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Existence and regularity of time-dependent global attractors for the nonclassical reaction-diffusion equations with lower forcing term

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Abstract

Based on the notation of time-dependent attractors, we prove the existence and the regularity of time-dependent global attractors for a class of nonclassical reaction-diffusion equations when the forcing term $g(x) \in H^{-1}(\Omega)$ and the nonlinear function satisfies the critical exponent growth, which is weaker than the conditions used in (Jing and Liu in Appl. Anal. 94(7):1439-1449, 2015).

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1 Introduction

Recently, Conti and Di Plinio *et al.* [2–4] presented the notation of time-dependent global attractors and studied the long-time behavior of the wave equations and oscillation equations in the topology space equipped with the norm related to the time, respectively. Motivated by these results we investigate the existence and regularity of time-dependent global attractors for a class of nonclassical reaction-diffusion equations

$$\begin{cases} u_t - \varepsilon(t)\triangle u_t - \triangle u + \lambda u = f(u) + g(x), & x \in \Omega, \\ u|_{\partial\Omega} = 0, & t \in \mathbb{R}, \\ u(x,\tau) = u_\tau(x), & t \geq \tau. \end{cases} \tag{1.1}$$

Here Ω is a bounded set of $\mathbb{R}^n$ ($n \geq 3$) with smooth boundary $\partial\Omega$. $\lambda > 0$, $\tau \in \mathbb{R}$, and $\varepsilon(t)$ is a decreasing bounded function satisfying

$$\lim_{t\to+\infty} \varepsilon(t) = 0, \tag{1.2}$$

and there exists $\nu > 0$ such that

$$\sup_{t\in\mathbb{R}}\bigl[\bigl|\varepsilon(t)\bigr| + \bigl|\varepsilon'(t)\bigr|\bigr] \leq \nu. \tag{1.3}$$

The nonlinearity $f \in C^1(\mathbb{R})$ with $f(0) = 0$, is assumed to satisfy the following conditions:

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1, \quad \forall s \in \mathbb{R}, \tag{1.4}$$

$$\left|f'(s)\right| \le C\left(1 + |s|^{\frac{4}{n-2}}\right), \quad \forall s \in \mathbb{R}, \tag{1.5}$$

where λ_1 is the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$, C is a positive constant.

The nonclassical reaction-diffusion equation arises as a mathematical model to describe physical phenomena, such as non-Newtonian flows, solid mechanics, and heat conduction [5–7]. Aifantis provides a quite general approach for obtaining these equations (see [5, 8]).

When $\varepsilon(t)$ in (1.1) is only a positive constant, the long-time behavior of solutions for (1.1) has been extensively studied by several authors in [9–19] and the references therein. For instance, some authors obtained the existence of global(pullback) attractors of solutions for both the autonomous case [9, 10, 12, 14, 16] and the nonautonomous case [15, 17, 19]. Anh and Toan [18] investigated the existence and upper semicontinuity of uniform attractor in $H^1(\mathbb{R}^{\mathbb{N}})$ for this problem; besides, they also considered the case of singularly oscillating external forces on $\mathbb{R}^{\mathbb{N}}$ [20]. The existence of exponential attractors was obtained in [11, 13, 15]. In the general case of a time dependence, to the best of our knowledge, only Ding and Liu [1] proved the existence and regularity of time-dependent global attractors of (1.1) when the force term $g \in L^2(\Omega)$ ($\Omega \subset \mathbb{R}^3$) and the nonlinear term f satisfies the following conditions:

(i) $\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1, \quad \forall s \in \mathbb{R}$,

(ii) $\left|f''(s)\right| \le C\left(1 + |s|\right), \quad \forall s \in \mathbb{R}$.

In this paper, following the general lines of the approach used in [2–4], we investigate the existence and regularity of the time-dependent attractors for the process $U(t,\tau)$ generated by (1.1) under weaker conditions than [1].

