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Nonplanar periodic solutions for spatial restricted 3-body and 4-body problems

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Abstract

In this paper, by using variational methods, we study the existence of nonplanar periodic solutions for the following spatial restricted 3-body and 4-body problems: for $N = 2$ or 3, N mass points with positive masses $m_1, \ldots, m_N$ move in a central configuration (for $N = 2$, two bodies are in a Euler configuration; for $N = 3$, three bodies are in a Lagrange configuration), and they move in the plane of N circular obits; the $N + 1$th mass point, called the zero mass point, moves on the perpendicular axis passing through the center of the masses.

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1 Introduction and main results

In this paper, we study the spatial circular restricted 3-body and 4-body problems. For $N = 2$ or 3, suppose N mass points with positive masses $m_1, \ldots, m_N$ move in the plane of their circular orbits $q_1(t), \ldots, q_N(t)$, the radius $r_1, \ldots, r_N$ of orbits are all positive and the center of masses is at the origin; suppose the $N+1$th mass point, called the zero mass point, does not influence the motion of the given N mass points, and moves on the vertical axis of the plane for the first N mass points, here the vertical axis passes through the center of masses.

It is known that $q_1(t), \ldots, q_N(t)$ ($N = 2$ or 3) satisfy the Newtonian equations:

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}, \quad i = 1, \ldots, N, \tag{1.1}$$

where

$$U = \sum_{1 \le i < j \le N} \frac{m_i m_j}{|q_i - q_j|}. \tag{1.2}$$

The orbit $q(t) = (0, 0, z(t)) \in R^3$ for zero mass point is governed by the gravitational forces of $m_1, \ldots, m_N$ ($N = 2$ or 3) and therefore it satisfies the following equation:

$$\ddot{q} = \sum_{i=1}^{N} \frac{m_i (q_i - q)}{|q_i - q|^3}, \quad N = 2 \text{ or } 3. \tag{1.3}$$

For $N \geq 2$, there are many papers concerned with the restricted N-body problem; see [1–6] and the references therein. In [7], Sitnikov considered the following model: two mass points of equal mass $m_1 = m_2 = \frac{1}{2}$ move in the plane of their elliptic orbits and the center of the masses is at rest, the third mass point which does not influence the motion of the first two moves on the line perpendicular to the plane containing the first two mass points and goes through the center of mass, and he used geometrical methods to prove the existence of the oscillatory parabolic orbit of

$$\ddot{z}(t) = \frac{-z(t)}{(|z(t)|^2 + |r(t)|^2)^{3/2}}, \tag{1.4}$$

where $r(t) = r(t + 2\pi) > 0$ is the distance from the center of mass to one of the first two mass points. McGehee [4] used stable and unstable manifolds to study the homoclinic orbits (parabolic orbits) of (1.4). In [5], Mathlouthi studied the periodic solutions for the spatial circular restricted 3-body problems by minimax variational methods. Li *et al.* [1] used Jacobi's necessary conditions for the variational minimizers to study the existence of nonplanar periodic solutions for spatial restricted $N + 1$-body problems with a zero mass moving on the vertical axis of the plane for N equal masses.

In this article, we study the spatial circular restricted 3-body and 4-body problems with a zero mass point moving on the perpendicular axis of the circular orbits plane for given masses $m_1, \ldots, m_N$ ($N = 2$ or 3) in their respective central configuration.

