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Positive Green's function and triple positive solutions of a second-order impulsive differential equation with integral boundary conditions and a delayed argument

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Abstract

In this paper, we first establish the expression of positive Green's function for a second-order impulsive differential equation with integral boundary conditions and a delayed argument. Furthermore, applying Legget-William's fixed point theorem and Hölder's inequality, we obtain the existence results of at least three positive solutions under three cases: $p = 1$, $1 < p < +\infty$, and $p = +\infty$. We discuss our problem with impulsive effects and a delayed argument. In this case, our results cover second-order boundary value problems without impulsive effects and delayed arguments and are compared with some recent results. Finally, we give an example to illustrate our main results.

Keywords: differential equations with impulsive effects and delayed arguments; positive Green's function; three positive solutions; Legget-William's fixed point theorem; Hölder's inequality

1 Introduction

Functional differential equations with impulses are characterized by the fact that per sudden changing of their state the processes under consideration depend on their prehistory at each moment of time. They are used in many models of optimal control, physics, chemical technology, population dynamics, biology, biotechnology, industrial robotic, pharmacokinetics, etc. [1–5]. Therefore, the study of impulsive functional differential equations has gained prominence, and it is a rapidly growing field; see Zhang and Feng [6, 7], Nieto and López [8], Yan and Shen [9], Li and Shen [10], Feng and Qiu [11], Liu [12], Liu [13], He and Yu [14], Ding, Han, and Mi [15], and the references therein. We note that the difficulties solving such problems are that they have deviating arguments and their states are discontinuous. So, the results on impulsive functional differential equations are fewer than those on differential equations without impulses and deviating arguments.

Moreover, boundary value problems with deviating arguments constitute a very interesting and important class of problems. The existence and multiplicity of positive solutions for such problems have received a great deal of attention; see, for example, [16–

22] and the references therein. In particular, we would like to mention some results of Jankowski [23], who discussed a three-point boundary value problem for second-order impulsive differential equations with advanced arguments:

$$\begin{cases} x''(t) + \omega(t)f(x(\alpha(t))) = 0, \quad t \in J', \\ \triangle x'|_{t=t_k} = Q_k(x(t_k)), \quad k = 1, 2, \dots, m, \\ x(0) = 0, \qquad x(1) = \beta x(\eta), \end{cases}$$

where $J' = (0,1)/\{t_1, t_2, \dots, t_m\}$, $\triangle x'|_{t=t_k} = x'(t_k^+) - x'(t_k^-)$ with $x'(t_k^+)$ and $x'(t_k^-)$ representing the right- and left-hand limits of $x'(t)$ at $t = t_k$. By employing fixed point index theory the author obtained the existence of positive solutions.

However, to the best of our knowledge, there are almost no papers on the existence of three positive solutions for second-order impulsive differential equations with integral boundary conditions and a delayed argument, especially for L^p-integrable ω; for example, see [23, 24] and the references therein.

In this paper, we investigate the existence of three positive solutions for a second-order boundary value problem with impulsive effects and a delayed argument of the form

$$\begin{cases} u''(t) + \omega(t)f(t, u(\alpha(t))) = 0, \quad 0 < t < 1, t \neq t_k, k = 1, 2, \dots, n, \\ -\triangle u'|_{t=t_k} = I_k(u(t_k)), \quad k = 1, 2, \dots, n, \\ u'(0) = 0, \qquad au(1) + bu'(1) = \int_0^1 g(t)u(t)\,dt, \end{cases} \tag{1.1}$$

where $f \in C(J \times R^+, R^+)$, $\omega \in L^p[0,1]$ for some $1 \leq p \leq +\infty$. $I_k \in C(R^+, R^+)$, $R^+ = [0, +\infty)$, $J = [0,1]$, t_k $(k = 1, 2, \dots, n)$ are fixed points with $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_n < t_{n+1} = 1$, $\triangle u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-)$, $a, b > 0$, and $\alpha(t) \not\equiv t$ on $J = [0,1]$. In addition, ω, f, and g satisfy the following:

(H_1) $\omega \in L^p[0,1]$ for some $1 \leq p \leq +\infty$, and there exists $N > 0$ such that $\omega(t) \geq N$ a.e. on J;
(H_2) $f \in C(J \times R^+, R^+)$, $\alpha \in C(J, J)$ with $\alpha(t) \leq t$ on J, $I_k \in C(R^+, R^+)$;
(H_3) $g \in L^1[0,1]$ is nonnegative with $\mu \in [0, a)$, where $\mu = \int_0^1 g(s)\,ds$.

Motivated by the results mentioned, in this paper, we study the existence of three positive solutions for problem (1.1) by overcoming difficulties arising from the appearances of $\alpha(t) \not\equiv t$, $I_k \neq 0$, and L^p-integrable ω. The arguments are based upon a fixed point theorem due to Leggett and Williams, which deals with fixed points of a cone-preserving operator defined on an ordered Banach space. Another contribution of this paper is to study the expression and properties of Green's function associated with problem (1.1). It is interesting to point out that Green's function associated with problem (1.1) is positive, differently from [25].

The organization of this paper is as follows. In Section 2, we present an expression and properties of positive Green's function associated with problem (1.1). In Section 3, we state some necessary definitions and lemmas. In Section 4, we use Leggett-Williams' fixed point theorem to obtain the existence of three positive solutions for problem (1.1). Finally, in Section 5, we give an example to illustrate the main results.