2 Preliminaries

Without loss of generality, denote $H = L^2(\Omega)$ with the inner products $\langle\cdot,\cdot\rangle$ and norms $\|\cdot\|$. For $0 \le \sigma \le 2$, we define the hierarchy of compactly nested Hilbert spaces

$$H_\sigma = D\left(A^{\frac{\sigma}{2}}\right), \qquad \langle w, v\rangle_\sigma = \left\langle A^{\frac{\sigma}{2}} w, A^{\frac{\sigma}{2}} v\right\rangle, \qquad \|w\|_\sigma = \left\|A^{\frac{\sigma}{2}} w\right\|.$$

Then, for $t \in \mathbb{R}$ and $-1 \le \sigma \le 1$, we introduce the time-dependent spaces

$$\mathcal{H}_t^\sigma = H_{\sigma+1},$$

endowed with the time-dependent norms

$$\|u\|^2_{\mathcal{H}_t^\sigma} = \|u\|^2_\sigma + \varepsilon(t)\|u\|^2_{\sigma+1}.$$

The symbol σ is always omitted whenever zero. In particular, the time-dependent phase space where we settle the problem is

$$\mathcal{H}_t^{-1} = H_0 = L^2, \qquad \mathcal{H}_t = H_1, \quad \text{with } \|u\|_{\mathcal{H}_t}^2 = \|u\|^2 + \varepsilon(t)\|u\|_1^2,$$

then we have the compact embeddings

$$\mathcal{H}_t^{\sigma} \Subset \mathcal{H}_t, \quad -1 \leq \sigma < 1,$$

with injection constants independent of $t \in \mathbb{R}$. Note that the spaces $\mathcal{H}_t$ are all the same as linear spaces; besides, since $\varepsilon(t)$ is a decreasing function of t, for every $u \in H_1$ and $t \geq \tau \in \mathbb{R}$ we have

$$\|u\|_{\mathcal{H}_t}^2 \leq \|u\|_{\mathcal{H}_\tau}^2 \leq \max\left\{1, \frac{\varepsilon(\tau)}{\varepsilon(t)}\right\}\|u\|_{\mathcal{H}_t}^2.$$

Hence the norms $\|u\|_{\mathcal{H}_t}^2$ and $\|u\|_{\mathcal{H}_\tau}^2$ are equivalent for any fixed $t, \tau \in \mathbb{R}$, but the equivalent constant blows up when $t \to +\infty$.

3 The main results

3.1 *A priori* estimates

Under the assumptions of (1.2)-(1.5), if $g \in H^{-1}(\Omega)$, then using the standard Galerkin approximation method ([21]), we can obtain the result concerning the existence and uniqueness of solution for the problem (1.1); see, for example, [6, 9, 10]. Thus, based on the subsequent Lemma 3.2 we get the following results.

Lemma 3.1 *Assume that* (1.2)-(1.5) *hold, for any* $u_\tau \in \mathcal{H}_\tau$, *there is a unique solution* u *of* (1.1) *satisfying*

$$u \in \mathcal{C}([\tau, t], H_1).$$

Furthermore, let $u_i(\tau) \in \mathcal{H}_\tau$ *be two initial conditions such that* $\|u_i(\tau)\|_{\mathcal{H}_\tau} \leq R$ $(i = 1, 2)$ *and denote by* $u_i(t)$ *the corresponding solutions to the problem* (1.1). *Then the following estimate holds*:

$$\|u_1(t) - u_2(t)\|_{\mathcal{H}_t} \leq e^{K(t-\tau)}\|u_1(\tau) - u_2(\tau)\|_{\mathcal{H}_\tau}, \quad \forall t \geq \tau, \tag{3.1}$$

for some constant $K = K(R) > 0$.

Proof We only need to prove the estimate (3.1). Let C be a generic positive constant depending on R but independent of $u_i(\tau)$. We first observe that the energy estimate in Lemma 3.2 below ensures

$$\|U(t,\tau)u_i(\tau)\|_{\mathcal{H}_t} \leq C, \quad i = 1, 2. \tag{3.2}$$

We write $u_i(t) = U(t,\tau)u_i(\tau)$, $\overline{u} = U(t,\tau)u_1(\tau) - U(t,\tau)u_2(\tau)$. Then the difference between the two solutions satisfies

$$\overline{u}_t - \varepsilon(t)\Delta\overline{u}_t - \Delta\overline{u} + \lambda\overline{u} = f(u_1) - f(u_2),$$

with initial datum $u(\tau) = u_1(\tau) - u_2(\tau)$. Multiplying by $2\overline{u}$ in $L^2(\Omega)$ we obtain

$$\frac{d}{dt}\|\overline{u}\|^2_{\mathcal{H}_t} + 2\lambda\|\overline{u}\| + \bigl(2 - \varepsilon'(t)\bigr)\|\overline{u}\|_1^2 = 2\bigl\langle f(u_1) - f(u_2), \overline{u}\bigr\rangle.$$

