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# Asymptotic behavior of solutions for the time-delayed equations of Benjamin-Bona-Mahony's type

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**Abstract**
In this paper, we investigate the asymptotic behavior of the solutions for the equations of Benjamin-Bona-Mahony's type with a time delay. We prove the global existence of solutions and energy decay. By using the Liapunov function method, we shall show that the solution is exponentially decay if the delay parameter $\tau$ is sufficiently small.

**MSC:** 35R10; 35B35; 35Q53

**Keywords:** Benjamin-Bona-Mahony equation; time delay; exponential decay; Liapunov function

## 1 Introduction

In this paper, we consider a family of dispersive equations of Benjamin-Bona-Mahony's type under the effect of dissipation, and we will investigate the asymptotic behavior of the solutions. Our model can be written in the abstract form

$$Mu_t(x,t) + \alpha Lu(x,t) + u(x,t-\tau)u_x(x,t) = 0, \tag{1.1}$$

$$u(x,t) = u(x+1,t), \tag{1.2}$$

$$u(x,s) = u_0(x,s), \tag{1.3}$$

where $x \in \mathbb{R}^1$, $t > 0$, $-\tau \leq s \leq 0$, $\alpha > 0$, and we set $\Omega = (0,1)$. The operators $M$ and $L$ can be differential operators or pseudo-differential operators, and the orders of pseudo-differential operators $M$ and $L$ are $\mu$ and $s$ with $s \geq \mu \geq 2$.

In the simplest case, when $M$ and $L$ are the differential operators $M = I - \frac{\partial}{\partial x^2}$, $L = -\frac{\partial}{\partial x^2}$, (1.1) is the well-known Benjamin-Bona-Mahony model [1]:

$$u_t(x,t) - u_{xxt}(x,t) - \alpha u_{xx}(x,t) + u(x,t)u_x(x,t) = 0 \tag{1.4}$$

which describes the unidirectional propagation of weakly nonlinear dispersive long waves where Burger's type dissipation is considered. The existence of global solutions and asymptotic behavior in time have been studied by several authors. The asymptotic behavior of solutions to the generalized Korteweg-de Vries-Burgers and Benjamin-Bona-Mahony-Burgers equations in one space dimension was studied by Amick, Bona and

Schonbek in [2], by Wang and Yang in [3], and by Bona and Luo in [4]. These results were generalized by Zhang [5] to multiple spatial dimensions. In [6–8] the authors considered a family of equations of KdV and BBM's type described by pseudo-differential operators, and studied the asymptotic behavior in one space dimension. In [9] study the Cauchy problem for a class of nonlinear dissipative equations of Benjamin-Bona-Mahony's type and discuss the existence of a global attractor and estimate its Hausdorff and fractal dimensions. But few of the equations involving delay.

To explain our motivation of introducing a time delay into Benjamin-Bona-Mahony equation, we consider the rate of change of $u$, which we denote by $\frac{Du}{Dt}$, is

$$\begin{aligned}\frac{Du}{Dt} &= \frac{d}{dt}u\bigl[x(t),t\bigr] = \frac{\partial}{\partial t}u(x,t) + \frac{dx(t)}{dt}\frac{\partial}{\partial x}u(x,t)\\ &= \frac{\partial}{\partial t}u(x,t) + u(x,t)\frac{\partial}{\partial x}u(x,t),\end{aligned}$$

where $x(t)$ is understood to change with time at $u = \frac{dx}{dt}$. However, we might have a delay $\tau$ to $u$. In this case the rate of change of $u$ with the delay $\tau$ should be

$$\begin{aligned}\frac{Du}{Dt} &= \frac{d}{dt}u\bigl[x(t-\tau),t\bigr] = \frac{\partial}{\partial t}u(x,t) + \frac{dx(t-\tau)}{dt}\frac{\partial}{\partial x}u(x,t)\\ &= \frac{\partial}{\partial t}u(x,t) + u(x,t-\tau)\frac{\partial}{\partial x}u(x,t).\end{aligned}$$

This clearly shows how we obtain the time-delay term $u(x,t-\tau)u_x(x,t)$ in Benjamin-Bona-Mahony equation (1.4). There is literature about delay reaction-diffusion equations [10–12], on which our work is based. Recently, the asymptotic behavior of solutions of time-delayed Burgers equation was studied by Liu in [13]. Subsequently, exponentially decay rate of solutions for Benjamin-Bona-Mahony equation (1.4) with a time decay was studied in [14], and in [15] author have obtained the exponentially decay rate of solutions for the time-Delayed Kuramoto-Sivashinsky equation. Here, by using the Liapunov function method, we shall show that the solution of problems (1.1)-(1.3) is exponentially decay if the delay parameter $\tau$ is sufficiently small.