Define

$$W^{1,2}(R/TZ, R) = \{u(t) | u(t), u'(t) \in L^2(R, R), u(t + T) = u(t)\}.$$

The inner product and the norm of $W^{1,2}(R/TZ, R)$ are

$$\langle u, v\rangle = \int_0^T (uv + u' \cdot v')\, dt, \tag{1.5}$$

$$\|u\| = \left[\int_0^T |u|^2\, dt\right]^{\frac{1}{2}} + \left[\int_0^T |u'|^2\, dt\right]^{\frac{1}{2}}. \tag{1.6}$$

We consider the Lagrangian functional of (1.3)

$$\begin{aligned} f(q) &= \int_0^T \left[\frac{1}{2}|\dot{q}|^2 + \sum_{i=1}^N \frac{m_i}{|q - q_i|}\right] dt \\ &= \int_0^T \left[\frac{1}{2}|z'|^2 + \frac{m_1}{\sqrt{r_1^2 + z^2}} + \cdots + \frac{m_N}{\sqrt{r_N^2 + z^2}}\right] dt \triangleq f(z), \quad N = 2 \text{ or } 3 \end{aligned} \tag{1.7}$$

on Λ_j, $j = 1, 2$, where

$$\Lambda_1 = \left\{q(t) = (0, 0, z(t)) \,\middle|\, z(t) \in W^{1,2}(R/TZ, R), z\left(t + \frac{T}{2}\right) = -z(t)\right\}$$

and

$$\Lambda_2 = \{q(t) = (0, 0, z(t)) | z(t) \in W^{1,2}(R/TZ, R), z(-t) = -z(t)\}.$$

Our main results are the following.

Theorem 1.1 *For $N = 2$, let the mass points of m_1, m_2 be in a Euler configuration, then the minimizer of $f(q)$ on the closure $\bar{\Lambda}_i$ of Λ_i ($i = 1,2$) exists and it is a nonplanar and noncollision periodic solution of* (1.3).

Theorem 1.2 *For $N = 3$, let the mass points of m_1, m_2, m_3 be in a Lagrange configuration, then the minimizer of $f(q)$ on the closure $\bar{\Lambda}_i$ of Λ_i ($i = 1,2$) exists and it is a nonplanar and noncollision periodic solution of* (1.3).

Remark 1.1 Obviously, the nonplanar periodic solutions we got in the above two theorems are collisionless.

2 Preliminaries

In this section, we will list some basic lemmas and inequality for proving our Theorems 1.1 and 1.2.

Lemma 2.1 (Poincaré-Wirtinger inequality [8]) *Let $q \in W^{1,2}(R/TZ, R^K)$ and $\int_0^T q(t)\,dt = 0$, then*

$$\int_0^T |q(t)|^2\,dt \le \frac{T^2}{4\pi^2}\int_0^T |\dot{q}(t)|^2\,dt. \tag{2.1}$$

Lemma 2.2 (Tonelli [3]) *Let X be a reflexive Banach space, S be a weakly closed subset of X, $f : S \to R \cup +\infty$. If $f \not\equiv +\infty$ is weakly lower semi-continuous and coercive ($f(x) \to +\infty$ as $\|x\| \to +\infty$), then f attains its infimum on S.*

Lemma 2.3 (Palais' symmetry principle [9]) *Let σ be an orthogonal representation of a finite or compact group G, H be a real Hilbert space, $f : H \to R$ satisfies $f(\sigma \cdot x) = f(x)$, $\forall\sigma \in G$, $\forall x \in H$.*

Set $F = \{x \in H \mid \sigma \cdot x = x, \forall\sigma \in G\}$. Then the critical point of f in F is also a critical point of f in H.

Remark 2.1 By Palais' symmetry principle, we know that the critical point of $f(q)$ in $\bar{\Lambda}_i = \Lambda_i$ ($i = 1,2$) is a periodic solution of Newtonian equation (1.3).

Lemma 2.4 *$f(q)$ in* (1.7) *attains its infimum on $\bar{\Lambda}_i = \Lambda_i$ ($i = 1,2$).*