Combining with the embedding $H_0^1(\Omega) \hookrightarrow L^{2n/(n-2)}$ $(n \geq 3)$, and according to (1.5) and (3.2), we have

$$\begin{aligned}
&2\bigl|\bigl\langle f(u_1) - f(u_1), \overline{u}\bigr\rangle\bigr| \\
&\quad = 2\int_\Omega \bigl|\bigl(f(u_1) - f(u_2)\bigr)\overline{u}\bigr|\,dx = 2\int_\Omega \bigl|\bigl(f'\bigl(\theta u_1 + (1-\theta)u_2\bigr)\overline{u}\bigr)\overline{u}\bigr|\,dx \\
&\quad \leq C\int_\Omega \bigl(1 + |u|^{\frac{4}{n-2}}\bigr)|\overline{u}|^2\,dx \leq C\biggl[\int_\Omega \bigl(1 + |u|^{\frac{4}{n-2}}\bigr)^{\frac{n}{2}}\biggr]^{\frac{2}{n}} \cdot \biggl[\int_\Omega |\overline{u}|^{2\cdot\frac{n}{n-2}}\biggr]^{\frac{n-2}{n}} \\
&\quad \leq C\bigl(1 + \|u\|^{4/(n-2)}_{L^{2n/(n-2)}}\bigr)\|\overline{u}\|^2_{L^{2n/(n-2)}} \leq C\|\overline{u}\|_1^2.
\end{aligned}$$

Thus, we end up with the differential inequality

$$\frac{d}{dt}\|\overline{u}\|^2_{\mathcal{H}_t} \leq C\|\overline{u}\|^2_{\mathcal{H}_t},$$

and an application of the Gronwall lemma on $[\tau, t]$ completes the proof. □

By means of the Lemma 3.1, a family of maps with $t \geq \tau \in \mathbb{R}$

$$U(t,\tau) : H_\tau \to H_t \quad \text{acting as } U(t,\tau)u_\tau = u(t),$$

define a strongly continuous process on a family of spaces $\{\mathcal{H}_t\}_{t\in\mathbb{R}}$.

Lemma 3.2 *Assume that* (1.2)-(1.5) *hold. For any* $u_\tau \in \mathcal{H}_\tau$, $t \geq \tau$, *let* $U(t,\tau)u_\tau$ *be the solution of* (1.1) *with initial value* u_τ. *Then there is a positive constant* K, *such that*

$$\bigl\|U(t,\tau)u_\tau\bigr\|_{\mathcal{H}_t} \leq K, \quad \forall t \geq \tau.$$

Proof Multiplying (1.1) by $2u + 2u_t$ in H we obtain

$$\begin{aligned}
&\frac{d}{dt}\bigl[(1+\lambda)\|u\|^2 + \bigl(1 + \varepsilon(t)\bigr)\|u\|_1^2 - 2\bigl\langle F(u), 1\bigr\rangle - 2\langle g, u\rangle\bigr] \\
&\quad + 2\lambda\|u\|^2 + \bigl(2 - \varepsilon'(t)\bigr)\|u\|_1^2 - 2\bigl\langle f(u), u\bigr\rangle - 2\langle g, u\rangle + 2\|u_t\|^2 + 2\varepsilon(t)\|u_t\|_1^2 = 0. \qquad (3.3)
\end{aligned}$$

Let

$$\begin{aligned}
E(t) &= (1+\lambda)\|u\|^2 + \bigl(1 + \varepsilon(t)\bigr)\|u\|_1^2 - 2\bigl\langle F(u), 1\bigr\rangle - 2\langle g, u\rangle, \\
I(t) &= 2\lambda\|u\|^2 + \bigl(2 - \varepsilon'(t)\bigr)\|u\|_1^2 - 2\bigl\langle f(u), u\bigr\rangle - 2\langle g, u\rangle,
\end{aligned}$$

it yields

$$\frac{d}{dt}E(t) + I(t) \leq 0,$$

namely

$$E(t) \leq -\int_{\tau}^{t} I(s)\,ds + E(\tau),$$

where

$$E(\tau) = (1+\lambda)\|u_\tau\|^2 + \bigl(1+\varepsilon(\tau)\bigr)\|u_\tau\|_1^2 - 2\bigl\langle F(u_\tau),1\bigr\rangle - 2\langle g,u_\tau\rangle.$$