2 Positive Green's function and its properties

Lemma 2.1 *Assume that* (H_3) *holds. Then, for any* $y \in C[0,1]$, *the boundary value problem*

$$\begin{cases} u''(t) + y(t) = 0, & t \in J, t \neq t_k, k = 1, 2, \dots, n, \\ -\triangle u'|_{t=t_k} = I_k(u(t_k)), & k = 1, 2, \dots, n, \\ u'(0) = 0, & au(1) + bu'(1) = \int_0^1 g(t)u(t)\,dt \end{cases} \tag{2.1}$$

has a unique solution u given by

$$u(t) = \int_0^1 H(t,s)y(s)\,ds + \sum_{k=1}^{n} H(t,t_k)I_k\bigl(u(t_k)\bigr), \tag{2.2}$$

where

$$H(t,s) = G(t,s) + \frac{1}{a-\mu}\int_0^1 G(\tau,s)g(\tau)\,d\tau, \tag{2.3}$$

$$G(t,s) = \begin{cases} 1 - t + \frac{b}{a}, & 0 \le s \le t \le 1, \\ 1 - s + \frac{b}{a}, & 0 \le t \le s \le 1. \end{cases} \tag{2.4}$$

Proof First, suppose that u is a solution of problem (2.1). It is easy to see by integration of problem (2.1) that

$$u'(t) - u'(0) = -\int_0^t y(s)\,ds - \sum_{t_k<t} I_k\bigl(u(t_k)\bigr). \tag{2.5}$$

Integrating again, we get

$$u(t) = u(0) + u'(0)t - \int_0^t (t-s)y(s)\,ds - \sum_{t_k<t} I_k\bigl(u(t_k)\bigr)(t-t_k). \tag{2.6}$$

Letting $t = 1$ in (2.6) and (2.5), we find

$$\begin{aligned} u(1) &= u(0) + u'(0) - \int_0^1 (1-s)y(s)\,ds - \sum_{t_k<1} I_k\bigl(u(t_k)\bigr)(1-t_k), \\ u'(1) &= -\int_0^1 y(s)\,ds - \sum_{t_k<1} I_k\bigl(u(t_k)\bigr). \end{aligned} \tag{2.7}$$

Substituting the boundary condition $u'(0) = 0$, $au(1) + bu'(1) = \int_0^1 g(t)u(t)\,dt$ and (2.7) into (2.6), we obtain

$$\begin{aligned} u(t) &= \frac{1}{a}\int_0^1 g(s)u(s)\,ds + \int_0^1 (1-s)y(s)\,ds + \sum_{t_k<1} I_k\bigl(u(t_k)\bigr)(1-t_k) + \frac{b}{a}\int_0^1 y(s)\,ds \\ &\quad + \frac{b}{a}\sum_{t_k<1} I_k\bigl(u(t_k)\bigr) - \int_0^t (t-s)y(s)\,ds - \sum_{t_k<t} I_k\bigl(u(t_k)\bigr)(t-t_k) \\ &= \int_0^1 G(t,s)y(s)\,ds + \sum_{k=1}^{n} G(t,t_k)I_k\bigl(u(t_k)\bigr) + \frac{1}{a}\int_0^1 g(s)u(s)\,ds, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned}\int_0^1 g(s)u(s)\,ds &= \int_0^1 g(s)\biggl[\int_0^1 G(s,\tau)y(\tau)\,d\tau + \sum_{k=1}^{n} G(s,t_k)I_k\bigl(u(t_k)\bigr)\\ &\quad + \frac{1}{a}\int_0^1 g(\tau)u(\tau)\,d\tau\biggr]ds\\ &= \frac{1}{a}\int_0^1 g(s)\,ds\int_0^1 g(\tau)u(\tau)\,d\tau\\ &\quad + \int_0^1 g(s)\biggl[\int_0^1 G(s,\tau)y(\tau)\,d\tau + \sum_{k=1}^{n} G(s,t_k)I_k\bigl(u(t_k)\bigr)\biggr]ds.\end{aligned}$$

Therefore, we have

$$\int_0^1 g(s)u(s)\,ds = \frac{a}{a-\mu}\int_0^1 g(s)\biggl[\int_0^1 G(s,\tau)y(\tau)\,d\tau + \sum_{k=1}^{n} G(s,t_k)I_k\bigl(u(t_k)\bigr)\biggr]ds$$

and

$$\begin{aligned}u(t) &= \int_0^1 G(t,s)y(s)\,ds + \sum_{k=1}^{n} G(t,t_k)I_k\bigl(u(t_k)\bigr)\\ &\quad + \frac{1}{a-\mu}\int_0^1 g(s)\biggl[\int_0^1 G(s,\tau)y(\tau)\,d\tau + \sum_{k=1}^{n} G(s,t_k)I_k\bigl(u(t_k)\bigr)\biggr]ds\\ &= \int_0^1 G(t,s)y(s)\,ds + \sum_{k=1}^{n} G(t,t_k)I_k\bigl(u(t_k)\bigr)\\ &\quad + \frac{1}{a-\mu}\int_0^1\biggl[\int_0^1 G(\tau,s)g(\tau)\,d\tau\biggr]y(s)\,ds\\ &\quad + \frac{1}{a-\mu}\int_0^1\biggl[\sum_{k=1}^{n} G(\tau,t_k)g(\tau)I_k\bigl(u(t_k)\bigr)\biggr]d\tau. \qquad (2.9)\end{aligned}$$

Let

$$H(t,s) = G(t,s) + \frac{1}{a-\mu}\int_0^1 G(\tau,s)g(\tau)\,d\tau. \qquad (2.10)$$

Then

$$u(t) = \int_0^1 H(t,s)y(s)\,ds + \sum_{k=1}^{n} H(t,t_k)I_k\bigl(u(t_k)\bigr). \qquad (2.11)$$

The proof of sufficiency is complete.

Conversely, let $u(t)$ be a solution of (2.1). Direct differentiation of (2.3) implies, for $t \neq t_k$,

$$u'(t) = -\int_0^t y(s)\,ds - \sum_{t_k<t} I_k\bigl(u(t_k)\bigr).$$

Evidently,

$$
\begin{aligned}
&u''(t) = -y(t),\\
&-\triangle u'|_{t=t_k} = I_k\big(u(t_k)\big) \quad (k = 1, 2, \ldots, n),\\
&u'(0) = 0, \qquad au(1) + bu'(1) = \int_0^1 g(t)u(t)\,dt.
\end{aligned}
$$

The lemma is proved. □

From (2.3) and (2.4) we can prove that $H(t,s)$, $G(t,s)$ have the following properties.