Proof By using Lemma 2.1, for $\forall z \in \Lambda_i$, $i = 1,2$, the equivalent norm of (1.6) in Λ_i ($i = 1,2$) is

$$\|z\| \cong \left[\int_0^T |z'|^2\,dt\right]^{\frac{1}{2}}. \tag{2.2}$$

Hence by the definitions of $f(q)$, it is easy to see that f is C^1 and coercive on Λ_i ($i = 1,2$). In order to get Lemma 2.4, we only need to prove that f is weakly lower semi-continuous on Λ_i ($i = 1,2$). In fact, for $\forall z_n \in \Lambda_i$, if $z_n \rightharpoonup z$ weakly, by compact embedding theorem, we have uniform convergence:

$$\max_{0\le t\le T}\left|z_n(t)-z(t)\right|\to 0,\quad n\to\infty,\tag{2.3}$$

which implies

$$\begin{aligned}&\int_0^T \frac{m_1}{\sqrt{r_1^2+z_n^2}}+\cdots+\frac{m_N}{\sqrt{r_N^2+z_n^2}}\,dt\\&\quad\to\int_0^T \frac{m_1}{\sqrt{r_1^2+z^2}}+\cdots+\frac{m_N}{\sqrt{r_N^2+z^2}}\,dt,\quad N=2\text{ or }3.\end{aligned}\tag{2.4}$$

It is well known that the norm and its square are weakly lower semi-continuous. Therefore, combined with (2.4), we obtain

$$\liminf_{n\to\infty} f(z_n)\ge f(z),$$

that is, f is weakly lower semi-continuous on Λ_i ($i=1,2$). By Lemma 2.2, we can see that $f(q)$ in (1.7) attains its infimum on $\bar{\Lambda}_i=\Lambda_i$ ($i=1,2$). □

3 Proof of Theorem 1.1

In this section, we consider the spatial circular restricted 3-body problem with a zero mass point moving on the vertical axis of the moving plane for two mass points with arbitrary given positive masses m_1, m_2 in a Euler configuration. Suppose the planar circular orbits are

$$q_1(t)=\left(r_1\cos\frac{2\pi}{T}t,r_1\sin\frac{2\pi}{T}t,0\right),\tag{3.1}$$

$$q_2(t)=\left(-r_2\cos\frac{2\pi}{T}t,-r_2\sin\frac{2\pi}{T}t,0\right);\tag{3.2}$$

here the radii r_1, r_2 are positive constants depending on m_i ($i=1,2$) and T (see Lemma 3.1). We also assume that

$$m_1q_1(t)+m_2q_2(t)=0.\tag{3.3}$$

We consider the Lagrangian functional of (1.3)

$$\begin{aligned}f(q)&=\int_0^T\left[\frac{1}{2}|\dot{q}|^2+\frac{m_1}{|q-q_1|}+\frac{m_2}{|q-q_2|}\right]dt\\&=\int_0^T\left[\frac{1}{2}\left|z'\right|^2+\frac{m_1}{\sqrt{r_1^2+z^2}}+\frac{m_2}{\sqrt{r_2^2+z^2}}\right]dt\triangleq f(z)\end{aligned}\tag{3.4}$$

on Λ_i, $i=1,2$.

Lemma 3.1 *The radii r_1, r_2 of the planar circular orbits for the masses m_1, m_2 are*

$$r_1=\left(\frac{T}{2\pi(m_1+m_2)}\right)^{\frac{2}{3}}m_2,\qquad r_2=\left(\frac{T}{2\pi(m_1+m_2)}\right)^{\frac{2}{3}}m_1.$$

Proof Substituting (3.1), (3.2) into (3.3), it is easy to get

$$r_2 = \frac{m_1}{m_2} r_1. \tag{3.5}$$

It follows from (1.1) and (1.2) that

$$\ddot{q}_1 = m_2 \frac{q_2 - q_1}{|q_2 - q_1|^3}. \tag{3.6}$$

Then by (3.1), (3.2), and (3.5), we have

$$-\frac{4\pi^2}{T^2} q_1 = m_2 \frac{(-\frac{m_1}{m_2} - 1) q_1}{r_1^3 | - \frac{m_1}{m_2} - 1|^3}, \tag{3.7}$$

