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Blow up of positive initial-energy solutions for coupled nonlinear wave equations with degenerate damping and source terms

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Abstract
In this work, we consider coupled nonlinear wave equations with degenerate damping and source terms. We will show the blow up of solutions in finite time with positive initial energy. This improves earlier results in the literature.
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1 Introduction

In this work, we consider the following initial-boundary value problem:

$$\begin{cases} u_{tt} + (|u|^k + |v|^l)|u_t|^{p-1}u_t = \operatorname{div}(\rho(|\nabla u|^2)\nabla u) + f_1(u,v), & (x,t) \in \Omega \times (0,T), \\ v_{tt} + (|v|^\theta + |u|^\varrho)|v_t|^{q-1}v_t = \operatorname{div}(\rho(|\nabla v|^2)\nabla v) + f_2(u,v), & (x,t) \in \Omega \times (0,T), \\ u(x,t) = v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n ($n = 1,2,3$); $p,q \geq 1$, $k,l,\theta,\varrho \geq 0$; $f_i(\cdot,\cdot) : R^2 \longrightarrow R$ are given functions to be specified later.

In the case of $\rho = 1$, equation (1.1) takes the form

$$\begin{cases} u_{tt} - \triangle u + (|u|^k + |v|^l)|u_t|^{p-1}u_t = f_1(u,v), \\ v_{tt} - \triangle v + (|v|^\theta + |u|^\varrho)|v_t|^{q-1}v_t = f_2(u,v). \end{cases} \tag{1.2}$$

In [1] Rammaha and Sakuntasathien studied the global well posedness of the solution of problem (1.2). Agre and Rammaha [2] studied the global existence and the blow up of the solution of problem (1.2) for $k = l = \theta = \varrho = 0$, and also Alves *et al.* [3] investigated the existence, uniform decay rates and blow up of the solution to systems. After that, the blow up result was improved by Houari [4]. Also, Houari [5] showed that the local solution obtained in [2] is global and decay of solutions.

When $k = l = \theta = \varrho = 0$, equation (1.1) reduces to the following form:

$$\begin{cases} u_{tt} + |u_t|^{p-1}u_t = \operatorname{div}(\rho(|\nabla u|^2)\nabla u) + f_1(u,v), \\ v_{tt} + |v_t|^{q-1}v_t = \operatorname{div}(\rho(|\nabla v|^2)\nabla v) + f_2(u,v). \end{cases} \tag{1.3}$$

Wu *et al.* [6] obtained the global existence and blow up of the solution of problem (1.3) under some suitable conditions. Also, Fei and Hongjun [7] considered problem (1.3) and improved the blow up result obtained in [6] for a large class of initial data in positive initial energy using some techniques as in Payne and Sattinger [8] and some estimates used firstly by Vitillaro [9]. Recently, Pişkin and Polat [10] studied the local and global existence, energy decay and blow up of the solution of problem (1.3).

In this work, we analyze the influence of degenerate damping terms and source terms on the solutions of problem (1.1). Blow up of the solution with positive initial energy was proved for $2(r+2) > \max\{k+p+1, l+p+1, \theta+q+1, \varrho+q+1\}$ by using the technique of [9] with a modification in the energy functional.

This work is organized as follows. In Section 2, we present some lemmas and the local existence theorem. In Section 3, the blow up of the solution is given.

2 Preliminaries

In this section, we shall give some assumptions and lemmas which will be used throughout this work. Let $\|\cdot\|$ and $\|\cdot\|_p$ denote the usual $L^2(\Omega)$ norm and $L^p(\Omega)$ norm, respectively.

Next, we give assumptions for problem (1.1).

(A1) ρ is a positive C^1 function satisfying

$$\rho(s) = b_1 + b_2 s^m, \quad m \geq 0,$$

where b_1, b_2 are nonnegative constants and $b_1 + b_2 > 0$.

(A2) For the nonlinearity, we suppose that

$$\begin{cases} p, q \geq 1 & \text{if } n = 1, 2, \\ 1 \leq p, q \leq 5 & \text{if } n = 3. \end{cases}$$

Concerning the functions $f_1(u,v)$ and $f_2(u,v)$, we take

$$f_1(u,v) = a|u+v|^{2(r+1)}(u+v) + b|u|^r u|v|^{r+2},$$
$$f_2(u,v) = a|u+v|^{2(r+1)}(u+v) + b|v|^r v|u|^{r+2},$$

where $a, b > 0$ are constants and r satisfies

$$\begin{cases} -1 < r & \text{if } n = 1, 2, \\ -1 < r \leq 1 & \text{if } n = 3. \end{cases} \tag{2.1}$$

According to the above equalities they can easily verify that

$$uf_1(u,v) + vf_2(u,v) = 2(r+2)F(u,v), \quad \forall (u,v) \in R^2, \tag{2.2}$$

where

$$F(u,v) = \frac{1}{2(r+2)}\left[a|u+v|^{2(r+2)} + 2b|uv|^{r+2}\right]. \tag{2.3}$$

We have the following result.