Lemma 2.2 *Let $\xi \in (0,1)$. If $\mu \in [0,a)$, then we have*

$$H(t,s) > 0, \qquad G(t,s) > 0, \quad \forall t,s \in J. \tag{2.12}$$

$$\frac{b}{a} \leq G(t,s) \leq G(s,s) \leq \left(1 + \frac{b}{a}\right), \quad \forall t,s \in J. \tag{2.13}$$

$$\rho_1 \leq H(t,s) \leq \frac{a}{a-\mu} G(s,s) \leq \rho_2, \quad \forall t,s \in J. \tag{2.14}$$

$$G(t,s) \geq \delta G(s,s), \qquad H(t,s) \geq \frac{a\delta}{a-\mu} G(s,s) \geq \delta\rho_1, \quad \forall t \in [0,\xi], s \in J, \tag{2.15}$$

where

$$\delta = \frac{1 - \xi + \frac{b}{a}}{1 + \frac{b}{a}}, \qquad \rho_1 = \frac{b}{a-\mu}, \qquad \rho_2 = \frac{a+b}{a-\mu}. \tag{2.16}$$

Proof It is obvious that (2.12), (2.13), and (2.14) hold by the definition of $G(t,s)$ and $H(t,s)$. Now, we show that (2.15) also holds.

In fact, for $t \in [0,\xi]$ and $s \in J$, we have the following.

Case 1. If $s \leq t$, then

$$\frac{G(t,s)}{G(s,s)} = \frac{1 - t + \frac{b}{a}}{1 - s + \frac{b}{a}} \geq \frac{1 - \xi + \frac{b}{a}}{1 + \frac{b}{a}}.$$

Case 2. If $t \leq s$, then

$$\frac{G(t,s)}{G(s,s)} = \frac{1 - s + \frac{b}{a}}{1 - s + \frac{b}{a}} = 1.$$

This shows that

$$G(t,s) \geq \delta G(s,s), \quad \forall t \in [0,\xi], s \in J.$$

Similarly, we can prove that

$$H(t,s) \geq \delta H(s,s), \quad \forall t \in [0,\xi], s \in J.$$

This gives the proof of Lemma 2.2. □

Remark 2.1 Noticing that (2.12), it is easy to see that Green's function associated with problem (1.1) is positive.

Remark 2.2 By the definition of δ and ρ_2, we obtain that

$$0 < \delta < 1, \qquad \rho_2 > 1.$$

3 Preliminaries

In this section, we provide some background materials from the theory of cones in Banach spaces, and then we state Hölder's inequality, the Arzelà-Ascoli theorem and Legget-Williams' fixed point theorem. The following definitions can be found in the book by Deimling [26] and in the book by Guo and Lakshmikantham [27].

Definition 3.1 Let E be a real Banach space over R. A nonempty closed set $K \subset E$ is said to be a cone if

(i) $c'u + d'v \in K$ for all $u, v \in P$ and all $c' \geq 0$, $d' \geq 0$.
(ii) $u, -u \in K$ implies $u = 0$.

Note that every cone $K \subset E$ induces an ordering in E given by $u \geq v$ if and only if $v - u \in K$.

Definition 3.2 A map Λ is said to be a nonnegative continuous concave functional on a cone K of a real Banach space E if $\Lambda : K \to R^+$ is continuous and

$$\Lambda\bigl(tx + (1-t)y\bigr) \geq t\Lambda(x) + (1-t)\Lambda(y)$$

for all $x, y \in K$ and $t \in J$.

Definition 3.3 An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

Lemma 3.1 (Arzelà-Ascoli) *A set $M \subset C(J,R)$ is said to be a precompact set if the following two conditions are satisfied*:

(i) *All the functions in the set M are uniformly bounded, that is, there exists a constant $r > 0$ such that $|u(t)| \leq r$ for all $t \in J$, $u \in M$*;
(ii) *All the functions in the set M are equicontinuous, that is, for every $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$, which is independent of the functions $u \in M$, such that*

$$\bigl|u(t_1) - u(t_2)\bigr| < \varepsilon$$

whenever $|t_1 - t_2| < \delta$, $t_1, t_2 \in J$.

Let $J' = J \setminus \{t_1, t_2, \ldots, t_n\}$, $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, n$, and

$$PC^1[0,1] = \bigl\{x \in C[0,1] : x' \in C(t_k, t_{k+1}), x'\bigl(t_k^-\bigr) = x'(t_k), \exists x'\bigl(t_k^+\bigr), k = 1, 2, \ldots, n\bigr\}.$$

Then $PC^1[0,1]$ is a real Banach space with norm

$$\|u\|_{PC^1} = \max\bigl\{\|u\|_\infty, \bigl\|u'\bigr\|_\infty\bigr\},$$

where $\|u\|_\infty = \sup_{t\in J} |u(t)|$, $\|u'\|_\infty = \sup_{t\in J} |u'(t)|$.

To establish the existence of positive solutions to problem (1.1), we construct the cone

$$K = \left\{ u \in PC^1[0,1] : u(t) \geq 0 \text{ on } J \text{ and } \min_{t\in[0,\xi]} u(t) \geq \frac{\delta\rho_1}{\rho_2} \|u\|_{PC^1}, t \in J \right\}. \tag{3.1}$$

It is easy to see that K is a closed convex cone of $PC^1[0,1]$.

Define $T : K \to PC^1[0,1]$ by

$$(Tu)(t) = \int_0^1 H(t,s)\omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds + \sum_{k=1}^n H(t,t_k)I_k\bigl(u(t_k)\bigr). \tag{3.2}$$

It follows from (3.2) and Lemma 2.1 that the following lemma holds.

Lemma 3.2 *Suppose that* (H_1)-(H_3) *hold. Then* $u \in PC^1[0,1]$ *is a solution of problem* (1.1) *if and only if* u *is a fixed point of operator* T.