which implies

$$r_1 = \left(\frac{T}{2\pi (m_1 + m_2)} \right)^{\frac{2}{3}} m_2. \tag{3.8}$$

Hence by (3.5), we obtain

$$r_2 = \left(\frac{T}{2\pi (m_1 + m_2)} \right)^{\frac{2}{3}} m_1. \tag{3.9}$$

□

Proof of Theorem 1.1 Clearly, $q(t) = (0, 0, 0)$ is a critical point of $f(q)$ on $\bar{\Lambda}_i = \Lambda_i$ $(i = 1, 2)$. The second variation of (3.4) in the sphere neighborhood of $q(t) = (0, 0, 0)$ (coordinate origin O) is given by

$$\delta^2 f(O, \varphi) = \int_0^T \left(\varphi_3'^2 - \left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} \right) \varphi_3^2 \right) dt, \quad \forall \varphi = (0, 0, \varphi_3) \in \Lambda_i, i = 1, 2. \tag{3.10}$$

Let

$$\tilde{\varphi}(t) = \left(0, 0, \sin \frac{2\pi}{T} t \right). \tag{3.11}$$

Obviously, $\tilde{\varphi} \in \Lambda_i$ $(i = 1, 2)$ and

$$\dot{\tilde{\varphi}}(t) = \left(0, 0, \frac{2\pi}{T} \cos \frac{2\pi}{T} t \right). \tag{3.12}$$

Since

$$m_1^6 + m_2^6 \geq 2\sqrt{m_1^6 \cdot m_2^6} = 2 m_1^3 m_2^3 > m_1^3 m_2^3, \tag{3.13}$$

it is easy to see that

$$\begin{aligned} \left(m_1^4 + m_2^4 \right) (m_1 + m_2)^2 &> m_1^6 + m_2^6 \\ &> m_1^3 m_2^3, \end{aligned} \tag{3.14}$$

which implies

$$\frac{\sqrt{m_1^3 m_2^3}}{\sqrt{m_1^4+m_2^4}(m_1+m_2)} < 1, \tag{3.15}$$

that is

$$\sqrt{\frac{m_1^4+m_2^4}{m_1^3 m_2^3}}(m_1+m_2) > 1. \tag{3.16}$$

It follows from (3.8), (3.9), and (3.16) that

$$\begin{aligned}\sqrt{\frac{m_1}{r_1^3}+\frac{m_2}{r_2^3}} &= \sqrt{\frac{m_1^4+m_2^4}{m_1^3 m_2^3}}(m_1+m_2)\cdot\frac{2\pi}{T} \\ &> \frac{2\pi}{T},\end{aligned} \tag{3.17}$$

that is

$$\frac{m_1}{r_1^3}+\frac{m_2}{r_2^3} > \frac{4\pi^2}{T^2}. \tag{3.18}$$

By (3.18), we have

$$\begin{aligned}\delta^2 f(O,\tilde{\varphi}) &= \int_0^T \left[\frac{4\pi^2}{T^2}\cos^2\frac{2\pi}{T}t - \left(\frac{m_1}{r_1^3}+\frac{m_2}{r_2^3}\right)\sin^2\frac{2\pi}{T}t\right]dt \\ &= \frac{2\pi^2}{T} - \left(\frac{m_1}{r_1^3}+\frac{m_2}{r_2^3}\right)\cdot\frac{T}{2} \\ &= \left[\frac{4\pi^2}{T^2} - \left(\frac{m_1}{r_1^3}+\frac{m_2}{r_2^3}\right)\right]\cdot\frac{T}{2} \\ &< 0,\end{aligned} \tag{3.19}$$

which implies $q(t) = (0,0,0)$ is not a local minimum for $f(q)$ on Λ_i ($i = 1,2$). Hence the minimizers of $f(q)$ on Λ_i ($i = 1,2$) are not always at the center of the masses, they must oscillate periodically on the vertical axis, that is, the minimizers are not always co-planar, therefore, we get nonplanar periodic solutions.