In view of the condition (1.5), there are $0<\nu<1$ and $c\geq 0$, such that

$$2\bigl\langle F(u),1\bigr\rangle \leq (1-\nu)\|u\|_1^2 + c, \tag{3.4}$$

$$\bigl\langle f(u),u\bigr\rangle \leq (1-\nu)\|u\|_1^2 + c, \quad \forall u\in H_1. \tag{3.5}$$

Thus, combining with (1.4), (3.4), and (3.5), there exist two positive constants M_1 and M_2, such that

$$\begin{aligned} E(t) &\geq (1+\lambda)\|u\|^2 + \bigl(1+\varepsilon(t)\bigr)\|u\|_1^2 - (1-\nu)\|u\|_1^2 - c - \frac{2}{\nu}\|g\|_{H^{-1}}^2 - \frac{\nu}{2}\|u\|_1^2 \\ &\geq \lambda\|u\|^2 + \left(\frac{\nu}{2}+\varepsilon(t)\right)\|u\|_1^2 - M_1, \end{aligned}$$

and

$$\begin{aligned} I(t) &\geq 2\lambda\|u\|^2 + \bigl(2-\varepsilon'(t)\bigr)\|u\|_1^2 - 2(1-\nu)\|u\|_1^2 - 2c - \frac{2}{\nu}\|g\|_{H^{-1}}^2 - \frac{\nu}{2}\|u\|_1^2 \\ &\geq \lambda\|u\|^2 + \left(\frac{\nu}{2}+\varepsilon(t)\right)\|u\|_1^2 - M_2. \end{aligned}$$

So we deduce that

$$\begin{aligned} &\lambda\bigl\|u(t)\bigr\|^2 + \left(\frac{\nu}{2}+\varepsilon(t)\right)\bigl\|u(t)\bigr\|_1^2 - M_1 \\ &\quad\leq -\int_{\tau}^{t}\left(\lambda\bigl\|u(s)\bigr\|^2 + \left(\frac{\nu}{2}+\varepsilon(s)\right)\bigl\|u(s)\bigr\|_1^2 - M_2\right)ds + E(\tau). \end{aligned}$$

Therefore, for any $K>M_2$, there exists $t_0>\tau$ such that

$$\lambda\bigl\|u(t_0)\bigr\|^2 + \left(\frac{\nu}{2}+\varepsilon(t_0)\right)\bigl\|u(t_0)\bigr\|_1^2 \leq K.$$

As a result, if u is a solution of the systems (1.1), if we let $B_t = \bigcup_{t\geq\tau} U(t,\tau)B_\tau$, where

$$B_\tau = \left\{u_\tau \in \mathbb{B}_\tau(R) : \lambda\|u_\tau\|^2 + \left(\frac{\nu}{2}+\varepsilon(\tau)\right)\|u_\tau\|_1^2 \leq K\right\},$$

then B_t is a bounded time-dependent absorbing set of $\{U(t,\tau)\}_{t\geq\tau}$. Moreover, B_t is positively invariant. □

On the other hand, from the above discussion, for every $R \geq 0$ there exist positive constants μ and $t_0 = t_0(R)$ such that

$$\lambda \|u(t)\|^2 + (1 + \varepsilon(t)) \|u(t)\|_1^2 \leq \mu^2, \quad \forall \tau \leq t - t_0. \tag{3.6}$$

3.2 The time-dependent global attractors and regularity

The main result concerning the asymptotic behavior of problem (1.1) is contained in the following theorem.

Theorem 3.3 *The process $U(t,\tau)$ generated by problem* (1.1) *admits an invariant time-dependent global attractor $\mathfrak{U} = \{A_t\}_{t\in\mathbb{R}}$ in $\mathcal{H}_t$. Besides, A_t is bounded in $\mathcal{H}_t^1$, with a bound independent of t.*

In order to show that the process is asymptotically compact, we shall exhibit a pullback attracting family of (non-void) compact sets. For this purpose, we exploit a suitable decomposition of the process in the sum of a decaying part and of a compact one.