Lemma 3.3 *Suppose that* (H_1)-(H_3) *hold. Then* $T(K) \subset K$, *and* $T : K \to K$ *is completely continuous.*

Proof For all $u \in K$, $Tu \geq 0$ on J, and it follows from (2.14) and (3.2) that

$$\begin{aligned}(Tu)(t) &= \int_0^1 H(t,s)\omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds + \sum_{k=1}^n H(t,t_k)I_k\bigl(u(t_k)\bigr) \\ &\leq \rho_2\left(\omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds + \sum_{k=1}^n I_k\bigl(u(t_k)\bigr)\right), \quad t \in J. \end{aligned} \tag{3.3}$$

It is obvious that

$$H'_t(t,s) = G'_t(t,s) = \begin{cases} -1, & 0 \leq s \leq t \leq 1, \\ 0, & 0 \leq t \leq s \leq 1, \end{cases} \tag{3.4}$$

and

$$\max_{t,s\in J, t\neq s} \bigl|H'_t(t,s)\bigr| = \max_{t,s\in J, t\neq s} \bigl|G'_t(t,s)\bigr| = 1.$$

Then

$$\begin{aligned}\bigl|(Tu)'(t)\bigr| &\leq \int_0^1 \bigl|H'_t(t,s)\bigr|\omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds + \sum_{k=1}^n \bigl|H'_t(t,t_k)\bigr|I_k\bigl(u(t_k)\bigr) \\ &\leq \int_0^1 \omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds + \sum_{k=1}^n I_k\bigl(u(t_k)\bigr), \quad t \in J. \end{aligned} \tag{3.5}$$

It follows from (3.3), (3.5), and $\rho_2 > 1$ that

$$\|Tu\|_{PC^1} \leq \rho_2\left(\int_0^1 \omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds + \sum_{k=1}^n I_k\bigl(u(t_k)\bigr)\right). \tag{3.6}$$

By (2.15), (3.2), and (3.6) we have

$$\begin{aligned}
&\min_{t\in[0,\xi]}(Tu)\\
&\quad=\min_{t\in[0,\xi]}\left(\int_0^1 H(t,s)\omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds+\sum_{k=1}^n H(t,t_k)I_k\bigl(u(t_k)\bigr)\right)\\
&\quad\ge\delta\rho_1\left(\int_0^1\omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds+\sum_{k=1}^n I_k\bigl(u(t_k)\bigr)\right)\\
&\quad=\frac{\delta\rho_1}{\rho_2}\rho_2\left(\int_0^1\omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds+\sum_{k=1}^n I_k\bigl(u(t_k)\bigr)\right)\\
&\quad\ge\frac{\delta\rho_1}{\rho_2}\|Tu\|_{PC^1}.
\end{aligned}\tag{3.7}$$

Thus, $T(K)\subset K$.

Next, we prove that $T:K\to K$ is completely continuous.

It is obvious that T is continuous.

Let $B_r=\{u\in PC^1[0,1]\mid\|u\|_{PC^1}\le r\}$ be bounded set. Then, for all $u\in B_r$, by the definition of $\|Tu\|_\infty$, $\|Tu'\|_\infty$, and $\|Tu\|_{PC^1}$ and by (3.3) and (3.5) we have

$$\begin{aligned}
\|Tu\|_\infty&=\sup_{t\in J}\bigl|Tu(t)\bigr|\\
&\le\rho_2\left(\int_0^1\omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds+\sum_{k=1}^n I_k\bigl(u(t_k)\bigr)\right)\\
&\le\rho_2\bigl(\|\omega\|_1L+nB\bigr)\\
&=\Gamma_0,\\
\bigl\|Tu'\bigr\|_\infty&=\sup_{t\in J}\bigl|Tu'(t)\bigr|\\
&\le\left(\int_0^1\omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds+\sum_{k=1}^n I_k\bigl(u(t_k)\bigr)\right)\\
&\le\bigl(\|\omega\|_1L+nB\bigr)\\
&=\Gamma_1,
\end{aligned}$$

and

$$\|Tu\|_{PC^1}=\max\bigl\{\|Tu\|_\infty,\bigl\|Tu'\bigr\|_\infty\bigr\}\le\max\{\Gamma_0,\Gamma_1\}=\Gamma_0,$$

where $L=\max_{t\in J,u\in K,\|u\|_{PC^1}\le r}f(t,u)$, $B=\max_{u\in K,\|u\|_{PC^1}\le r}I_k(u)$.

Therefore, $T(B_r)$ is uniformly bounded.

On the other hand, for all $t_1,t_2\in J_k$ with $t_1<t_2$, we have

$$\bigl|(Tu)(t_1)-(Tu)(t_2)\bigr|=\left|\int_{t_1}^{t_2}(Tu)'(t)\,dt\right|\le\Gamma_1|t_1-t_2|\to0\quad(t_1\to t_2).$$

Noting (3.4), we know that $H'(t,s)$ is a constant and

$$\begin{aligned}\bigl|(Tu)'(t_1)-(Tu)'(t_2)\bigr| = {}&\Biggl|\int_0^1 \bigl[H_t'(t_1,s)-H_t'(t_2,s)\bigr]\omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds \\ &+\sum_{k=1}^{n}\bigl[H_t'(t_1,t_k)-H_t'(t_1,t_k)\bigr]I_k\bigl(u(t_k)\bigr)\Biggr| \to 0 \quad (t_1\to t_2).\end{aligned}$$

Then $T(B_r)$ is equicontinuous. Lemma 3.1 shows that $T:K\to K$ is completely continuous, and the lemma is proved. □

Definition 3.4 The map β is said to be a nonnegative continuous concave functional on a cone K of a real Banach space E if $\beta: K\to[0,+\infty)$ is continuous and

$$\begin{aligned}&\beta\bigl(tx+(1-t)y\bigr)\ge t\beta(x)+(1-t)\beta(y)\\ &\quad \forall x,y\in K, 0\le t\le 1.\end{aligned}$$

For positive numbers $0<c<d$, we define the convex sets K_c, $\bar{K}_c$, $K(\beta,c,d)$ by

$$K_c=\bigl\{u\in K \mid \|u\|_{PC^1}<c\bigr\}, \tag{3.8}$$

$$\bar{K}_c=\bigl\{u\in K \mid \|u\|_{PC^1}\le c\bigr\}, \tag{3.9}$$

$$K(\beta,c,d)=\bigl\{u \mid u\in K, c\le\beta(u), \|u\|_{PC^1}\le d\bigr\}. \tag{3.10}$$

It is easy to see that $K(\beta,c,d)$ is a bounded closed convex set.