We shall use standard notation. By $L^p(\Omega)$ we shall denote the space of functions in $\Omega$ whose $p$th power is integrable, with the norm $\|g\|_{L^p}^p = \int_\Omega |g(x)|^p\,dx$, $1 \le p < +\infty$. The norm in $L^2(\Omega)$ we will denote by $\|\cdot\|_{L^2} = \|\cdot\|$. By $L^\infty(\Omega)$ we denote the space of measurable essentially bounded functions in $\Omega$ with the norm

$$\|g\|_{L^\infty} = \operatorname*{ess\,sup}_{x\in\Omega}\bigl|g(x)\bigr|.$$

For each $\sigma \in \mathbb{R}$ we shall denote by $H^\sigma(\Omega)$ the usual Sobolev space of order $\sigma$. By $H^\sigma_{\mathrm{per}}(\Omega)$, $\sigma \ge 0$ we shall indicate the space of functions periodic in the sense of (1.2). If $g \in H^\sigma_{\mathrm{per}}(\Omega)$ then $g$ has an expansion in a Fourier series,

$$g(x) = \sum_{k\in\mathbb{Z}} g_k \exp(2ki\pi x).$$

The norm of $g$ in $H^\sigma_{\mathrm{per}}(\Omega)$ will be denoted by

$$\|g\|_{H^\sigma}^2 = \sum_{k\in\mathbb{Z}}\bigl(1+|k|^2\bigr)^\sigma |g_k|^2,$$

which is equivalent to $H^\sigma(\Omega)$ norm, $\sigma \geq 0$, according to Temam [16]. We shall denote by $\dot{L}^2(\Omega)$ and $\dot{H}^\sigma(\Omega)$ the space of functions $g \in L^2(\Omega)$ or $H^\sigma(\Omega)$ such that

$$\int_\Omega g(x)\,dx = 0. \tag{1.5}$$

The space $\dot{H}^\sigma(\Omega)$, $\sigma \in \mathbb{R}_+$, is the space of functions $g \in L^2(\Omega))$ such that $g$ satisfies (1.5) and

$$\sum_{k\in\mathbb{Z}} \left(1+|k|^2\right)^\sigma |g_k|^2 < +\infty.$$

In $\dot{H}^1_{\mathrm{per}}(\Omega)$ the Poincaré inequality holds, that is, if $g \in \dot{H}^1_{\mathrm{per}}(\Omega)$ then

$$\|g\| \leq C(\Omega)\left\|g'\right\|. \tag{1.6}$$

The inequality (1.6) shows that $\dot{H}^1_{\mathrm{per}}(\Omega)$ is a Hilbert space with scalar product of $H^1_0(\Omega)$, and $\|u\|_{H^1} = (u,u)^{1/2}_{H^1}$ is a norm on this space equivalent to that induced by $H^1(\Omega)$.

The operators $M$ and $L$ of (1.1) are pseudo-differential operators of orders $\mu$ and $s$, respectively, with

$$M : \dot{H}^\mu_{\mathrm{per}}(\Omega) \to \dot{L}^2_{\mathrm{per}}(\Omega), \quad \mu \geq 1, \mu \in \mathbb{R},$$
$$L : \dot{H}^s_{\mathrm{per}}(\Omega) \to \dot{L}^2_{\mathrm{per}}(\Omega), \quad s \geq 0, s \in \mathbb{R},$$

and

$$Mg(x) = \sum_{k\in\mathbb{Z}} m(k) g_k \exp(2ki\pi x),$$
$$Lg(x) = \sum_{k\in\mathbb{Z}} l(k) g_k \exp(2ki\pi x),$$

where $m$ and $l$ are the principal symbols of the operators $M$ and $L$, respectively. We assume from now on that the symbols $m$ and $l$ are even functions of $k$ that satisfy the growth conditions:

