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Solutions of semiclassical states for perturbed *p*-Laplacian equation with critical exponent

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Abstract

In this paper, we study semiclassical states for perturbed p-Laplacian equations. Under some given conditions and minimax methods, we show that this problem has at least one positive solution provided that $\varepsilon \le \mathcal{E}$; for any $m \in \mathbb{N}$, it has m pairs of solutions if $\varepsilon \le \mathcal{E}_m$, where $\mathcal{E}, \mathcal{E}_m$ are sufficiently small positive numbers. Moreover, these solutions $u_\varepsilon \to 0$ in $W^{1,p}(\mathbb{R}^N)$ as $\varepsilon \to 0$.

Keywords: semiclassical states; positive solutions; critical exponent

1 Introduction and main results

In this paper, we consider the existence and multiplicity of semiclassical solutions of the following perturbed p-Laplacian equation:

$$\begin{cases} -\varepsilon^p \Delta_p u + V(x)|u|^{p-2}u - \varepsilon^p \Delta_p(|u|^{2\varpi})|u|^{2\varpi-2}u \\ \quad = K(x)|u|^{2\varpi p^*-2}u + h(x,u), \quad x \in \mathbb{R}^N, \\ u \to 0, \quad \text{as } |x| \to \infty, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator with $1 < p < N$, $\varpi \ge 1$, $p^* = \frac{Np}{N-p}$ is the Sobolev critical exponent, $V(x)$ is a nonnegative potential, $K(x)$ is bounded positive coefficient, and $h(x,u)$ is a p-superlinear but subcritical function.

Such types of equations have been derived as models of several physical phenomena and have been the subject of extensive study in recent years. For example, solutions to (1.1) for $p = 2$, $\varpi = 1$ are related to the solitary wave solutions for quasilinear Schrödinger equations,

$$i\hbar\partial_t\psi = -\hbar^2\Delta\psi + W(x)\psi - \widetilde{h}(x,|\psi|^2)\psi - \hbar^2\kappa\Delta[\rho(|\psi|^2)]\rho'(|\psi|^2)\psi, \tag{1.2}$$

where $\psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential, κ, $\hbar$ are real constants and ρ, $\widetilde{h}$ are real functions. The quasilinear equation (1.2) appears more naturally in mathematical physics and has been derived as models of several physical phenomena corresponding to various types of $\rho(s)$. In the case $\rho(s) = s$, (1.2) models the superfluid film equation in fluid mechanics by Kurihara [1]. In the case $\rho(s) = (1+s)^{1/2}$, (1.2) models the self-channeling of a high-power ultra short laser in matter (see [2–5]). For more physical motivations and more references dealing with applications, we can refer to [6–10] and references therein.

Taking $\psi(t,x) = \exp(-\frac{iEt}{\hbar})u(x)$ in (1.2), E is some real constant. It is clear that $\psi(t,x)$ solves (1.2) if and only if $u(x)$ solves the following elliptic equation:

$$-\varepsilon^2 \Delta u + V(x)u - \varepsilon^2 \kappa \Delta\big[\rho\big(|u|^2\big)\big]\rho'\big(|u|^2\big)u = g(x,u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

with $V(x) = W(x) - E$, $\varepsilon^2 = \hbar^2$ and $g(x,u) = \widetilde{h}(x,|u|^2)u$.

When $\kappa = 0$, the semilinear problem has been studied extensively under various hypotheses on the potential and the nonlinearities. See, for example, [11–24] and the references therein.

When $\varepsilon = 1$, $\varpi = 1$, $\rho(s) = s$, $\kappa = 1$, we can refer to [9, 25–29], and so on. Here positive or sign-changing solutions were obtained by using a constrained minimization argument, or a Nehari method, or a technique of changing variables. We remark that among the above three methods, the last one, which was first proposed in [28], is most effective for the power nonlinearity case since this argument can transform the quasilinear problem to a semilinear one and an Orlicz space framework was used as the working space.

It is worth pointing out that the critical exponent case was mentioned as an open problem in [29], where the authors observed that the number 22^* behaves like a critical exponent for (1.3). In [30], for $N = 2$, the authors treated the case where the nonlinearity $h : \mathbb{R} \to \mathbb{R}$ has critical exponential growth, that is, h behaves like $\exp(4\pi s^4) - 1$ as $|s| \to \infty$. For $N \geq 3$, when $V(x)$ satisfies radially symmetrical, periodic, and some geometric conditions, Moameni [31] obtained the existence of nonnegative solutions for (1.3) with the critical growth case; when $V(x)$ satisfied asymptotic and periodic condition. In [24, 32], the authors prove the existence of ground state solutions for (1.3) with $\varepsilon = 1$ or $\kappa = 0$. In the present paper, we will consider a class of quasilinear Schrödinger equations with a nonperiodic potential function $V(x)$ in $\mathbb{R}^N$, $N \geq 3$. In fact, we will investigate the existence of solutions for the critical growth case when the parameter ε goes to zero, *i.e.*, the semiclassical problems for the critical quasilinear Schrödinger equation (1.1). It is well known that in this case the laws of quantum mechanics must reduce to those of classical mechanics, and it describes the transition between quantum mechanics and classical mechanics. As far as we know, there are few papers considering the existence and concentration of semiclassical states for quasilinear Schrödinger equations. For instance, in [33, 34], using a suitable Trudinger-Moser inequality in $\mathbb{R}^2$ and a penalization technique, the authors established the existence of semiclassical solutions for the critical exponent case via the mountain pass lemma.