3.2.1 The decomposition

Under the conditions (1.4)-(1.5), like in [15] we write $f = f_0 + f_1$, where $f_0, f_1 \in C^1(\mathbb{R})$ fulfill, respectively, for some $k \geq 0$,

$$|f_0(s)| \leq C(1 + |s|^{\frac{n+2}{n-2}}), \quad \forall s \in \mathbb{R}, \tag{3.7}$$

$$f_0(s)s \leq 0, \quad \forall s \in \mathbb{R}, \tag{3.8}$$

$$|f_1(s)| \leq C(1 + |s|^{\gamma}), \quad \forall s \in \mathbb{R}, 0 < \gamma < \frac{n+2}{n-2}, \tag{3.9}$$

$$\limsup_{|s|\to\infty} \frac{f_1(s)}{s} < \lambda_1, \quad \forall s \in \mathbb{R}. \tag{3.10}$$

Since the injection $i : L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is dense, we know that for every $g \in H^{-1}(\Omega)$ and any $\eta > 0$, there is a $g^\eta \in L^2(\Omega)$ which depends on g and η such that

$$\|g - g^\eta\|_{H^{-1}} < \eta. \tag{3.11}$$

Let $\mathfrak{B} = \{\mathbb{B}_t(R_0)\}_{t\in\mathbb{R}}$ be a time-dependent absorbing set as in Lemma 3.2 and let $\tau \in \mathbb{R}$ be fixed. Then, for any $u_\tau \in \mathbb{B}_\tau(R_0)$, we divide $U(t,\tau)u_\tau$ into the sum

$$U(t,\tau)u_\tau = u(t) = U_0(t,\tau)u_\tau + U_1(t,\tau)u_\tau,$$

where

$$U_0(t,\tau)u_\tau = v^\eta(t), \qquad U_1(t,\tau)u_\tau = w^\eta(t),$$

respectively, solve the following systems:

$$\begin{cases} v_t^\eta + \varepsilon(t)Av_t^\eta + Av^\eta + \lambda v^\eta = f_0(v^\eta) + g - g^\eta, \quad x \in \Omega, \\ U_0(\tau,\tau) = u_\tau, \end{cases} \tag{3.12}$$

and

$$\begin{cases} w_t^\eta + \varepsilon(t)Aw_t^\eta + Aw^\eta + \lambda w^\eta = f(u) - f_0(v^\eta) + g^\eta, & x \in \Omega. \\ U_1(\tau,\tau) = 0, \end{cases} \tag{3.13}$$

In the following, the generic constant $C \geq 0$ depends only on $\mathfrak{B}$.

Lemma 3.4 *Under the conditions* (1.2)-(1.5), *there exist two constants* $\delta > 0$ *and* $K_1 > 0$ *such that*

$$\|U_0(t,\tau)u_\tau\|_{\mathcal{H}_t} \leq Ce^{-\delta(t-\tau)} + K_1, \quad \forall t \geq \tau.$$

Proof Multiplying (3.12) by $2v^\eta$ in H we obtain

$$\frac{d}{dt}\big[\|v^\eta\|^2 + \varepsilon(t)\|v^\eta\|_1^2\big] + 2\lambda\|v^\eta\|^2 + \big(2 - \varepsilon'(t)\big)\|v^\eta\|_1^2 = 2\big\langle f_0(v^\eta) + g - g^\eta, v^\eta\big\rangle.$$

By (3.5), we have

$$2\big\langle f_0(v^\eta), v^\eta\big\rangle = 2\int_\Omega f_0(v^\eta)v^\eta\,dx \leq 0,$$

and using the Cauchy and Young inequalities we get

$$2\big\langle g - g^\eta, v^\eta\big\rangle \leq 2\|g - g^\eta\|_{H^{-1}}\|v^\eta\|_{H^1} \leq \|g - g^\eta\|_{H^{-1}}^2 + \|v^\eta\|_1^2.$$

In view of (1.3) we get $1 - \varepsilon'(t) \geq \varepsilon(t) > 0$, thus we find

$$\frac{d}{dt}\big(\varepsilon(t)\|v^\eta\|^2 + \|v^\eta\|_1^2\big) + 2\lambda\|v^\eta\|^2 + \varepsilon(t)\|v^\eta\|_1^2 \leq \eta^2.$$

Taking $\delta = \min\{2\lambda, 1\} > 0$, then

$$\frac{d}{dt}\|U_0(t,\tau)u_\tau\|_{\mathcal{H}_t}^2 + \delta\|U_0(t,\tau)u_\tau\|_{\mathcal{H}_t}^2 \leq \eta^2.$$

