{aligned} &\text{if } \bigl(t, z(t)\bigr) \in [0, 2\pi] \times R^n \setminus B_{R_0}, \\ &\quad \text{then } \frac{1}{|z(t)|^{2p}} < +\infty \quad \text{and} \quad \left|\operatorname{grad}_z \frac{1}{|z(t)|^{2p}}\right| < +\infty. \end{aligned} \tag{3.1}$$

We also note that there exists a neighborhood Z of $\bar{N}_\epsilon(\theta)$ such that

$$\frac{1}{|z(t)|^{2p}} \geq \frac{C}{d^2(z,Z)} \quad \text{for } (t,z) \in [0,2\pi] \times Z. \tag{3.2}$$

It follows that there exists a constant $\gamma_0 > 0$ such that

$$\int_0^{2\pi} \left[\left|\ddot{z}(t)\right|^2 - c\left|\dot{z}(t)\right|^2 \right] dt \leq \gamma_0,$$

i.e., we have

$$\left\|\dot{z}(t)\right\|_E^2 - c \int_0^{2\pi} \left|\dot{z}(t)\right|^2 dt \leq \gamma_0.$$

Since the number of elements of the set $\{\lambda_i - c \mid \lambda_i - c < 0\}$ is finite and $\lambda_i - c \to \infty$ as $i \in \infty$, there exists a constant $\gamma_1 > 0$ such that

$$\int_0^{2\pi} \left|\dot{z}(t)\right|^2 dt \leq \gamma_1. \tag{3.3}$$

By Lemma 2.2, there exists $\epsilon_0 = \epsilon(\gamma, \gamma_1)$ such that

$$d\bigl(z, \bar{N}_\epsilon(\theta)\bigr) \geq \epsilon_0 \quad \forall z \in J_\gamma, \forall t \in S^1.$$

Let us choose an integer $M = M_\gamma > 2\pi \frac{\gamma_1^{\frac{1}{2}}}{\epsilon_0}$ and let

$$t_i = \frac{2\pi i}{M}, \quad i = 1, 2, \ldots, M.$$

Let us define a broken line

$$\bar{z}(t) = \left(1 - \frac{1}{2\pi} M(t - t_{i-1})\right) z(t_{i-1}) + \frac{1}{2\pi} M(t - t_{i-1}) z(t_i),$$

$\forall t \in [t_{i-1}, t_i]$, $i = 0, 1, 2, \ldots, M$, $\forall x \in J_\gamma$. Let

$$\Omega = \bigl\{\bar{z}(t) \mid z \in J_\gamma\bigr\}.$$

The corresponding $\bar{z} \mapsto (z(t_1), z(t_2), \ldots, z(t_M))$ define a homeomorphism between Ω and a certain open subset of the M-fold product $D \times D \times \cdots \times D$. We first claim that $\Omega \subset \Lambda D$. In fact, $\forall z \in J_\gamma$, for $t_2 > t_1$, by (3.3), we have

$$\begin{aligned} \left\|z(t_2) - z(t_1)\right\|_{R^n} &\leq \int_{t_1}^{t_2} \left|\dot{z}(t)\right| dt \\ &\leq \left(\int_0^{2\pi} \left|\dot{z}(t)\right|^2 dt\right)^{\frac{1}{2}} |t_2 - t_1|^{\frac{1}{2}} \\ &\leq \gamma_1^{\frac{1}{2}} |t_2 - t_1|. \end{aligned}$$

Therefore

$$\begin{aligned} d\bigl(\bar{z}(t),\bar{N}_\epsilon(\theta)\bigr) &\geq d\bigl(z(t_i),\bar{N}_\epsilon(\theta)\bigr) - \left(1-\frac{1}{2\pi}M(t-t_{i-1})\right)\bigl\|z(t_i)-z(t_{i-1})\bigr\|_{R^n} \\ &\geq \epsilon_0 - 2\pi M^{-1}\gamma_1^{\frac{1}{2}} > 0 \end{aligned}$$

$\forall s \in [t_{i-1}, t_i]$, $i = 0,1,2,\ldots,M$. We next claim that there exists $\nu \in C([0,1]\times J_\gamma, \Lambda D)$ such that $\nu(0,\cdot) = \mathrm{id}$, and $\nu(1,J_\gamma) = \Omega$. In fact, let us choose $z(t) \in \Lambda D$ and let us define ν as follows:

$$\nu(s,z)(t) = \begin{cases} z(t) & \text{for } t \geq 2\pi s, \\ (1-\frac{t-t_{i-1}}{2\pi s - t_{i-1}})z(t_{i-1}) + \frac{t-t_{i-1}}{2\pi s-t_{i-1}}z(2\pi s) & \text{for } t_{i-1} < t < 2\pi s, \\ \bar{z}(t) & \text{for } t \leq t_{i-1} \leq 2\pi s \leq t_i. \end{cases}$$

Then $\nu(0,\cdot) = \mathrm{id}$, and $\nu(1,J_\gamma) = \Omega$. Thus we prove that J_γ is deformed into Ω in the loop space ΛD. Thus the lemma is proved. □

Proof of Theorem 1.1 (*Existence of a weak solution*) We shall show that the functional $J(z)$ has a critical value by the generalized mountain pass theorem. Thus we first shall show that $J(z)$ satisfies the geometric assumptions of the generalized mountain pass theorem.