Lemma 2.1 [11] *There exist two positive constants c_0 and c_1 such that*

$$c_0\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right) \leq 2(r+2)F(u,v) \leq c_1\left(|u|^{2(r+2)} + |v|^{2(r+2)}\right) \tag{2.4}$$

is satisfied.

Lemma 2.2 (Sobolev-Poincaré inequality) [12] *Let q be a number with $2 \leq q < \infty$ ($n = 1,2$) or $2 \leq q \leq 2n/(n-2)$ ($n \geq 3$), then there is a constant $C_* = C_*(\Omega, q)$ such that*

$$\|u\|_q \leq C_*\|\nabla u\| \quad \text{for } u \in H_0^1(\Omega).$$

Lemma 2.3 [13] *Suppose that*

$$p \leq 2\frac{n-1}{n-2}, \quad n \geq 3$$

holds. Then there exists a positive constant $C > 1$ depending on Ω only such that

$$\|u\|_p^s \leq C\left(\|\nabla u\|^2 + \|u\|_p^p\right)$$

for any $u \in H_0^1(\Omega)$, $2 \leq s \leq p$.

Lemma 2.4 *$E(t)$ is a nonincreasing function for $t \geq 0$ and*

$$\frac{d}{dt}E(t) = -\int_\Omega \left(|u|^k + |v|^l\right)|u_t|^{p+1}\,dx - \int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v_t|^{q+1}\,dx. \tag{2.5}$$

Proof Multiplying the first equation of (1.1) by u_t, the second equation by v_t, and integrating them over Ω, then adding them together and integrating by parts, we obtain

$$E(t) - E(0) = -\int_0^t \int_\Omega \left(\left(|u|^k + |v|^l\right)|u_\tau|^{p+1} + \left(|v|^\theta + |u|^\varrho\right)|v_\tau|^{q+1}\right)dx\,d\tau \quad \text{for } t \geq 0. \tag{2.6}$$

□

Next, we state the local existence theorem that can be established by combining arguments of [1, 10]. Firstly, we give the definition of a weak solution to problem (1.1).

Definition 2.1 A pair of functions (u,v) is said to be a weak solution of (1.1) on $[0,T]$ if $u, v \in C([0,T]; W_0^{1,2(m+1)}(\Omega) \cap L^{r+1}(\Omega))$, $u_t \in C([0,T]; L^2(\Omega)) \cap L^{p+1}(\Omega \times (0,T))$ and $v_t \in C([0,T]; L^2(\Omega)) \cap L^{q+1}(\Omega \times (0,T))$. In addition, (u,v) satisfies

$$\begin{aligned}
&\int_\Omega u'(t)\phi\,dx - \int_\Omega u_1(t)\phi\,dx + \int_\Omega \left(\rho\left(|\nabla u|^2\right)\nabla u\right)\nabla\phi\,dx \\
&\quad + \int_0^t \int_\Omega \left(|u|^k + |v|^l\right)\left|u'\right|^{p-1}u'\phi\,dx\,d\tau \\
&= \int_0^t \int_\Omega f_1\left(u(\tau), v(\tau)\right)\phi\,dx\,d\tau,
\end{aligned} \tag{2.7}$$

$$
\begin{aligned}
&\int_\Omega v'(t)\varphi\,dx - \int_\Omega v_1(t)\varphi\,dx + \int_\Omega \bigl(\rho\bigl(|\nabla v|^2\bigr)\nabla v\bigr)\nabla\varphi\,dx \\
&\qquad + \int_0^t \int_\Omega \bigl(|v|^\theta + |u|^\varrho\bigr)\bigl|v'\bigr|^{q-1} v'\varphi\,dx\,d\tau \\
&\quad = \int_0^t \int_\Omega f_2\bigl(u(\tau), v(\tau)\bigr)\varphi\,dx\,d\tau
\end{aligned}
\tag{2.8}
$$

for all test functions $\phi \in W_0^{1,2(m+1)}(\Omega) \cap L^{p+1}(\Omega)$, $\varphi \in W_0^{1,2(m+1)}(\Omega) \cap L^{q+1}(\Omega)$ and for almost all $t \in [0, T]$.

Theorem 2.1 (Local existence) *Assume that* (A1), (A2) *and* (2.1) *hold. Then, for any initial data* $u_0, v_0 \in W_0^{1,2(m+1)}(\Omega) \cap L^{r+1}(\Omega)$ *and* $u_1, v_1 \in L^2(\Omega)$, *there exists a unique local weak solution* (u, v) *of problem* (1.1) (*in the sense of Definition* 2.1) *defined in* $[0, T]$ *for some* $T > 0$, *and satisfies the energy identity*

$$
E(t) + \int_0^t \int_\Omega \bigl(\bigl(|u|^k + |v|^l\bigr)|u_\tau|^{p+1} + \bigl(|v|^\theta + |u|^\varrho\bigr)|v_\tau|^{q+1}\bigr)\,dx\,d\tau = E(0),
\tag{2.9}
$$

where

$$
E(t) = \frac{1}{2}\bigl(\|u_t\|^2 + \|v_t\|^2\bigr) + \frac{1}{2}\int_\Omega \bigl(P\bigl(|\nabla u|^2\bigr) + P\bigl(|\nabla v|^2\bigr)\bigr)\,dx - \int_\Omega F(u, v)\,dx,
\tag{2.10}
$$

where $P(s) = \int_0^s \rho(\xi)\,d\xi$, $s \geq 0$.