(i) There exist constants $c_1, c_2 > 0$ such that

$$c_1\left(1+|k|^2\right)^\mu \leq m(k) \leq c_2\left(1+|k|^2\right)^\mu. \tag{1.7}$$

(ii) There exist constants $c_3, c_4 > 0$ such that

$$c_3|k|^s \leq l(k) \leq c_4|k|^s. \tag{1.8}$$

The domains of operators $M$ and $L$ are given by

$$D(M) = \left\{ g \in \dot{H}^\mu_{\mathrm{per}}(\Omega), \sum_{k\in\mathbb{Z}} \left|m(k)\right|^2 |g_k|^2 < \infty \right\},$$

$$D(L) = \left\{ g \in \dot{H}^s_{\mathrm{per}}(\Omega), \sum_{k\in\mathbb{Z}} \left|l(k)\right|^2 |g_k|^2 < \infty \right\}.$$

Let $X$ be a Banach space and $a < b$. We denote by $C^n([a,b];X)$ the space of $n$ times continuously differentiable functions defined on $[a,b]$ with values in $X$ with the supremum norm and we write $C([a,b];X)$ for $C^0([a,b];X)$.

The main result of this paper is stated as follows.

**Theorem 1.1** *For any initial condition* $u_0 = u_0(x,s) \in C([-\tau,0],\dot{H}^{\frac{\mu}{2}}_{\mathrm{per}}(\Omega))$, $\mu \geq 2, s \geq 0$, *and suppose* $M$ *and* $L$ *satisfy the assumptions* (1.7), (1.8). *Then problem* (1.1)-(1.3) *has a unique global mild solution* $u$ *on* $[-\tau,\infty)$ *with*

$$u \in C\big([-\tau,+\infty),\dot{H}^{\frac{\mu}{2}}_{\mathrm{per}}(\Omega)\big).$$

**Theorem 1.2** *Under the assumption of Theorem* 1.1, *if* $\mu = s$, *there are* $\tau_0,\omega,K > 0$ *such that, for* $\tau < \tau_0$, *the solution of* (1.1)-(1.3) *satisfies*

$$\big\|M^{1/2}u(t)\big\|^2 \leq \frac{K^2}{4}\exp\{-\omega t\}, \quad t \geq 0. \tag{1.9}$$

This paper is organized as follows. In next section, we prove the existence of the solution. Furthermore, we show that the solution is exponentially decay by using the Liapunov function method.

## 2 Exponential decay estimates

Firstly, we briefly show that problem (1.1)-(1.3) is well posed. To conveniences, we denote $\|u\| \leq \lambda_1\|L^{\frac{1}{2}}u\|$, $\|u_x\| \leq \lambda_2\|L^{\frac{1}{2}}u\|$, $\|u\| \leq \lambda_3\|M^{\frac{1}{2}}u\|$, $\|u_x\| \leq \lambda_4\|M^{\frac{1}{2}}u\|$, $\|M^{\frac{1}{2}}u\| \leq \lambda_5\|L^{\frac{1}{2}}u\|$.

*Proof of Theorem* 1.1 By the standard methods as in [9], it is easy to prove that for every initial value $u_0 = u_0(x,s) \in C([-\tau,0],\dot{H}^{\frac{\mu}{2}}_{\mathrm{per}}(\Omega))$, there exists a $T = T(u_0) > 0$ such that problem (1.1)-(1.3) has a unique mild solution $u$ on $[-\tau,T]$ with

$$u(x,t) \in C\big([-\tau,T],\dot{H}^{\frac{\mu}{2}}_{\mathrm{per}}(\Omega)\big).$$