Let

$$\Lambda D^+ = \Lambda D \cap E^+, \qquad \Lambda D^- = \Lambda D \cap E^-.$$

Then

$$\Lambda D = \Lambda D^+ \oplus \Lambda D^-.$$

Let $z \in \Lambda D^+$. Then we have

$$\begin{aligned} J(z) &= \frac{1}{2}\bigl\|P^+z(t)\bigr\|_E^2 - \frac{1}{2}\bigl\|P^-z(t)\bigr\|_E^2 + \int_0^{2\pi}\frac{1}{|z(t)|^{2p}}\,dt \\ &= \frac{1}{2}\bigl\|P^+z(t)\bigr\|_E^2 + \int_0^{2\pi}\frac{1}{|z(t)|^{2p}}\,dt. \end{aligned}$$

Since $\frac{1}{|z(t)|^{2p}}$ is positive and bounded, if $z \in \Lambda D^+$, then there exists a number $r > 0$ such that if $z \in \partial B_r \cap \Lambda D^+$, then $J(z) > 0$. Thus $\inf_{z\in\partial B_r\cap\Lambda D^+} J(z) > 0$. We note that by (3.1), there exists $R > R_0$ such that

$$\text{if} \quad \bigl(t,z(t)\bigr) \in [0,2\pi]\times R^n \setminus B_R, \quad \text{then } \frac{1}{|z(t)|^{2p}} < +\infty \quad \text{and} \quad \left|\mathrm{grad}_z \frac{1}{|z(t)|^{2p}}\right| < +\infty,$$

and by (3.2), there exists a neighborhood Z of $\bar{N}_\epsilon(\theta)$ such that

$$\frac{1}{|z(t)|^{2p}} \geq \frac{C}{d^2(z,Z)} \quad \text{for } (t,z) \in [0,2\pi]\times Z.$$

Let us choose $e \in B_1 \cap \Lambda D^+$. Let $z \in \Lambda D^- \oplus \{\rho e \mid \rho > 0\}$. Then $z = x + y$, $x \in \Lambda D^-$, $y = \rho e$. Then we have

$$\begin{aligned} J(z) &= \frac{1}{2}\|P^+ z(t)\|_E^2 - \frac{1}{2}\|P^- z(t)\|_E^2 + \int_0^{2\pi} \frac{1}{|z(t)|^{2p}}\,dt \\ &= \frac{1}{2}\rho^2 - \frac{1}{2}\|P^- x\|^2 + \int_\Omega \frac{1}{|x+\rho e|^{2p}}\,dt. \end{aligned}$$

By (3.1), there exists constant $R_0 > 0$ such that if $(t, z(t)) \in [0, 2\pi] \times R^n \setminus B_{R_0}$, then $|\frac{1}{|z(t)|^{2p}}| < +\infty$ and $|\operatorname{grad}_z \frac{1}{|z(t)|^{2p}}| < +\infty$. Thus there exist a large number $R > R_0$ and a small number $\rho > 0$ such that if $z = x + \rho e \in \partial Q = \partial(((\bar{B}_R \cap \Lambda D^-) \oplus \{re \mid e \in B_1 \cap \Lambda D^+, 0 < r < R\}) \setminus B_{R_0})$, then $J(z) < 0$. Thus we have $\sup_{z \in \partial Q} J(z) < 0$. By Lemma 2.1, $J(z)$ is continuous and Fréchet differentiable in ΛD and, moreover, $DJ \in C$. By Lemma 2.4, $J(z)$ satisfies the $(P.S.)$ condition. Thus by the generalized mountain pass theorem [7], $J(z)$ possesses a critical value $c > 0$, which is characterized as

$$c = \inf_{h \in \Gamma} \sup_{z \in Q} J\bigl(h(z)\bigr),$$

where

$$\Gamma = \bigl\{h \in C(\bar{Q}, \Lambda D) \mid h = \mathrm{id} \text{ on } \partial Q\bigr\}.$$

Thus (1.1) has at least one nontrivial weak solution. Thus we prove Theorem 1.1. □

Proof of Theorem 1.2 (*Existence of infinitely many weak nontrivial solutions*) By contradiction, we assume that $J(z)$ has only finitely many critical points $z_1, z_2, \ldots, z_l$ such that by the process of the proof of Theorem 1.1, we can obtain $J(z_j) > 0$, $1 \le j \le l$. Let us set

$$K = \{z_1, z_2, \ldots, z_l\}.$$

We note that $\dim\ker(D^2 J(z_j)) \le 2n$, for all j. Letting

$$b^* > \max\bigl\{nM_0, \operatorname{ind}(J, z_j) + \dim\ker\bigl(D^2 J(z_j)\bigr) \mid 1 \le j \le l\bigr\},$$

where M_γ is defined in the proof of Lemma 3.2, and

$$\tau > \max\bigl\{0, J(z_j) \mid 1 \le j \le l\bigr\},$$

we have

$$C_b(J, z_j) = 0 \quad \forall b \ge b^*, j = 1, 2, \ldots,$$

and

$$H_*(\Lambda D, J_0) = H_*(J_\tau, J_0).$$

It follows that

$$H_b(\Lambda D, J_0) = 0 \quad \forall b > b^*.$$

Since

$$i_* : H_b(\Lambda D) \longrightarrow H_b(\Lambda D, J_0) \quad \text{is injective for } b \geq b^*,$$
$$H_b(\Lambda D) = 0 \quad \text{for } b \geq b^*,$$

which is a contradiction to Lemma 3.1. Thus $J(z)$ has infinitely many critical points z_j, $j = 1, 2, \ldots$, in ΛD. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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