3 Blow up of solutions

In this section, we are going to consider the blow up of the solution for problem (1.1).

Lemma 3.1 *Suppose that* (2.1) *holds. Then there exists* $\eta > 0$ *such that for any* $(u, v) \in (H^{2m}(\Omega) \cap H_0^m(\Omega)) \times (H^{2m}(\Omega) \cap H_0^m(\Omega))$ *the inequality*

$$
\|u + v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2} \leq \eta\left(\int_\Omega \bigl(P\bigl(|\nabla u|^2\bigr) + P\bigl(|\nabla v|^2\bigr)\bigr)\right)^{r+2}
\tag{3.1}
$$

holds.

Proof The proof is almost the same as that of [11], so we omit it here. □

In order to state and prove our result and for the sake of simplicity, we take $a = b = 1$. We introduce the following:

$$
\begin{aligned}
&B = \eta^{\frac{1}{2(r+2)}}, \qquad \alpha_1 = B^{-\frac{r+2}{r+1}}, \\
&E_1 = \left(\frac{1}{2} - \frac{1}{2(r+2)}\right)\alpha_1^2, \qquad E_2 = \left(\frac{1}{2(m+1)} - \frac{1}{2(r+2)}\right)\alpha_1^2,
\end{aligned}
\tag{3.2}
$$

where η is the optimal constant in (3.1).

The following lemma will play an essential role in the proof of our main result, and it is similar to the lemma used firstly by Vitillaro [9].

Lemma 3.2 [7] *Assume that* (A1) *and* (2.1) *hold. Let* (u,v) *be a solution of* (1.1). *Assume further that* $E(0) < E_1$ *and*

$$\left(\int_\Omega \big(P\big(|\nabla u_0|^2\big) + P\big(|\nabla v_0|^2\big)\big)\,dx\right)^{\frac{1}{2}} > \alpha_1. \tag{3.3}$$

Then there exists a constant $\alpha_2 > \alpha_1$ *such that*

$$\left(\int_\Omega \big(P\big(|\nabla u|^2\big) + P\big(|\nabla v|^2\big)\big)\,dx\right)^{\frac{1}{2}} > \alpha_2 \quad \textit{for } t > 0, \tag{3.4}$$

$$\big(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2}\big)^{\frac{1}{2(r+2)}} \geq B\alpha_2 \quad \textit{for } t > 0. \tag{3.5}$$

Theorem 3.1 *Assume that* (A1), (A2) *and* (2.1) *hold. Assume further that*

$$2(r+2) > \max\{k+p+1, l+p+1, \theta+q+1, \varrho+q+1\}.$$

Then any solution of problem (1.1) *with initial data satisfying*

$$\left(\int_\Omega \big(P\big(|\nabla u_0|^2\big) + P\big(|\nabla v_0|^2\big)\big)\,dx\right)^{\frac{1}{2}} > \alpha_1, \qquad E(0) < E_2,$$

cannot exist for all time.

Proof We suppose that the solution exists for all time and we reach a contradiction.
Set

$$H(t) = E_2 - E(t). \tag{3.6}$$

By using (2.10) and (3.6) we get

$$\begin{aligned} 0 < H(0) \leq H(t) &= E_2 - \frac{1}{2}\big(\|u_t\|^2 + \|v_t\|^2\big) \\ &\quad - \frac{1}{2}\int_\Omega \big(P\big(|\nabla u|^2\big) + P\big(|\nabla v|^2\big)\big)\,dx + \int_\Omega F(u,v)\,dx. \end{aligned} \tag{3.7}$$

From (3.4) and (2.4) we have

$$\begin{aligned} &E_2 - \frac{1}{2}\big(\|u_t\|^2 + \|v_t\|^2\big) - \frac{1}{2}\int_\Omega \big(P\big(|\nabla u|^2\big) + P\big(|\nabla v|^2\big)\big)\,dx + \int_\Omega F(u,v)\,dx \\ &\quad \leq E_2 - \frac{1}{2}\alpha_1^2 + \frac{c_1}{2(r+2)}\big(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\big) \\ &\quad \leq E_1 - \frac{1}{2}\alpha_1^2 + \frac{c_1}{2(r+2)}\big(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\big) \\ &\quad \leq -\frac{1}{2(r+2)}\alpha_1^2 + \frac{c_1}{2(r+2)}\big(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\big) \\ &\quad \leq \frac{c_1}{2(r+2)}\big(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\big). \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) we have

$$0 < H(0) \leq H(t) \leq \frac{c_1}{2(r+2)} \big(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)} \big). \tag{3.9}$$