Combined with Lemma 2.4, the proof is completed. □

4 Proof of Theorem 1.2

In this section, we consider the spatial circular restricted 4-body problem with a zero mass point moving on the vertical axis of the moving plane for three mass points with arbitrary positive masses m_1, m_2, m_3 in a Lagrange configuration. Suppose there exist $\theta_1,\theta_2,\theta_3 \in [0,2\pi)$ such that the planar circular orbits are

$$q_1(t) = \left(r_1\cos\left(\frac{2\pi}{T}t+\theta_1\right), r_1\sin\left(\frac{2\pi}{T}t+\theta_1\right), 0\right), \tag{4.1}$$

$$q_2(t) = \left(r_2\cos\left(\frac{2\pi}{T}t+\theta_2\right), r_2\sin\left(\frac{2\pi}{T}t+\theta_2\right), 0\right), \tag{4.2}$$

$$q_3(t) = \left(r_3\cos\left(\frac{2\pi}{T}t + \theta_3\right), r_3\sin\left(\frac{2\pi}{T}t + \theta_3\right), 0\right), \tag{4.3}$$

here the radius r_1, r_2, r_3 are positive constants depending on m_i ($i = 1,2,3$) and T (see Lemma 4.2). We also assume that

$$m_1q_1(t) + m_2q_2(t) + m_3q_3(t) = 0 \tag{4.4}$$

and

$$|q_i - q_j| = l, \quad 1 \le i \ne j \le 3, \tag{4.5}$$

where the constant $l > 0$ depends on m_i ($i = 1,2,3$) and T (see Lemma 4.1). We consider the Lagrangian functional of equation (1.3)

$$\begin{aligned} f(q) &= \int_0^T \left[\frac{1}{2}|\dot{q}|^2 + \frac{m_1}{|q - q_1|} + \frac{m_2}{|q - q_2|} + \frac{m_3}{|q - q_3|}\right] dt \\ &= \int_0^T \left[\frac{1}{2}|z'|^2 + \frac{m_1}{\sqrt{r_1^2 + z^2}} + \frac{m_2}{\sqrt{r_2^2 + z^2}} + \frac{m_3}{\sqrt{r_3^2 + z^2}}\right] dt \triangleq f(z) \end{aligned} \tag{4.6}$$

on Λ_i, $i = 1,2$.

In order to get Theorem 1.2, we firstly prove Lemmas 4.1 and 4.2 as follows.

Lemma 4.1 *Let* $M = m_1 + m_2 + m_3$, *we have* $l = \sqrt[3]{\frac{MT^2}{4\pi^2}}$.

Proof It follows from (1.1) and (1.2) that

$$\ddot{q}_1 = m_2\frac{q_2 - q_1}{|q_2 - q_1|^3} + m_3\frac{q_3 - q_1}{|q_3 - q_1|^3}. \tag{4.7}$$

Then by (4.1), (4.4), and (4.5), we obtain

$$\begin{aligned} -\frac{4\pi^2}{T^2}q_1 &= \frac{1}{l^3}(m_2q_2 + m_3q_3 - m_2q_1 - m_3q_1) \\ &= \frac{1}{l^3}(-m_1q_1 - m_2q_1 - m_3q_1), \end{aligned} \tag{4.8}$$

which implies

$$l^3 = \frac{MT^2}{4\pi^2}, \tag{4.9}$$

that is,

$$l = \sqrt[3]{\frac{MT^2}{4\pi^2}}. \tag{4.10}$$

□

Lemma 4.2 *The radius* r_1, r_2, r_3 *of the planar circular orbits for the masses* m_1, m_2, m_3 *are*

$$r_1 = \frac{\sqrt{m_2^2 + m_2m_3 + m_3^2}}{M}l,$$

$$r_2 = \frac{\sqrt{m_1^2 + m_1 m_3 + m_3^2}}{M} l,$$
$$r_3 = \frac{\sqrt{m_1^2 + m_1 m_2 + m_2^2}}{M} l.$$