We state the well-known Leggett-William fixed point theorem [28].

Lemma 3.4 *Let K be a cone in a real Banach space E. Suppose that $A:\bar{K}_l\to\bar{K}_l$ is completely continuous, $\beta(u)$ be a nonnegative continuous concave functional on K satisfying $\beta(u)\le\|u\|$ for all $u\in\bar{K}_l$, and there exist positive numbers $0<m<c<d\le l$ such that*

(i) *$\{u \mid u\in K(\beta,c,d), \beta(u)>c\}\ne\emptyset$ and $\beta(Au)>c$ for $u\in K(\beta,c,d)$;*

(ii) *$\|Au\|<m$ for $\|u\|\le m$;*

(iii) *$\beta(Au)>c$ for $u\in K(\beta,c,l)$ and $\|Au\|>d$.*

Then, A has at least three fixed points u_1, u_2, and u_3 satisfying

$$\|u_1\|<m, \qquad c<\beta(u_2), \qquad \|u_3\|>m, \quad \text{and} \quad \beta(u_3)<c.$$

To obtain some of the norm inequalities in our main results, we employ Hölder's inequality.

Lemma 3.5 (Hölder) *Let $e\in L^p[a,b]$ with $p>1$, $h\in L^q[a,b]$ with $q>1$, and $\frac{1}{p}+\frac{1}{q}=1$. Then $eh\in L^1[a,b]$, and*

$$\|eh\|_1\le\|e\|_p\|h\|_q.$$

Let $e\in L^1[a,b]$ and $h\in L^\infty[a,b]$. Then $eh\in L^1[a,b]$, and

$$\|eh\|_1\le\|e\|_1\|h\|_\infty.$$

4 Main results

In this section, we establish the existence of triple positive solutions for problem (1.1). We consider the following three cases for $w \in L^p[0,1]$: $p > 1$, $p = 1$, and $p = \infty$. The case $p > 1$ is treated in the following theorem.

For convenience, we introduce the following notation:

$$\rho_3 = \max\left\{\frac{a}{a-\mu}, \frac{a}{b}\right\}, \qquad D = \rho_3\|G\|_q\|\omega\|_p, \qquad D_1 = n\rho_3\left(1+\frac{b}{a}\right), \qquad \delta^* = \frac{\delta\rho_1}{\rho_2},$$
$$f^\infty = \limsup_{u\to\infty}\max_{t\in J}\frac{f(t,u)}{u}, \qquad I^\infty(k) = \limsup_{u\to\infty}\frac{I_k(u)}{u}, \quad k = 1,2,\ldots,n.$$

Theorem 4.1 *Assume that* (H_1)-(H_3) *hold. Furthermore, suppose that there exist constants* $0 < m < c < \frac{c}{\delta^*} \le l$ *such that*

(H_4) $f^\infty < \frac{1}{2D}$, $I^\infty(k) < \frac{1}{2D_1}$, $k = 1,2,\ldots,n$;
(H_5) $f(t,u) \ge \frac{2c}{\xi\delta\rho_1 N}$, $\forall(t,u) \in [0,\xi] \times [c, \frac{c}{\delta^*}]$;
(H_6) $f(t,u) < \frac{m}{2D}$, $I_k(u) < \frac{m}{2D_1}$, $\forall(t,u) \in J \times [0,m]$, $k = 1,2,\ldots,n$.

Then problem (1.1) *has at least three positive solutions* u_1, u_2, *and* u_3 *satisfying*

$$\|u_1\|_{PC^1} < m, \qquad c < \beta(u_2), \qquad \|u_3\|_{PC^1} > m, \quad \text{and} \quad \beta(u_3) < c.$$

Proof Let $\beta(u) = \min_{0\le t\le\xi} u(t)$. It is clear that $\beta(u)$ is a nonnegative continuous concave functional on the cone K satisfying $\beta(u) \le \|u\|_{PC^1}$ for all $u \in K$. By (H_4) there exist $0 < \gamma < \frac{1}{2D}$, $0 < \gamma_1 < \frac{1}{2D_1}$, and $\rho' > 0$ such that

$$f(t,u) \le \gamma u, \qquad I_k(u) \le \gamma_1 u, \quad k = 1,2,\ldots,n, \forall t \in J, u \ge \rho'.$$

Let

$$\eta = \max_{(t,u)\in[0,1]\times[0,\rho']} f(t,u), \qquad \eta_1 = \max_{u\in[0,\rho']} I_k(u), \quad k = 1,2,\ldots,n.$$

Then

$$f\bigl(t,u(t)\bigr) \le \gamma u(t) + \eta, \qquad I_k(u) \le \gamma_1 u + \eta_1, \quad \forall t \in J, u \ge 0. \tag{4.1}$$

Since $0 \le \alpha(t) \le t \le 1$ on J, it follows from $u(t) \ge \rho'$ on J that

$$u\bigl(\alpha(t)\bigr) \ge \rho', \quad \forall t \in J. \tag{4.2}$$

Set $l > \max\{\frac{2D\eta}{1-2D\gamma}, \frac{2D_1\eta_1}{1-2D_1\gamma_1}, \frac{c}{\delta^*}\}$.