Furthermore, for any $\tau > 0$, the solution of (1.1)-(1.3) does not blow up in finite time. Indeed, multiplying (1.1) by $u$, integrating by parts, we obtain, for $0 \leq t \leq \tau$,

$$\begin{aligned}
&\frac{d}{dt}\big\|M^{1/2}u(t)\big\|^2 + 2\alpha\big\|L^{1/2}u(t)\big\|^2 \\
&\quad = -2\int_0^1 u(t)u(t-\tau)u_x(t)\,dx \\
&\quad \leq 2\|u_0\|_{C([-\tau,0],\dot{H}^1_{\mathrm{per}}(\Omega))}\int_0^1 \big|u(t)\big|\big|u_x(t)\big|\,dx \\
&\quad \leq 2\|u_0\|_{C([-\tau,0],\dot{H}^1_{\mathrm{per}}(\Omega))}\big\|u(t)\big\|\,\big\|u_x(t)\big\| \\
&\quad \leq 2\lambda_1\big\|L^{1/2}u(t)\big\|\,\|u_0\|_{C([-\tau,0],\dot{H}^1_{\mathrm{per}}(\Omega))}\big\|u_x(t)\big\|.
\end{aligned}$$

By the Young inequality, we get

$$\begin{aligned}\frac{d}{dt}\left\|M^{1/2}u(t)\right\|^2 &\le \frac{\lambda_1^2}{2\alpha}\|u_0\|^2_{C([-\tau,0],\dot{H}_{\mathrm{per}}^{\frac{\mu}{2}}(\Omega))}\left\|u_x(t)\right\|^2 \\ &\le \frac{\lambda_1^2\lambda_4^2}{2\alpha}\|u_0\|^2_{C([-\tau,0],\dot{H}_{\mathrm{per}}^{\frac{\mu}{2}}(\Omega))}\left\|M^{1/2}u(t)\right\|^2,\end{aligned}$$

which implies that

$$\left\|M^{1/2}u(t)\right\|^2 \le K_1\left(\|u_0\|_{C([-\tau,0],\dot{H}_{\mathrm{per}}^{\frac{\mu}{2}}(\Omega))}\right),$$

where $K_1(\|u_0\|_{C([-\tau,0],\dot{H}_{\mathrm{per}}^{\frac{\mu}{2}}(\Omega))})$ is a positive constant depending on $\|u_0\|_{C([-\tau,0],\dot{H}_{\mathrm{per}}^{\frac{\mu}{2}}(\Omega))}$. Repeating the above procedure, we can prove that, for $n\tau \le t \le (n+1)\tau$ $(n = 1, 2, \ldots)$,

$$\left\|M^{1/2}u(t)\right\|^2 \le K_1\left(n, \|u_0\|_{C([-\tau,0],\dot{H}_{\mathrm{per}}^{\frac{\mu}{2}}(\Omega))}\right).$$

In summary, we have proved Theorem 1.1. □

Secondly, we prove that our main result about the exponential stability. To this end, we introduce the following notations. For a given initial condition $u_0 = u_0(x,s) \in C([-\tau,0], \dot{H}_{\mathrm{per}}^{\frac{\mu}{2}}(\Omega))$, denote

$$K = \sup_{-\tau\le s\le 0}\left\|M^{1/2}u_0(s)\right\| + \sqrt{8\left\|M^{1/2}u_0(0)\right\|^2}. \tag{2.1}$$

Set

$$\sigma = \sup\left\{\delta > 0 : \left\|M^{1/2}u_0(0)\right\|^2 \le \frac{K^2}{4}, 0 \le \tau \le \delta\right\}, \tag{2.2}$$

and let $\tau_0$ be small enough, such that for any $\tau$, $0 \le \tau < \tau_0 \le \sigma$,

$$\omega = \frac{2\alpha}{\lambda_5^2} - \frac{2\lambda_2^2 K}{\lambda_5^2}\sqrt{\frac{\tau[\alpha + \frac{\lambda_4^4 K^2}{\varepsilon}\tau]}{2\lambda_3^{-2} - \varepsilon}} > 0. \tag{2.3}$$

*Proof of Theorem* 1.2 Let

$$T_0 = \sup\left\{\delta : \left\|M^{1/2}u(t)\right\|^2 \le K^2, 0 \le t \le \delta\right\}. \tag{2.4}$$

Since

$$\left\|M^{1/2}u(0)\right\|^2 \le K^2,$$

and $\|M^{1/2}u(t)\|$ is continuous, we have $T_0 > 0$. We shall prove that $T_0 = +\infty$. For this, we argue by *contradiction*. If $T_0 < +\infty$, then we have

$$\left\|M^{1/2}u(t)\right\|^2 \le K^2, \quad \forall -\tau \le t \le T_0 \tag{2.5}$$

and

$$\|M^{1/2}u(T_0)\|^2 = K^2. \tag{2.6}$$

Multiplying (1.1) by $u$, then integrating on $\Omega$ with respect to $x$, we obtain

$$\frac{d}{dt}\|M^{1/2}u(t)\|^2 + 2\alpha\|L^{1/2}u(t)\|^2 = -2\int_0^1 u(t)u(t-\tau)u_x(t)\,dx.$$