Proof Choose the geometrical center of the initial configuration $(q_1(0), q_2(0), q_3(0))$ as the origin of the coordinate (x, y, z). Without loss of generality, by (4.5), we may suppose the location coordinates of $q_1(0)$, $q_2(0)$, $q_3(0)$ are $A_1(\frac{\sqrt{3}l}{3}, 0, 0)$, $A_2(-\frac{\sqrt{3}l}{6}, \frac{l}{2}, 0)$, $A_3(-\frac{\sqrt{3}l}{6}, -\frac{l}{2}, 0)$. Then we can get the coordinate of the center of masses m_1, m_2, m_3 is $C(\frac{\frac{\sqrt{3}}{3}m_1 l - \frac{\sqrt{3}}{6}m_2 l - \frac{\sqrt{3}}{6}m_3 l}{M}, \frac{\frac{m_2}{2}l - \frac{m_3}{2}l}{M}, 0)$. To make sure assumption (4.4) holds, we introduce the new coordinates

$$\begin{cases} X = x - \frac{\frac{\sqrt{3}}{3}m_1 l - \frac{\sqrt{3}}{6}m_2 l - \frac{\sqrt{3}}{6}m_3 l}{M}, \\ Y = y - \frac{\frac{m_2}{2}l - \frac{m_3}{2}l}{M}, \\ Z = z. \end{cases}$$

Hence in the new coordinates (X, Y, Z), the location coordinates of $q_1(0)$, $q_2(0)$, $q_3(0)$ are $A_1(\frac{\frac{\sqrt{3}}{2}m_2 l + \frac{\sqrt{3}}{2}m_3 l}{M}, \frac{-\frac{m_2}{2}l + \frac{m_3}{2}l}{M}, 0)$, $A_2(-\frac{\frac{\sqrt{3}}{2}m_1 l}{M}, \frac{\frac{m_1}{2}l + m_3 l}{M}, 0)$, $A_3(-\frac{\frac{\sqrt{3}}{2}m_1 l}{M}, -\frac{\frac{m_1}{2}l + m_2 l}{M}, 0)$ and the center of the masses m_1, m_2, m_3 are at the origin $O(0, 0, 0)$. Then compared with (4.1)-(4.3), we have

$$r_1 = |A_1 O| = \frac{\sqrt{m_2^2 + m_2 m_3 + m_3^2}}{M} l, \tag{4.11}$$
$$r_2 = |A_2 O| = \frac{\sqrt{m_1^2 + m_1 m_3 + m_3^2}}{M} l, \tag{4.12}$$
$$r_3 = |A_3 O| = \frac{\sqrt{m_1^2 + m_1 m_2 + m_2^2}}{M} l \tag{4.13}$$

and

$$\sin\theta_1 = \frac{-m_2 + m_3}{2\sqrt{m_2^2 + m_2 m_3 + m_3^2}}, \qquad \cos\theta_1 = \frac{\sqrt{3}(m_2 + m_3)}{2\sqrt{m_2^2 + m_2 m_3 + m_3^2}}, \tag{4.14}$$
$$\sin\theta_2 = \frac{m_1 + 2m_3}{2\sqrt{m_1^2 + m_1 m_3 + m_3^2}}, \qquad \cos\theta_2 = -\frac{\sqrt{3}m_1}{2\sqrt{m_1^2 + m_1 m_3 + m_3^2}}, \tag{4.15}$$
$$\sin\theta_3 = -\frac{m_1 + 2m_2}{2\sqrt{m_1^2 + m_1 m_2 + m_2^2}}, \qquad \cos\theta_3 = -\frac{\sqrt{3}m_1}{2\sqrt{m_1^2 + m_1 m_2 + m_2^2}}. \tag{4.16}$$