We then define

$$\Psi(t) = H^{1-\sigma}(t) + \varepsilon \left(\int_\Omega u u_t \, dx + \int_\Omega v v_t \, dx \right), \tag{3.10}$$

where ε is small to be chosen later and

$$\begin{aligned} 0 < \sigma \leq \min \bigg\{ & \frac{r+1}{2(r+2)}, \frac{2r+3-(k+p)}{2p(r+2)}, \frac{2r+3-(l+p)}{2p(r+2)}, \\ & \frac{2r+3-(\varrho+q)}{2q(r+2)}, \frac{2r+3-(\theta+q)}{2q(r+2)} \bigg\}. \end{aligned} \tag{3.11}$$

Our goal is to show that $\Psi(t)$ satisfies a differential inequality of the form

$$\Psi'(t) \geq \xi \Psi^{\zeta}(t), \quad \zeta > 1. \tag{3.12}$$

This, of course, will lead to a blow up in finite time.

By taking a derivative of (3.10) and using equation (1.1) we obtain

$$\begin{aligned} \Psi'(t) = {} & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \big(\|u_t\|^2 + \|v_t\|^2 \big) \\ & - \varepsilon \int_\Omega \big(\rho\big(|\nabla u|^2\big) |\nabla u|^2 + \rho\big(|\nabla v|^2\big) |\nabla v|^2 \big) \, dx \\ & - \varepsilon \left(\int_\Omega u \big(|u|^k + |v|^l\big) u_t |u_t|^{p-1} \, dx + \int_\Omega v \big(|v|^\theta + |u|^\varrho\big) v_t |v_t|^{q-1} \, dx \right) \\ & + \varepsilon \int_\Omega \big(u f_1(u,v) + v f_2(u,v) \big) \, dx \\ = {} & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \big(\|u_t\|^2 + \|v_t\|^2 \big) - \varepsilon b_1 \big(\|\nabla u\|^2 + \|\nabla v\|^2 \big) \\ & - \varepsilon b_2 \big(\|\nabla u\|_{2(r+2)}^{2(r+2)} + \|\nabla v\|_{2(r+2)}^{2(r+2)} \big) + \varepsilon \big(\|u+v\|_{2(r+2)}^{2(r+2)} + 2 \|uv\|_{r+2}^{r+2} \big) \\ & - \varepsilon \left(\int_\Omega u \big(|u|^k + |v|^l\big) u_t |u_t|^{p-1} \, dx + \int_\Omega v \big(|v|^\theta + |u|^\varrho\big) v_t |v_t|^{q-1} \, dx \right). \end{aligned} \tag{3.13}$$

From the definition of $H(t)$, it follows that

$$\begin{aligned} & -b_2 \big(\|\nabla u\|_{2(r+2)}^{2(r+2)} + \|\nabla v\|_{2(r+2)}^{2(r+2)} \big) \\ & \quad = 2(m+1) H(t) - 2(m+1) E_2 + (m+1) \big(\|u_t\|^2 + \|v_t\|^2 \big) \\ & \qquad + (m+1) b_1 \big(\|\nabla u\|^2 + \|\nabla v\|^2 \big) - 2(m+1) \int_\Omega F(u,v) \, dx. \end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.13), we obtain

$$\begin{aligned} \Psi'(t) = {} & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon (m+2) \big(\|u_t\|^2 + \|v_t\|^2 \big) \\ & + \varepsilon b_1 m \big(\|\nabla u\|^2 + \|\nabla v\|^2 \big) \end{aligned}$$

$$+ 2\varepsilon(m+1)H(t) - 2\varepsilon(m+1)E_2 + \varepsilon\left(1 - \frac{m+1}{r+2}\right)\left(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2}\right)$$
$$- \varepsilon\left(\int_\Omega u\left(|u|^k + |v|^l\right)u_t|u_t|^{p-1}\,dx + \int_\Omega v\left(|v|^\theta + |u|^\varrho\right)v_t|v_t|^{q-1}\,dx\right). \tag{3.15}$$

In order to estimate the last two terms in (3.15), we make use of the following Young inequality:

$$XY \le \frac{\delta^k X^k}{k} + \frac{\delta^{-l}Y^l}{l},$$

where $X, Y \geq 0$, $\delta > 0$, $k, l \in R^+$ such that $\frac{1}{k} + \frac{1}{l} = 1$. Consequently, applying the above inequality we have

$$\int_\Omega uu_t|u_t|^{p-1}\,dx \le \frac{\delta_1^{p+1}}{p+1}\|u\|_{p+1}^{p+1} + \frac{p\delta_1^{-\frac{p+1}{p}}}{p+1}\|u_t\|_{p+1}^{p+1},$$

and therefore

$$\begin{aligned}\int_\Omega \left(|u|^k + |v|^l\right)uu_t|u_t|^{p-1}\,dx \le{}& \frac{\delta_1^{p+1}}{p+1}\int_\Omega \left(|u|^k + |v|^l\right)|u|^{p+1}\,dx \\ &+ \frac{p\delta_1^{-\frac{p+1}{p}}}{p+1}\int_\Omega \left(|u|^k + |v|^l\right)|u_t|^{p+1}\,dx.\end{aligned}$$