Consequently, for any $t \in J$ and $u \in \bar{K}_l$, (3.2) and (3.9) imply

$$
\begin{aligned}
(Tu)(t) &= \int_0^1 H(t,s)\omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds + \sum_{k=1}^{n} H(t,t_k)I_k\bigl(u(t_k)\bigr) \\
&\le \frac{a}{a-\mu}\left(\int_0^1 G(s,s)\omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds + \sum_{k=1}^{n} G(t,t_k)I_k\bigl(u(t_k)\bigr)\right) \\
&\le \frac{a}{a-\mu}\left(\|G\|_q\|\omega\|_p\int_0^1 f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds + \left(1+\frac{b}{a}\right)\sum_{k=1}^{n} I_k\bigl(u(t_k)\bigr)\right),
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
\bigl|(Tu)'(t)\bigr| &\le \int_0^1 \bigl|H_t'(t,s)\bigr|\omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds + \sum_{k=1}^{n}\bigl|H_t'(t,t_k)\bigr|I_k\bigl(u(t_k)\bigr) \\
&\le \int_0^1 \omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds + \sum_{k=1}^{n} I_k\bigl(u(t_k)\bigr) \\
&= \int_0^1 \frac{1}{G(s,s)}G(s,s)\omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds + \frac{1}{G(s,s)}\sum_{k=1}^{n} G(s,s)I_k\bigl(u(t_k)\bigr) \\
&\le \frac{a}{b}\left(\int_0^1 G(s,s)\omega(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds + \sum_{k=1}^{n} G(s,s)I_k\bigl(u(t_k)\bigr)\right) \\
&\le \frac{a}{b}\left(\|G\|_q\|\omega\|_p\int_0^1 f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds + \left(1+\frac{b}{a}\right)\sum_{k=1}^{n} I_k\bigl(u(t_k)\bigr)\right).
\end{aligned}
\tag{4.4}
$$

It follows from (4.3) and (4.4) that

$$
\begin{aligned}
\|Tu\|_{PC^1} &\le \rho_3\left(\|G\|_q\|\omega\|_p\int_0^1 f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds + \left(1+\frac{b}{a}\right)\sum_{k=1}^{n} I_k\bigl(u(t_k)\bigr)\right) \\
&\le \rho_3\|G\|_q\|\omega\|_p\bigl(\gamma\|u\|_{PC^1}+\eta\bigr) + \rho_3\left(1+\frac{b}{a}\right)n\bigl(\gamma_1\|u\|_{PC^1}+\eta_1\bigr) \\
&\le \rho_3\|G\|_q\|\omega\|_p(\gamma l+\eta) + \rho_3\left(1+\frac{b}{a}\right)n(\gamma_1 l+\eta_1) \\
&< \frac{l}{2}+\frac{l}{2} = l,
\end{aligned}
\tag{4.5}
$$

which implies that $Tu \in K_l$.

Hence, we have shown that if (H_4) holds, then the operator $T : \bar{K}_l \to \bar{K}_l$ is completely continuous.

Next, we verify that $\{u \mid u \in K(\beta, c, \frac{c}{\delta^*}), \beta(u) > c\} \neq \emptyset$ and $\beta(Tu) > c$ for all $u \in K(\beta, c, \frac{c}{\delta^*})$. Take $u_0(t) = \frac{\delta^*+1}{2\delta^*}c$, for $t \in J$. Then

$$
u_0 \in \left\{u \mid u \in K\left(\beta, c, \frac{c}{\delta^*}\right), \beta(u) > c\right\},
$$

which shows that

$$
\left\{u \mid u \in K\left(\beta, c, \frac{c}{\delta^*}\right), \beta(u) > c\right\} \neq \emptyset.
$$

Since $0 \le \alpha(t) \le t \le \xi$ on $[0,\xi]$, it follows from $c \le u(t) \le \frac{c}{\delta^*}$ on J that

$$c \le u(\alpha(t)) \le \frac{c}{\delta^*}, \quad \forall t \in [0,\xi].$$

Then, it follows from (H_5) that

$$\begin{aligned}
\beta(Tu) &= \min_{t\in[0,\xi]}(Tu) \\
&= \min_{t\in[0,\xi]} \int_0^1 H(t,s)\omega(s)f(s,u(\alpha(s)))\,ds + \sum_{k=1}^n H(t,t_k)I_k(u(t_k)) \\
&\ge \min_{t\in[0,\xi]} \int_0^1 H(t,s)\omega(s)f(s,u(\alpha(s)))\,ds \\
&\ge \delta\rho_1 \int_0^1 \omega(s)f(s,u(\alpha(s)))\,ds \\
&> \frac{1}{2}\delta\rho_1 \int_0^1 \omega(s)f(s,u(\alpha(s)))\,ds \\
&\ge \frac{1}{2}\delta\rho_1 \int_0^\xi N\frac{2c}{\xi\delta\rho_1 N}\,ds \\
&= c.
\end{aligned} \tag{4.6}$$

This implies that condition (i) of Lemma 3.4 holds.

Since $0 \le \alpha(t) \le t \le \xi$ on $[0,1]$, it follows from $0 \le \|u(t)\|_{PC^1} \le m$ on J that

$$0 \le \|u(\alpha(t))\|_{PC^1} \le m, \quad \forall t \in [0,1].$$

Then, for $u \in \bar{K}_m$, it follows from (H_6) and (4.5) that

$$\begin{aligned}
\|Tu\|_{PC^1} &\le \rho_3\left(\int_0^1 \|G\|_q\|\omega\|_p f(s,u(\alpha(s)))\,ds + \left(1+\frac{b}{a}\right)\sum_{k=1}^n I_k(u(t_k))\right) \\
&= \rho_3\|G\|_q\|\omega\|_p \int_0^1 f(s,u(\alpha(s)))\,ds + \rho_1\left(1+\frac{b}{a}\right)\sum_{k=1}^n I_k(u(t_k)) \\
&< \rho_3\|G\|_q\|\omega\|_p \int_0^1 \frac{m}{2D}\,ds + \rho_3\left(1+\frac{b}{a}\right)\sum_{k=1}^n \frac{m}{2D_1} \\
&= \frac{m}{2} + \frac{m}{2} = m.
\end{aligned} \tag{4.7}$$

This implies that condition (ii) of Lemma 3.4 holds.