Applying the Gronwall lemma on the interval $[\tau, t]$ with $t \geq \tau$, it follows that

$$\|U_0(t,\tau)u_\tau\|_{\mathcal{H}_t}^2 \leq \|u_\tau\|_{\mathcal{H}_\tau}^2 e^{-\delta(t-\tau)} + \eta^2/\delta.$$

The proof is complete. □

Summing up, the following uniform boundedness holds:

$$\sup_{t\geq\tau}\big[\|U(t,\tau)u_\tau\|_{\mathcal{H}_t} + \|U_0(t,\tau)u_\tau\|_{\mathcal{H}_t} + \|U_1(t,\tau)u_\tau\|_{\mathcal{H}_t}\big] \leq C. \tag{3.14}$$

In order to prove our further result, we also need the condition

$$\limsup_{|s|\to\infty} f'(s) < \lambda_1, \quad \forall s \in \mathbb{R}. \tag{3.15}$$

Lemma 3.5 *Under the conditions* (1.2)-(1.4) *and* (3.15), *there exists* $M = M(\mathfrak{B}) > 0$ *such that*

$$\sup_{t\geq\tau}\left\|U_1(t,\tau)u_\tau\right\|_{\mathcal{H}_t^{1/3}} \leq M.$$

Proof Multiplying (3.13) by $2A^{1/3}w^\eta$ in H we have

$$\begin{aligned}
&\frac{d}{dt}\left[\left\|w^\eta\right\|_{1/3}^2 + \varepsilon(t)\left\|w^\eta\right\|_{4/3}^2\right] + 2\lambda\left\|w^\eta\right\|_{1/3}^2 + \left(2-\varepsilon'(t)\right)\left\|w^\eta\right\|_{4/3}^2 \\
&\quad = 2\left\langle f(u) - f_0\left(v^\eta\right) + g^\eta, A^{1/3}w^\eta\right\rangle \\
&\quad = 2\left\langle f(u) - f\left(v^\eta\right), A^{1/3}w^\eta\right\rangle + 2\left\langle f_1\left(v^\eta\right), A^{1/3}w^\eta\right\rangle + 2\left\langle g^\eta, A^{1/3}w^\eta\right\rangle.
\end{aligned}$$

Using the Young equality, it leads to

$$\begin{aligned}
2\left\langle f(u) - f\left(v^\eta\right), A^{1/3}w^\eta\right\rangle &= 2\int_\Omega \left(f(u) - f\left(v^\eta\right)\right)A^{1/3}w^\eta\,dx \\
&\leq 2\int_\Omega \left|f'\left(u-\theta\left(u-v^\eta\right)\right)\right|\left|u-v^\eta\right|\left|A^{1/3}w^\eta\right|dx \\
&\leq C\int_\Omega \left|w^\eta\right|\left|A^{1/3}w^\eta\right|dx \leq C\left\|w^\eta\right\|\left\|A^{1/3}w^\eta\right\| \leq C + \frac{1}{4}\left\|w^\eta\right\|_{4/3}^2.
\end{aligned}$$

Since $\frac{3(n-2)\gamma}{3n+4} < 1$, from (3.9) it follows that

$$\begin{aligned}
&2\left|\left\langle f_1\left(v^\eta\right), A^{1/3}w^\eta\right\rangle\right| \\
&\quad \leq C\int_\Omega \left(1+\left|v^\eta\right|^\gamma\right)\left|A^{1/3}w^\eta\right|dx \leq C\left(\int_\Omega \left(1+\left|v^\eta\right|^\gamma\right)^{\frac{6n}{3n+4}}\right)^{\frac{3n+4}{6n}}\left(\int_\Omega \left|A^{1/3}w^\eta\right|^{\frac{6n}{3n-4}}\right)^{\frac{3n-4}{6n}} \\
&\quad \leq C\left(1+\left\|v^\eta\right\|_{L^{6n\gamma/(3n+4)}}^\gamma\right)\left\|A^{1/3}w^\eta\right\|_{L^{6n/(3n-4)}} \leq C\left(1+\left\|v^\eta\right\|_1^\gamma\right)\left\|A^{2/3}w^\eta\right\| \\
&\quad \leq C\left\|w^\eta\right\|_{4/3} \leq C + \frac{1}{4}\left\|w^\eta\right\|_{4/3}^2,
\end{aligned}$$