Similarly

$$\int_\Omega vv_t|v_t|^{q-1}\,dx \le \frac{\delta_2^{q+1}}{q+1}\|v\|_{q+1}^{q+1} + \frac{q\delta_2^{-\frac{q+1}{q}}}{q+1}\|v_t\|_{q+1}^{q+1},$$

and therefore

$$\begin{aligned}\int_\Omega v\left(|v|^\theta + |u|^\varrho\right)v_t|v_t|^{q-1}\,dx \le{}& \frac{\delta_2^{q+1}}{q+1}\int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v|^{q+1}\,dx \\ &+ \frac{q\delta_2^{-\frac{q+1}{q}}}{q+1}\int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v_t|^{q+1}\,dx,\end{aligned}$$

where δ_1, δ_2 are constants depending on the time t that will be specified later. Therefore, (3.15) becomes

$$\begin{aligned}\Psi'(t) \ge{}& (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon(m+2)\left(\|u_t\|^2 + \|v_t\|^2\right) + \varepsilon b_1 m\left(\|\nabla u\|^2 + \|\nabla v\|^2\right) \\ &+ 2\varepsilon(m+1)H(t) - 2\varepsilon(m+1)E_2 + \varepsilon\left(1 - \frac{m+1}{r+2}\right)\left(\|u+v\|_{2(r+2)}^{2(r+2)} + 2\|uv\|_{r+2}^{r+2}\right) \\ &- \varepsilon\frac{\delta_1^{p+1}}{p+1}\int_\Omega \left(|u|^k + |v|^l\right)|u|^{p+1}\,dx - \varepsilon\frac{p\delta_1^{-\frac{p+1}{p}}}{p+1}\int_\Omega \left(|u|^k + |v|^l\right)|u_t|^{p+1}\,dx \\ &- \varepsilon\frac{\delta_2^{q+1}}{q+1}\int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v|^{q+1}\,dx - \varepsilon\frac{q\delta_2^{-\frac{q+1}{q}}}{q+1}\int_\Omega \left(|v|^\theta + |u|^\varrho\right)|v_t|^{q+1}\,dx.\end{aligned} \tag{3.16}$$

Therefore, by taking δ_1 and δ_2 so that $\delta_1^{-\frac{p+1}{p}} = k_1 H^{-\sigma}(t)$, $\delta_2^{-\frac{q+1}{q}} = k_2 H^{-\sigma}(t)$ where $k_1, k_2 > 0$ are specified later, we get

$$\begin{aligned}\Psi'(t) \geq{}& \big((1-\sigma) - K\varepsilon\big)H^{-\sigma}(t)H'(t) + \varepsilon(m+2)\big(\|u_t\|^2 + \|v_t\|^2\big) + \varepsilon b_1 m\big(\|\nabla u\|^2 + \|\nabla v\|^2\big)\\ &+ 2\varepsilon(m+1)H(t) + \varepsilon c'\big(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\big)\\ &- \varepsilon\frac{k_1^{-p}H^{\sigma p}(t)}{p+1}\int_\Omega\big(|u|^k + |v|^l\big)|u|^{p+1}\,dx\\ &- \varepsilon\frac{k_2^{-q}H^{\sigma q}(t)}{q+1}\int_\Omega\big(|v|^\theta + |u|^\varrho\big)|v|^{q+1}\,dx, \end{aligned}\tag{3.17}$$

where $K = \frac{k_1 p}{p+1} + \frac{k_2 q}{q+1}$ and $c' = c_0(1 - \frac{m+1}{r+2} - 2(m+1)E_2(B\alpha_2)^{-2(r+2)})$. It is clear $c' > 0$ since $\alpha_2 > \alpha_1 = B^{-\frac{r+2}{r+1}}$.

Applying the Young inequality, we have

$$\begin{aligned}\int_\Omega\big(|u|^k + |v|^l\big)|u|^{p+1}\,dx &\leq \int_\Omega |u|^{k+p+1}\,dx + \int_\Omega |v|^l|u|^{p+1}\,dx\\ &\leq \int_\Omega |u|^{k+p+1}\,dx + \frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}}\int_\Omega |v|^{l+p+1}\,dx\\ &\quad+ \frac{p+1}{l+p+1}\delta_1^{-\frac{l+p+1}{p+1}}\int_\Omega |u|^{l+p+1}\,dx\\ &= \|u\|_{k+p+1}^{k+p+1} + \frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}}\|v\|_{l+p+1}^{l+p+1}\\ &\quad+ \frac{p+1}{l+p+1}\delta_1^{-\frac{l+p+1}{p+1}}\|u\|_{l+p+1}^{l+p+1} \end{aligned}\tag{3.18}$$