Finally, we assert that if $u \in K(\beta,c,l)$ and $\|Tu\|_{PC^1} > \frac{c}{\delta^*}$, then $\beta(Tu) > c$.

Suppose that $u \in K(\beta, c, l)$ and $\|Tu\|_{PC^1} > \frac{c}{\delta^*}$. Then it follows form (3.7) that

$$\begin{aligned}\beta(Tu) &= \min_{t\in[0,\xi]}(Tu)\\ &= \min_{t\in[0,\xi]}\left[\int_0^1 H(t,s)\omega(s)f\big(s,u\big(\alpha(s)\big)\big)\,ds + \sum_{k=1}^{n} H(t,t_k)I_k\big(u(t_k)\big)\right]\\ &\geq \frac{\delta\rho_1}{\rho_2}\|Tu\|_{PC^1}\\ &= \delta^*\|Tu\|_{PC^1} > c.\end{aligned} \tag{4.8}$$

This implies that condition (iii) of Lemma 3.4 holds.

To sum up, the hypotheses of Lemma 3.4 hold. Therefore, an application of Lemma 3.4 implies that problem (1.1) has at least three positive solutions u_1, u_2, and u_3 satisfying

$$\|u_1\|_{PC^1} < m, \qquad c < \beta(u_2), \qquad \|u_3\|_{PC^1} > m \quad \text{and} \quad \beta(u_3) < c.$$

The following results deal with the case $p = \infty$. □

Corollary 4.1 *Assume that* (H_1)-(H_6) *hold. Then problem* (1.1) *has at least three positive solutions* u_1, u_2, *and* u_3 *satisfying*

$$\|u_1\|_{PC^1} < m, \qquad c < \beta(u_2), \qquad \|u_3\|_{PC^1} > m, \quad \textit{and} \quad \beta(u_3) < c.$$

Proof Let $\|G\|_1\|\omega\|_\infty$ replace $\|G\|_q\|\omega\|_p$ and repeat the previous argument.

Finally, we consider the case of $p = 1$. Let

(H_4') $f^\infty < \frac{1}{2D'}$, $I^\infty(k) < \frac{1}{2D_1'}$, $k = 1, 2, \ldots, n$;
(H_6') $f(t,u) < \frac{m}{2D'}$, $I_k(u) < \frac{m}{2D_1'}$, $\forall (t,u) \in J \times [0,m]$, $k = 1, 2, \ldots, n$, where $D' = \rho_2\|\omega\|_1$, $D_1' = \rho_2 n$. □

Corollary 4.2 *Assume that* (H_1)-(H_3), (H_4'), (H_5), *and* (H_6') *hold. Then problem* (1.1) *has at least three positive solutions* u_1, u_2, *and* u_3 *satisfying*

$$\|u_1\|_{PC^1} < m, \qquad c < \beta(u_2), \qquad \|u_3\|_{PC^1} > m, \quad \textit{and} \quad \beta(u_3) < c.$$

Proof Set $l' > \max\{\frac{2D'\eta}{1-2D'\gamma'}, \frac{2D_1'\eta_1}{1-D_1'\gamma_1}, \frac{c}{\delta^*}\}$, where $0 < \gamma' < \frac{1}{2D'}$. If $u \in \bar{K}_{l'}$, then, by assumption (H_4'), from (4.1) and (4.2) we obtain

$$f\big(t,u\big(\alpha(t)\big)\big) \leq \gamma' u\big(\alpha(t)\big) + \eta.$$

Then, for $u \in \bar{K}_{l'}$, it follows from (3.6) and (3.9) that

$$\begin{aligned}
\|Tu\|_{PC^1} &\le \rho_2 \left(\int_0^1 \omega(s) f\big(s, u\big(\alpha(s)\big)\big)\,ds + \sum_{k=1}^{n} I_k\big(u(t_k)\big) \right) \\
&\le \rho_2 \|\omega\|_1 \int_0^1 f\big(s, u\big(\alpha(s)\big)\big)\,ds + \rho_2 \sum_{k=1}^{n} I_k\big(u(t_k)\big) \\
&\le \rho_2 \|\omega\|_1 \int_0^1 \big(\gamma' u\big(\alpha(s)\big) + \eta\big)\,ds + \rho_2 n(\gamma_1 u + \eta_1) \\
&\le \rho_2 \|\omega\|_1 \big(\gamma' \|u\|_{PC^1} + \eta\big) + \rho_2 n\big(\gamma_1 \|u\|_{PC^1} + \eta_1\big) \\
&\le \rho_2 \|\omega\|_1 \big(\gamma' l' + \eta\big) + \rho_2 n\big(\gamma_1 l' + \eta_1\big) \\
&< \frac{l'}{2} + \frac{l'}{2} \\
&= l',
\end{aligned}$$

which implies that $Tu \in K_{l'}$.

Hence, we have shown that if (H_4') holds, then the operator $T : \bar{K}_{l'} \to \bar{K}_{l'}$ is completely continuous.

If $u \in \bar{K}_m$, then it follows from (3.6) and (H_6') that

$$\begin{aligned}
\|Tu\|_{PC^1} &\le \rho_2 \left(\int_0^1 \omega(s) f\big(s, u\big(\alpha(s)\big)\big)\,ds + \sum_{k=1}^{n} I_k\big(u(t_k)\big) \right) \\
&\le \rho_2 \|\omega\|_1 \int_0^1 f\big(s, u\big(\alpha(s)\big)\big)\,ds + \rho_2 \sum_{k=1}^{n} I_k\big(u(t_k)\big) \\
&< \rho_2 \|\omega\|_1 \int_0^1 \frac{m}{2D'}\,ds + \rho_2 \sum_{k=1}^{n} \frac{m}{2D_1'} \\
&= \frac{m}{2} + \frac{m}{2} = m.
\end{aligned}$$

Similarly to the proof of Theorem 4.1, we can get Corollary 4.2. □

Remark 4.1 Comparing with Jankowski [23], the main features of this paper are as follows.