where we have used the embedding $H_1 = D(A^{\frac{1}{2}}) \hookrightarrow L^{\frac{2n}{n-2}}$ with the facts that $\frac{6n\gamma}{3n+4} \leq \frac{2n}{n-2}$ and $H_{2/3} = D(A^{1/3}) \hookrightarrow L^{\frac{6n}{3n-4}}$. Moreover, making use of the embedding $H_{2/3} \subset L^{18/5}(\Omega)$, we have

$$2\left\langle g^\eta, A^{1/3}w^\eta\right\rangle \leq 2\left\|g^\eta\right\|_{L^{18/13}}\left\|A^{1/3}w^\eta\right\|_{L^{18/5}} \leq C\left\|g^\eta\right\|\|w^\eta\|_{4/3} \leq C\left\|g^\eta\right\|^2 + \frac{1}{2}\|w^\eta\|_{4/3}^2.$$

As a result, we deduce

$$\frac{d}{dt}\left(\|w^\eta\|_{1/3}^2 + \varepsilon(t)\|w^\eta\|_{4/3}^2\right) + 2\lambda\|w^\eta\|_{1/3}^2 + \left(1-\varepsilon'(t)\right)\|w^\eta\|_{4/3}^2 \leq C.$$

Using $1 - \varepsilon'(t) \geq \varepsilon(t) > 0$, we conclude

$$\frac{d}{dt}\left(\|w^\eta\|_{1/3}^2 + \varepsilon(t)\|w^\eta\|_{4/3}^2\right) + 2\lambda\|w^\eta\|_{1/3}^2 + \varepsilon(t)\|w^\eta\|_{4/3}^2 \leq C.$$

Taking $\delta = \min\{2\lambda, 1\} > 0$, we have

$$\frac{d}{dt}\left\|U_1(t,\tau)u_\tau\right\|_{\mathcal{H}_t}^2 + \delta\left\|U_1(t,\tau)u_\tau\right\|_{\mathcal{H}_t}^2 \leq C.$$

Applying the Gronwall lemma on the interval $[\tau, t]$ with $t \geq \tau$ we obtain

$$\left\| U_1(t,\tau)u_\tau \right\|^2_{\mathcal{H}_t^{1/3}} \leq \|u_\tau\|^2_{\mathcal{H}_\tau^{1/3}} e^{-\delta(t-\tau)} + C/\delta.$$

The proof is complete. □

3.2.2 Existence of the invariant attractor

In line with the Lemma 3.5, we consider a family of $\mathcal{K} = \{K_t\}_{t\in\mathbb{R}}$, where

$$K_t = \left\{ u(t) \in \mathcal{H}_t^{1/3} : \left\| u(t) \right\|_{\mathcal{H}_t^{1/3}} \leq M \right\}.$$

It is clear that K_t is compact since embedding $\mathcal{H}_t^{1/3} \Subset \mathcal{H}_t$ is compact; besides, since the injection constants are independent of t, $\mathcal{K}$ is uniform. Finally, Lemma 3.2, Lemma 3.4, and Lemma 3.5 imply that $\mathcal{K}$ is pullback attracting; indeed,

$$\delta_t\left(\mathbb{B}_\tau(R_0), K_t\right) \leq Ce^{-\delta(t-\tau)}, \quad \forall t \geq \tau,$$

where $\delta_t(B, C)$ is the Hausdorff semidistance of two nonempty sets B, C.

Hence the process $U(t,\tau)$ is asymptotically compact, which proves the existence of the unique time-dependent global attractor $\mathcal{U} = \{A_t\}_{t\in\mathbb{R}}$. The invariance of $\mathcal{U}$ follows by the strong continuity of the process stated in Lemma 3.1.

3.2.3 Regularity of the attractor

The minimality of $\mathcal{U}$ in $\mathcal{K}$ establishes that $A_t \subset K_t$ for all $t \in \mathbb{R}$. Therefore, we immediately obtain the following regularity result.

Lemma 3.6 *A_t is bounded in $\mathcal{H}_t^{1/3}$ (with a bound independent of t).*