and

$$\begin{aligned}\int_\Omega\big(|v|^\theta + |u|^\varrho\big)|v|^{q+1}\,dx &\leq \int_\Omega |v|^{\theta+q+1}\,dx + \int_\Omega |u|^\varrho|v|^{q+1}\,dx\\ &\leq \int_\Omega |v|^{\theta+q+1}\,dx + \frac{\varrho}{\varrho+q+1}\delta_2^{\frac{\varrho+q+1}{\varrho}}\int_\Omega |u|^{\varrho+q+1}\,dx\\ &\quad+ \frac{q+1}{\varrho+q+1}\delta_2^{-\frac{\varrho+q+1}{q+1}}\int_\Omega |v|^{\varrho+q+1}\,dx\\ &= \|v\|_{\theta+q+1}^{\theta+q+1} + \frac{\varrho}{\varrho+q+1}\delta_2^{\frac{\varrho+q+1}{\varrho}}\|u\|_{\varrho+q+1}^{\varrho+q+1}\\ &\quad+ \frac{q+1}{\varrho+q+1}\delta_2^{-\frac{\varrho+q+1}{q+1}}\|v\|_{\varrho+q+1}^{\varrho+q+1}. \end{aligned}\tag{3.19}$$

Substituting (3.18) and (3.19) into (3.17), we have

$$\begin{aligned}\Psi'(t) \geq{}& \big((1-\sigma) - K\varepsilon\big)H^{-\sigma}(t)H'(t) + \varepsilon(m+2)\big(\|u_t\|^2 + \|v_t\|^2\big) + \varepsilon b_1 m\big(\|\nabla u\|^2 + \|\nabla v\|^2\big)\\ &+ 2\varepsilon(m+1)H(t) + \varepsilon c'\big(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\big)\\ &- \varepsilon\frac{k_1^{-p}H^{\sigma p}(t)}{p+1}\bigg(\|u\|_{k+p+1}^{k+p+1} + \frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}}\|v\|_{l+p+1}^{l+p+1} + \frac{p+1}{l+p+1}\delta_1^{-\frac{l+p+1}{p+1}}\|u\|_{l+p+1}^{l+p+1}\bigg) \end{aligned}$$

$$
\begin{aligned}
&- \varepsilon \frac{k_2^{-q} H^{\sigma q}(t)}{q+1} \left(\|v\|_{\theta+q+1}^{\theta+q+1} + \frac{\varrho}{\varrho+q+1} \delta_2^{\frac{\varrho+q+1}{\varrho}} \|u\|_{\varrho+q+1}^{\varrho+q+1} \right. \\
&\left. + \frac{q+1}{\varrho+q+1} \delta_2^{-\frac{\varrho+q+1}{q+1}} \|v\|_{\varrho+q+1}^{\varrho+q+1} \right). \qquad (3.20)
\end{aligned}
$$

Since $2(r+2) > \max\{k+p+1, l+p+1, \theta+q+1, \varrho+q+1\}$, we obtain

$$
H^{\sigma p}(t) \|u\|_{k+p+1}^{k+p+1} \leq C \big(\|u\|_{2(r+2)}^{2\sigma p(r+2)+k+p+1} + \|v\|_{2(r+2)}^{2\sigma p(r+2)} \|u\|_{k+p+1}^{k+p+1} \big), \qquad (3.21)
$$

$$
H^{\sigma q}(t) \|v\|_{\theta+q+1}^{\theta+q+1} \leq C \big(\|v\|_{2(r+2)}^{2\sigma q(r+2)+\theta+q+1} + \|u\|_{2(r+2)}^{2\sigma q(r+2)} \|v\|_{\theta+q+1}^{\theta+q+1} \big), \qquad (3.22)
$$

$$
\begin{aligned}
&\frac{l}{l+p+1} \delta_1^{\frac{l+p+1}{l}} H^{\sigma p}(t) \|v\|_{l+p+1}^{l+p+1} \\
&\quad \leq C \frac{l}{l+p+1} \delta_1^{\frac{l+p+1}{l}} \big(\|v\|_{2(r+2)}^{2\sigma p(r+2)+l+p+1} + \|u\|_{2(r+2)}^{2\sigma p(r+2)} \|v\|_{l+p+1}^{l+p+1} \big), \qquad (3.23)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\varrho}{\varrho+q+1} \delta_2^{\frac{\varrho+q+1}{\varrho}} H^{\sigma q}(t) \|u\|_{\varrho+q+1}^{\varrho+q+1} \\
&\quad \leq C \frac{\varrho}{\varrho+q+1} \delta_2^{\frac{\varrho+q+1}{\varrho}} \big(\|u\|_{2(r+2)}^{2\sigma q(r+2)+\varrho+q+1} + \|v\|_{2(r+2)}^{2\sigma q(r+2)} \|u\|_{\varrho+q+1}^{\varrho+q+1} \big). \qquad (3.24)
\end{aligned}
$$