□

Proof of Theorem 1.2 Clearly, $q(t) = (0, 0, 0)$ is a critical point of $f(q)$ on $\bar{\Lambda}_i = \Lambda_i$ $(i = 1, 2)$. The second variation of (4.6) in the sphere neighborhood of $q(t) = (0, 0, 0)$ (coordinate origin O) is given by

$$\delta^2 f(O, \varphi) = \int_0^T \left(\varphi_3'^2 - \left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} + \frac{m_3}{r_3^3}\right)\varphi_3^2\right) dt, \quad \forall \varphi = (0, 0, \varphi_3) \in \Lambda_i, i = 1, 2. \tag{4.17}$$

Let

$$\tilde{\varphi}(t) = \left(0, 0, \sin\frac{2\pi}{T}t\right) \tag{4.18}$$

and

$$\begin{aligned} A &= \frac{\sqrt{m_2^2 + m_2m_3 + m_3^2}}{M}, \\ B &= \frac{\sqrt{m_1^2 + m_1m_3 + m_3^2}}{M}, \\ C &= \frac{\sqrt{m_1^2 + m_1m_2 + m_2^2}}{M}. \end{aligned}$$

Obviously, $\tilde{\varphi} \in \Lambda_i$ $(i = 1, 2)$ and

$$\dot{\tilde{\varphi}}(t) = \left(0, 0, \frac{2\pi}{T}\cos\frac{2\pi}{T}t\right). \tag{4.19}$$

It is easy to check that

$$\begin{aligned} M^2 &> m_1^2 + m_1m_2 + m_2^2, \\ M^2 &> m_1^2 + m_1m_3 + m_3^2, \\ M^2 &> m_2^2 + m_2m_3 + m_3^2, \end{aligned} \tag{4.20}$$

which implies

$$\frac{m_1}{A^3} + \frac{m_2}{B^3} + \frac{m_3}{C^3} > m_1 + m_2 + m_3 = M. \tag{4.21}$$

Therefore

$$\frac{\sqrt{M}}{\sqrt{\frac{m_1}{A^3} + \frac{m_2}{B^3} + \frac{m_3}{C^3}}} < 1, \tag{4.22}$$

that is,

$$\frac{\sqrt{\frac{m_1}{A^3} + \frac{m_2}{B^3} + \frac{m_3}{C^3}}}{\sqrt{M}} > 1. \tag{4.23}$$

It follows from (4.10)-(4.13) and (4.23) that

$$\begin{aligned} \sqrt{\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} + \frac{m_3}{r_3^3}} &= \sqrt{\frac{m_1}{A^3} + \frac{m_2}{B^3} + \frac{m_3}{C^3}}\sqrt{\frac{1}{l^3}} \\ &= \sqrt{\frac{m_1}{A^3} + \frac{m_2}{B^3} + \frac{m_3}{C^3}} \cdot \frac{2\pi}{\sqrt{M}T} \\ &> \frac{2\pi}{T}, \end{aligned}$$

that is,

$$\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} + \frac{m_3}{r_3^3} > \frac{4\pi^2}{T^2}. \tag{4.24}$$

By (4.24), we have

$$\begin{aligned}
\delta^2 f(O,\tilde{\varphi}) &= \int_0^T \left[\frac{4\pi^2}{T^2}\cos^2\frac{2\pi}{T}t - \left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} + \frac{m_3}{r_3^3}\right)\sin^2\frac{2\pi}{T}t\right]dt \\
&= \frac{2\pi^2}{T} - \left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} + \frac{m_3}{r_3^3}\right)\cdot\frac{T}{2} \\
&= \left[\frac{4\pi^2}{T^2} - \left(\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3} + \frac{m_3}{r_3^3}\right)\right]\cdot\frac{T}{2} \\
&< 0,
\end{aligned} \tag{4.25}$$

which implies $q(t) = (0,0,0)$ is not a local minimum for $f(q)$ on Λ_i $(i = 1,2)$. Hence the minimizers of $f(q)$ on Λ_i $(i = 1,2)$ are not always at the center of the masses, they must oscillate periodically on the vertical axis, that is, the minimizers are not always co-planar, therefore, we get nonplanar periodic solutions.

Combined with Lemma 2.4, the proof is completed. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript, and they read and approved the final draft.

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