(i) A Green function, especially, a positive Green function, is available.

(ii) We consider integral boundary conditions.

(iii) $\omega(t)$ is L^p-integrable, not only $\omega(t) \in C[0,1]$ on $t \in J$.

5 An example

To illustrate how our main results can be used in practice, we present an example.

Example 5.1 Let $\xi = \frac{1}{2}$, $n = 1$, $t_1 = \frac{1}{2}$, $p = 2$. Consider the following boundary value problem:

$$\begin{cases} u''(t) + \omega(t) f(t, u(\alpha(t))) = 0, \quad 0 \le t \le 1, t \neq \frac{1}{2}, \\ -\triangle u'|_{t=\frac{1}{2}} = I_1(u(\frac{1}{2})), \\ u'(0) = 0, \qquad u(1) + u'(1) = \int_0^1 tu(t)\,dt, \end{cases} \tag{5.1}$$

where $\alpha \in C(J, J)$, $\alpha(t) \le t$ on J, and $\omega(t) = \frac{1}{|t-\frac{1}{5}|^{\frac{1}{4}}}$, $\alpha(t) = t^2$, $I_1(u) = \frac{u}{10}$, $g(t) = t$, $a = b = 1$,

$$f(t,u) = \begin{cases} \frac{1}{4}\sqrt{\frac{\sqrt{5}}{15}}m, & t \in J, u \in [0, m], \\ \frac{1}{4}\sqrt{\frac{\sqrt{5}}{15}}m \times \frac{c-u}{c-m} + 3\sqrt{\frac{2}{\sqrt{5}}}c \times \frac{u-m}{c-m}, & t \in J, u \in [m, c], \\ 3\sqrt{\frac{2}{\sqrt{5}}}c, & t \in J, u \in [c, \frac{c}{\delta^*}], \\ 3\sqrt{\frac{2}{\sqrt{5}}}c + \sqrt{t(u - \frac{c}{\delta^*})}, & t \in J, u \in [\frac{c}{\delta^*}, \infty). \end{cases}$$

Thus, it is easy to see by calculating that $\omega(t) \ge N = \sqrt{\frac{\sqrt{5}}{2}}$ for a.e. $t \in J$ and

$$\begin{aligned} &\mu = \int_0^1 g(t)\,dt = \frac{1}{2}, \qquad \rho_1 = \frac{b}{a-\mu} = 2, \\ &\delta = \frac{2-\xi}{2} = \frac{3}{4}, \qquad \rho_2 = \frac{a+b}{a-\mu} = 4, \\ &\rho_3 = \max\left\{\frac{a}{a-\mu}, \frac{a}{b}\right\} = \max\{2, 1\} = 2, \qquad \delta^* = \frac{\delta\rho_1}{\rho_2} = \frac{3}{8}. \end{aligned}$$

Therefore, it follows from the definitions of ω, f, and g that (H_1)-(H_3) hold.

On the other hand, it follows from $\omega(t) = \frac{1}{|t-\frac{1}{5}|^{\frac{1}{4}}}$ and $G(t,t) = 1 - t + \frac{b}{a}$ that

$$\begin{aligned} &\|\omega\|_2 = \left[\int_0^1 \left(\frac{1}{|t-\frac{1}{5}|^{\frac{1}{4}}}\right)^2 dt\right]^{\frac{1}{2}} = \sqrt{\frac{6\sqrt{5}}{5}} = \sqrt{\frac{6}{\sqrt{5}}}, \\ &\|G\|_2 = \left[\int_0^1 \left(1 - t + \frac{b}{a}\right)^2 dt\right]^{\frac{1}{2}} = \sqrt{\frac{7}{3}}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &D = \rho_3\|G\|_2\|\omega\|_2 = 2\sqrt{\frac{14}{\sqrt{5}}}, \qquad D_1 = n\rho_3\left(1 + \frac{b}{a}\right) = 4, \\ &\frac{1}{2D} = \frac{1}{4}\sqrt{\frac{\sqrt{5}}{14}}, \qquad \frac{1}{2D_1} = \frac{1}{8}. \end{aligned}$$

Choosing $0 < m < c < \frac{8}{3}c \le l$, we have

$$f^\infty = \limsup_{u\to\infty} \max_{t\in J} \frac{f(t,u)}{u} = 0 < \frac{1}{4}\sqrt{\frac{\sqrt{5}}{14}} = \frac{1}{2D}, \qquad I^\infty(1) = \frac{1}{10} < \frac{1}{8} = \frac{1}{2D_1},$$

$$f(t,u) = 3\sqrt{\frac{2}{\sqrt{5}}}c > \frac{8}{3}\sqrt{\frac{2}{\sqrt{5}}}c = \frac{2c}{\xi\delta\rho_1 N}, \quad \forall(t,u) \in \left[0, \frac{1}{2}\right] \times \left[c, \frac{8}{3}c\right],$$

$$f(t,u) = \frac{1}{4}\sqrt{\frac{\sqrt{5}}{15}}m < \frac{1}{4}\sqrt{\frac{\sqrt{5}}{14}}m = \frac{m}{2D}, \qquad I_1(u) = \frac{u}{10} \le \frac{m}{10} < \frac{m}{8} = \frac{m}{2D_1},$$
$$\forall(t,u) \in [0,1] \times [0,m],$$

which show that (H_4), (H_5), and (H_6) hold.

By Theorem 4.1, problem (5.1) has at lest three positive solutions u_1, u_2, and u_3 satisfying

$$\|u_1\|_{PC^1} < m, \qquad c < \beta(u_2), \qquad \|u_3\|_{PC^1} > m, \quad \text{and} \quad \beta(u_3) < c.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All results belong to GL and MF. Both authors read and approved the final manuscript.

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