By using (3.11) and the algebraic inequality

$$
z^{\upsilon} \leq z+1 \leq \left(1 + \frac{1}{a}\right)(z+a), \quad \forall z \geq 0, 0 < \upsilon \leq 1, a \geq 0, \qquad (3.25)
$$

we have, for all $t \geq 0$,

$$
\begin{aligned}
\|u\|_{2(r+2)}^{2\sigma p(r+2)+k+p+1} &\leq d \big(\|u\|_{2(r+2)}^{2(r+2)} + H(0) \big) \\
&\leq d \big(\|u\|_{2(r+2)}^{2(r+2)} + H(t) \big), \qquad (3.26)
\end{aligned}
$$

$$
\|v\|_{2(r+2)}^{2\sigma q(r+2)+\theta+q+1} \leq d \big(\|v\|_{2(r+2)}^{2(r+2)} + H(t) \big), \qquad (3.27)
$$

where $d = 1 + \frac{1}{H(0)}$. Similarly

$$
\|u\|_{2(r+2)}^{2\sigma q(r+2)+\varrho+q+1} \leq d \big(\|u\|_{2(r+2)}^{2(r+2)} + H(t) \big), \qquad (3.28)
$$

$$
\|v\|_{2(r+2)}^{2\sigma p(r+2)+l+p+1} \leq d \big(\|v\|_{2(r+2)}^{2(r+2)} + H(t) \big). \qquad (3.29)
$$

Also, since $(a+b)^{\lambda} \leq C(a^{\lambda} + b^{\lambda})$, $a, b > 0$, by the Young inequality and using (3.11) and (3.25), we conclude that

$$
\begin{aligned}
\|v\|_{2(r+2)}^{2\sigma p(r+2)} \|u\|_{k+p+1}^{k+p+1} &\leq C \big(\|v\|_{2(r+2)}^{2(r+2)} + \|u\|_{k+p+1}^{2(r+2)} \big) \\
&\leq C \big(\|v\|_{2(r+2)}^{2(r+2)} + \|u\|_{2(r+2)}^{2(r+2)} \big), \qquad (3.30)
\end{aligned}
$$

$$
\|u\|_{2(r+2)}^{2\sigma q(r+2)} \|v\|_{\theta+q+1}^{\theta+q+1} \leq C \big(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)} \big), \qquad (3.31)
$$

$$
\|u\|_{2(r+2)}^{2\sigma p(r+2)} \|v\|_{l+p+1}^{l+p+1} \leq C \big(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)} \big) \qquad (3.32)
$$

and

$$\|v\|_{2(r+2)}^{2\sigma q(r+2)}\|u\|_{\varrho+q+1}^{\varrho+q+1} \le C\big(\|v\|_{2(r+2)}^{2(r+2)} + \|u\|_{2(r+2)}^{2(r+2)}\big). \tag{3.33}$$

Combining (3.21)-(3.24) and (3.26)-(3.33) into (3.20), we have

$$\begin{aligned}
\Psi'(t) \ge{}& \big((1-\sigma) - K\varepsilon\big)H^{-\sigma}(t)H'(t) + \varepsilon(m+2)\big(\|u_t\|^2 + \|v_t\|^2\big) + \varepsilon b_1 m\big(\|\nabla u\|^2 + \|\nabla v\|^2\big) \\
&+ \varepsilon\bigg[2(m+1) - Ck_1^{-p}\bigg(1 + \frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}} + \frac{p+1}{l+p+1}\delta_1^{-\frac{l+p+1}{p+1}}\bigg) \\
&- Ck_2^{-q}\bigg(1 + \frac{\varrho}{\varrho+q+1}\delta_2^{\frac{\varrho+q+1}{\varrho}} + \frac{q+1}{\varrho+q+1}\delta_2^{-\frac{\varrho+q+1}{q+1}}\bigg)\bigg]H(t) \\
&+ \varepsilon\bigg[c' - Ck_1^{-p}\bigg(1 + \frac{l}{l+p+1}\delta_1^{\frac{l+p+1}{l}} + \frac{p+1}{l+p+1}\delta_1^{-\frac{l+p+1}{p+1}}\bigg) \\
&- Ck_2^{-q}\bigg(1 + \frac{\varrho}{\varrho+q+1}\delta_2^{\frac{\varrho+q+1}{\varrho}} + \frac{q+1}{\varrho+q+1}\delta_2^{-\frac{\varrho+q+1}{q+1}}\bigg)\bigg] \\
&\times \big(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\big).
\end{aligned} \tag{3.34}$$