Since $\int_0^1 u^2(t)u_x(t)\,dx = 0$, we have

$$\frac{d}{dt}\|M^{1/2}u(t)\|^2 + 2\alpha\|L^{1/2}u(t)\|^2 = \Phi, \tag{2.7}$$

where

$$\Phi = -2\int_0^1 u(t)\big[u(t-\tau) - u(t)\big]u_x(t)\,dx.$$

We now majorize $\Phi$ in the right hand side of (2.7). Firstly, since

$$\big|u(x,t)\big| \le \big\|u_x(x,t)\big\|, \quad 0 \le x \le 1,$$

we have

$$\begin{aligned}
\Phi &\le 2\int_0^1 \big|u(t-\tau) - u(t)\big|\big|u(t)\big|\big|u_x(t)\big|\,dx \\
&\le 2\int_0^1 \big|u(t-\tau) - u(t)\big|\big\|u_x(t)\big\|\big|u_x(t)\big|\,dx \\
&\le 2\big\|u_x(t)\big\|\int_0^1 \big|u(t-\tau) - u(t)\big|\big|u_x(t)\big|\,dx \\
&\le 2\big\|u_x(t)\big\|\left(\int_0^1 \big|u(t-\tau) - u(t)\big|^2\,dx\right)^{\frac{1}{2}}\left(\int_0^1 \big|u_x(t)\big|^2\,dx\right)^{\frac{1}{2}} \\
&= 2\big\|u_x(t)\big\|^2\left(\int_0^1 \big|u(t-\tau) - u(t)\big|^2\,dx\right)^{\frac{1}{2}} \\
&= 2\big\|u_x(t)\big\|^2\left(\int_0^1 \left|\int_{t-\tau}^t u_s(s)\,ds\right|^2\,dx\right)^{\frac{1}{2}} \\
&\le 2\big\|u_x(t)\big\|^2\left(\int_0^1 \left(\int_{t-\tau}^t u_s^2(s)\,ds\int_{t-\tau}^t ds\right)\,dx\right)^{\frac{1}{2}} \\
&= 2\sqrt{\tau}\big\|u_x(t)\big\|^2\left(\int_0^1\int_{t-\tau}^t u_s^2(s)\,ds\,dx\right)^{\frac{1}{2}}.
\end{aligned}$$

Let

$$\Psi = \left(\int_0^1\int_{t-\tau}^t u_s^2(s)\,ds\,dx\right)^{\frac{1}{2}},$$

we have

$$\Phi \le 2\sqrt{\tau}\left\|u_x(t)\right\|^2\Psi \tag{2.8}$$

and

$$\Psi^2 \le \lambda_3^2\int_{t-\tau}^{t}\left\|M^{1/2}u_s(s)\right\|^2 ds. \tag{2.9}$$

We now want to estimate $\Psi$. To this end, multiplying (1.1) by $u_t$, then integrating on $\Omega$ with respect to $x$, we obtain

$$\left\|M^{1/2}u_t(t)\right\|^2 + \frac{\alpha}{2}\frac{d}{dt}\left\|L^{1/2}u(t)\right\|^2 + \int_0^1 u_t(t)u(t-\tau)u_x(t)\,dx = 0.$$

Integrating on $[t-\tau,t]$, $0 \le t \le T_0$, we obtain

$$2\int_{t-\tau}^{t}\left\|M^{1/2}u_s(s)\right\|^2 ds + \alpha\left\|L^{1/2}u(t)\right\|^2 - \alpha\left\|L^{1/2}u(t-\tau)\right\|^2 + H = 0,$$

where

$$H = 2\int_{t-\tau}^{t}\int_0^1 u_s(s)u(s-\tau)u_x(s)\,dx\,ds,$$

which implies that

$$2\int_{t-\tau}^{t}\left\|M^{1/2}u_s(s)\right\|^2 ds \le \alpha\left\|L^{1/2}u(t-\tau)\right\|^2 - H. \tag{2.10}$$