To prove that A_t is uniformly bounded in $\mathcal{H}_t^1$, as claimed in Theorem 3.3, we argue as follows. Fix $\tau \in \mathbb{R}$, for $u_\tau \in A_\tau$, we split the solution $U(t,\tau)u_\tau = u(t)$ into the sum $U_0(t,\tau)u_\tau + U_1(t,\tau)u_\tau$, where $U_0(t,\tau)u_\tau = v^\eta(t)$ and $U_1(t,\tau)u_\tau = w^\eta(t)$, instead of (3.12)-(3.13), solving, respectively,

$$\begin{cases} v_t^\eta + \varepsilon(t)Av_t^\eta + Av^\eta + \lambda v^\eta = g - g^\eta, & x \in \Omega, \\ U_0(\tau,\tau) = u_\tau, \end{cases} \tag{3.16}$$

$$\begin{cases} w_t^\eta + \varepsilon(t)Aw_t^\eta + Aw^\eta + \lambda w^\eta = f(u) + g^\eta, & x \in \Omega, \\ U_1(\tau,\tau) = 0. \end{cases} \tag{3.17}$$

As a particular case of Lemma 3.4, we know that

$$\left\| U_0(t,\tau)u_\tau \right\|_{\mathcal{H}_t^1} \leq Ce^{-\delta(t-\tau)} + \eta^2/\delta, \quad \forall t \geq \tau. \tag{3.18}$$

Lemma 3.7 *Under the assumptions* (1.2)-(1.5), *the following estimate holds*:

$$\sup_{t\geq\tau} \left\| U_1(t,\tau)u_\tau \right\|_{\mathcal{H}_t^1} \leq M_1$$

for some $M_1 = M_1(\mathcal{U}) > 0$.

Proof Multiplying (3.17) by $2Aw^\eta$ in H we obtain

$$\frac{d}{dt}\left[\|w^\eta\|_1^2 + \varepsilon(t)\|w^\eta\|_2^2\right] + 2\lambda\|w^\eta\|_1^2 + \left(2 - \varepsilon'(t)\right)\|w^\eta\|_2^2 = 2\left\langle f(u) + g^\eta, Aw^\eta\right\rangle.$$

Together with embeddings $H_1(\Omega) \hookrightarrow L^6(\Omega)$, Lemma 3.2, and (3.15), we have

$$\begin{aligned}
2\left|\left\langle f(u), Aw^\eta\right\rangle\right| &= 2\int_\Omega \left|f'(u)\nabla u \cdot A^{1/2}w^\eta\right| dx \le 2\lambda_1 \int_\Omega \left|\nabla u \cdot A^{1/2}w^\eta\right| dx \\
&\le 2\lambda_1 \|\nabla u\| \left\|A^{1/2}w^\eta\right\| \le C + \frac{1}{2}\|w^\eta\|_2^2, \\
2\left\langle g^\eta, Aw^\eta\right\rangle &\le 2\|g^\eta\|\|Aw^\eta\| \le 4\|g^\eta\|^2 + \frac{1}{2}\|w^\eta\|_2^2.
\end{aligned}$$

Therefore, we conclude

$$\frac{d}{dt}\left(\|w^\eta\|_1^2 + \varepsilon(t)\|w^\eta\|_2^2\right) + 2\lambda\|w^\eta\|_1^2 + \left(1 - \varepsilon'(t)\right)\|w^\eta\|_2^2 \le C.$$

From (1.3) we have $1 - \varepsilon'(t) \ge \varepsilon(t) > 0$, so we deduce

$$\frac{d}{dt}\left\|U_1(t,\tau)u_\tau\right\|_{\mathcal{H}_t^1}^2 + \delta\left\|U_1(t,\tau)u_\tau\right\|_{\mathcal{H}_t^1}^2 \le C.$$

Applying the Gronwall lemma on the interval $[\tau, t]$ with $t \ge \tau$, we obtain

$$\left\|U_1(t,\tau)u_\tau\right\|_{\mathcal{H}_t^1}^2 \le \|u_\tau\|_{\mathcal{H}_\tau^1}^2 e^{-\delta(t-\tau)} + C/\delta.$$

The proof is complete. □

Proof of Theorem 3.3 For all $t \in \mathbb{R}$, inequality (3.18) and Lemma 3.7 imply that

$$\lim_{\tau \to -\infty} \delta_t\left(U(t,\tau)A_\tau, K_t^1\right) = 0,$$

where

$$K_t^1 = \left\{u(t) \in \mathcal{H}_t^1 : \left\|u(t)\right\|_{\mathcal{H}_t^1} \le M_1\right\}.$$

Since $\mathcal{U}$ is invariant, this means

$$\delta_t\left(A_t, K_t^1\right) = 0.$$

Hence, $A_t \subset \overline{K_t^1} = K_t^1$; that is, A_t is bounded in $\mathcal{H}_t^1$ with a bound independent of $t \in \mathbb{R}$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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