At this point, and for large values of k_1 and k_2, we can find positive constants K_1 and K_2 such that (3.34) becomes

$$\begin{aligned}
\Psi'(t) \ge{}& \big((1-\sigma) - K\varepsilon\big)H^{-\sigma}(t)H'(t) + \varepsilon(m+2)\big(\|u_t\|^2 + \|v_t\|^2\big) + \varepsilon b_1 m\big(\|\nabla u\|^2 + \|\nabla v\|^2\big) \\
&+ \varepsilon K_1 H(t) + \varepsilon K_2\big(\|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\big) \\
\ge{}& \beta\big(\|u_t\|^2 + \|v_t\|^2 + H(t) + \|\nabla u\|^2 + \|\nabla v\|^2 + \|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)}\big),
\end{aligned} \tag{3.35}$$

where $\beta = \min\{\varepsilon(m+2), \varepsilon b_1 m, \varepsilon K_1, \varepsilon K_2\}$, and we pick ε small enough so that $(1-\sigma) - K\varepsilon \ge 0$. Consequently, we have

$$\Psi(t) \ge \Psi(0) = H^{1-\sigma}(0) + \varepsilon\bigg(\int_\Omega u_0 u_1\,dx + \int_\Omega v_0 v_1\,dx\bigg) > 0, \quad \forall t \ge 0. \tag{3.36}$$

On the other hand, applying the Hölder inequality, we obtain

$$\begin{aligned}
\bigg|\int_\Omega u u_t\,dx + \int_\Omega v v_t\,dx\bigg|^{\frac{1}{1-\sigma}} &\le \|u\|^{\frac{1}{1-\sigma}}\|u_t\|^{\frac{1}{1-\sigma}} + \|v\|^{\frac{1}{1-\sigma}}\|v_t\|^{\frac{1}{1-\sigma}} \\
&\le C\big(\|u\|_{2(r+2)}^{\frac{1}{1-\sigma}}\|u_t\|^{\frac{1}{1-\sigma}} + \|v\|_{2(r+2)}^{\frac{1}{1-\sigma}}\|v_t\|^{\frac{1}{1-\sigma}}\big).
\end{aligned} \tag{3.37}$$

The Young inequality gives

$$\bigg|\int_\Omega u u_t\,dx + \int_\Omega v v_t\,dx\bigg|^{\frac{1}{1-\sigma}} \le C\big(\|u\|_{2(r+2)}^{\frac{\mu_1}{1-\sigma}} + \|u_t\|^{\frac{\mu_2}{1-\sigma}} + \|v\|_{2(r+2)}^{\frac{\mu_1}{1-\sigma}} + \|v_t\|^{\frac{\mu_2}{1-\sigma}}\big) \tag{3.38}$$

for $\frac{1}{\mu_1} + \frac{1}{\mu_2} = 1$. We take $\mu_2 = 2(1-\sigma)$ to get $\mu_1 = \frac{2(1-\sigma)}{1-2\sigma}$ by (3.11). Therefore (3.38) becomes

$$\bigg|\int_\Omega u u_t\,dx + \int_\Omega v v_t\,dx\bigg|^{\frac{1}{1-\sigma}} \le C\big(\|u_t\|^2 + \|v_t\|^2 + \|u\|_{2(r+2)}^{\frac{2}{1-2\sigma}} + \|v\|_{2(r+2)}^{\frac{2}{1-2\sigma}}\big). \tag{3.39}$$

By using Lemma 2.3, we obtain

$$\left|\int_\Omega uu_t\,dx + \int_\Omega vv_t\,dx\right|^{\frac{1}{1-\sigma}} \leq C\bigl(\|u_t\|^2 + \|v_t\|^2 + \|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)} + \|\nabla u\|^2 + \|\nabla v\|^2\bigr). \tag{3.40}$$

Thus

$$\begin{aligned}\Psi^{\frac{1}{1-\sigma}}(t) &= \left[H^{1-\sigma}(t) + \varepsilon\left(\int_\Omega uu_t\,dx + \int_\Omega vv_t\,dx\right)\right]^{\frac{1}{1-\sigma}} \\ &\leq 2^{\frac{\sigma}{1-\sigma}}\left(H(t) + \varepsilon^{\frac{1}{1-\sigma}}\left|\int_\Omega uu_t\,dx + \int_\Omega vv_t\,dx\right|^{\frac{1}{1-\sigma}}\right) \\ &\leq C\bigl(\|u_t\|^2 + \|v_t\|^2 + H(t) + \|u\|_{2(r+2)}^{2(r+2)} + \|v\|_{2(r+2)}^{2(r+2)} + \|\nabla u\|^2 + \|\nabla v\|^2\bigr). \end{aligned} \tag{3.41}$$

By combining (3.35) and (3.41), we arrive at

$$\Psi'(t) \geq \xi\Psi^{\frac{1}{1-\sigma}}(t), \tag{3.42}$$

where ξ is a positive constant.

A simple integration of (3.42) over $(0,t)$ yields $\Psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\xi\sigma t}{1-\sigma}}$, which implies that the solution blows up in a finite time T^* with

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}. \qquad \square$$

Competing interests
The author declares that they have no competing interests.

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