We now majorize $H$:

$$\begin{aligned}
H &\le 2\int_{t-\tau}^{t}\int_0^1 \left|u_s(s)\right|\left|u(s-\tau)\right|\left|u_x(s)\right| dx\,ds \\
&\le 2\int_{t-\tau}^{t}\int_0^1 \left\|u_x(s-\tau)\right\|\left(\left|u_s(s)\right|\left|u_x(s)\right|\right) dx\,ds \\
&\le 2\lambda_4 K\int_{t-\tau}^{t}\int_0^1 \left|u_s(s)\right|\left|u_x(s)\right| dx\,ds \\
&\le 2\lambda_4 K\left(\int_{t-\tau}^{t}\int_0^1 u_s^2(s)\,dx\,ds\right)^{\frac{1}{2}}\left(\int_{t-\tau}^{t}\int_0^1 u_x^2(s)\,dx\,ds\right)^{\frac{1}{2}} \\
&\le \varepsilon\Psi^2 + \frac{\lambda_4^2K^2}{\varepsilon}\int_{t-\tau}^{t}\int_0^1 u_x^2(s)\,dx\,ds \\
&\le \varepsilon\Psi^2 + \frac{\lambda_4^2K^2}{\varepsilon}\int_{t-\tau}^{t} K^2\,ds \\
&\le \varepsilon\Psi^2 + \frac{\lambda_4^2K^4}{\varepsilon}\tau,
\end{aligned} \tag{2.11}$$

with $0 < \varepsilon < 2\lambda_3^{-2}$. Thus by (2.10)-(2.11), we have

$$2\int_{t-\tau}^{t}\left\|M^{1/2}u_s(s)\right\|^2 ds \le \alpha K^2 + \varepsilon\Psi^2 + \frac{\lambda_4^2K^4}{\varepsilon}\tau, \tag{2.12}$$

furthermore, by (2.9), we get

$$\Psi \le K\sqrt{\frac{[\alpha + \frac{\lambda_4^4 K^2}{\varepsilon}\tau]}{2\lambda_3^{-2} - \varepsilon}}, \quad \forall 0 \le t \le T_0. \tag{2.13}$$

Then (2.8) and (2.13) implies that

$$\begin{aligned}\Phi &\le 2\lambda_2^2 K\sqrt{\frac{\tau[\alpha + \frac{\lambda_4^4 K^2}{\varepsilon}\tau]}{2\lambda_3^{-2} - \varepsilon}}\|u_x(t)\|^2 \\ &\le 2\lambda_2^2 K\sqrt{\frac{\tau[\alpha + \frac{\lambda_4^4 K^2}{\varepsilon}\tau]}{2\lambda_3^{-2} - \varepsilon}}\|L^{1/2}u(t)\|^2.\end{aligned} \tag{2.14}$$

Thus, by (2.7) and (2.14), we obtain

$$\frac{d}{dt}\|M^{1/2}u(t)\|^2 + \|L^{1/2}u(t)\|^2\left(2\alpha - 2\lambda_2^2 K\sqrt{\frac{\tau[\alpha + \frac{\lambda_4^4 K^2}{\varepsilon}\tau]}{2\lambda_3^{-2} - \varepsilon}}\right) \le 0,$$

furthermore,

$$\frac{d}{dt}\|M^{1/2}u(t)\|^2 + \omega\|M^{1/2}u(t)\|^2 \le 0, \tag{2.15}$$

where $\omega$ is defined by (2.3). Solving the above inequality gives

$$\|M^{1/2}u(t)\|^2 \le \|M^{1/2}u_0(x,0)\|^2 e^{-\omega t} \le \frac{K^2}{4}e^{-\omega t}, \quad 0 \le t \le T_0. \tag{2.16}$$

Hence

$$\|M^{1/2}u(t)(T_0)\|^2 \le K^2 e^{-\omega T_0}, \tag{2.17}$$

which is in contradiction with (2.6). Therefore, we have proved that $T_0 = +\infty$ and then (1.9) follows from (2.16). Thus we have completed the proof of Theorem 1.2. □

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
FW gave access to information, PZ formulated the outline and ZC wrote the essay. All authors read and approved the final manuscript.

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