However, it seems that there is almost no work on the existence of semiclassical solutions to the quasilinear problem on $\mathbb{R}^N$ involving critical nonlinearities and generalized potential $V(x)$. Fortunately, Ding and Lin [35] have been concerned with the existence and multiplicity of semiclassical solutions of the following perturbed nonperiodic quasilinear Schrödinger equation:

$$\begin{cases} -\varepsilon^2 \Delta u + V(x)u = K(x)|u|^{22^*-2}u + h(x,u), & x \in \mathbb{R}^N, \\ u \to 0, \quad \text{as } |x| \to \infty. \end{cases} \tag{1.4}$$

Later, Yang and Ding [36] extended (1.4) to the following quasilinear Schrödinger equation:

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u - \varepsilon^2\Delta(|u|^2)u = K(x)|u|^{22^*-2}u + h(x,u), & x \in \mathbb{R}^N, \\ u \to 0, \quad \text{as } |x| \to \infty. \end{cases} \tag{1.5}$$

Inspired by [36], we will extend the existence and multiplicity of solutions for (1.5) to the general case for (1.5) with $N > p > 1$, $\varpi \geq 1$. Moreover, the corresponding problem becomes more complicated: first, $W^{1,p}(\mathbb{R}^N)$ is not a Hilbert space when $p \neq 2$; secondly, the weak continuity of operator $A_i(u) = |\nabla u|^{p-2}\partial u/\partial x_i$ in $W^{1,p}(\mathbb{R}^N)$ is difficulty to establish.

In this paper, we make the following assumptions:

(V_1) $V(x) \in C(\mathbb{R}^N)$ and there is $b > 0$ such that the set $V^b = \{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure.

(V_2) $0 = V(0) = \min V \leq V(x) < M$.

(K) $K(x) \in C(\mathbb{R}^N)$, $0 < \inf K \leq \sup K < \infty$.

(h_1) $H(x,u) = \int_0^u h(x,s)\,ds$, $h \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R}^+)$, $h(x,u) = o(|u|^{p-1})$ uniformly in x as $u \to 0$.

(h_2) There are $c_0 > 0$ and $p < q < p^*$ such that

$$\bigl|h(x,u)\bigr| \leq c_0\bigl(1 + |u|^{2\varpi q-1}\bigr) \quad \text{for all } (x,u).$$

(h_3) There are $\widetilde{c}_0 > 0$, $p < l$, $\mu < p^*$ such that $|H(x,u)| \geq \widetilde{c}_0(|u|^{2\varpi} + |u|)^l$ and $2\varpi\mu H(x,u) \leq h(x,u)u$.

A typical example satisfying (h_1)-(h_3) is the function $h(x,u) = P(x)(|u|^{2\varpi l-2} + |u|^{l-2})u$ with $p < l < p^*$ and $P(x)$ being positive and bounded.

Our main results of this paper are as follows.

Theorem 1.1 *Let* (V_1)-(V_2), (K), *and* (h_1)-(h_3) *hold. Then for any* $\sigma > 0$ *there is* $\mathcal{E}_\sigma > 0$ *such that if* $\varepsilon \leq \mathcal{E}_\sigma$ *then problem* (1.1) *has at least one positive solution* u_ε *satisfying*

(i)

$$\frac{\mu - p}{p}\int_{\mathbb{R}^N} H(x,u_\varepsilon) + \frac{1}{2\varpi N}\int_{\mathbb{R}^N} K(x)|u_\varepsilon|^{2\varpi p^*} \leq \sigma\varepsilon^N$$

and

(ii)

$$\frac{\mu - p}{p\mu}\int_{\mathbb{R}^N}\bigl[\varepsilon^p\bigl(1 + (2\varpi)^{p-1}|u_\varepsilon|^{p(2\varpi-1)}\bigr)|\nabla u_\varepsilon|^p + V(x)|u_\varepsilon|^p\bigr] \leq \sigma\varepsilon^N.$$

Moreover, $u_\varepsilon \to 0$ *in* $W^{1,p}(\mathbb{R}^N)$ *as* $\varepsilon \to 0$.

Theorem 1.2 *Assume that* (V_1)-(V_2), (K), *and* (h_1)-(h_3) *hold, and* $h(x,-u) = -h(x,u)$. *Then for any* $m \in \mathbb{N}$ *and* $\sigma > 0$ *there is* $\mathcal{E}_\sigma > 0$ *such that if* $\varepsilon \leq \mathcal{E}_\sigma$, *problem* (1.1) *has at least m pairs of solutions* $u_{\varepsilon,i}$, $-u_{\varepsilon,i}$, $i = 1, 2, \ldots, m$, *which satisfy the estimates* (i) *and* (ii) *in Theorem* 1.1. *Moreover,* $u_\varepsilon \to 0$ *in* $W^{1,p}(\mathbb{R}^N)$ *as* $\varepsilon \to 0$.

These results are new for the p-Laplacian equation and are a generalization of the results in [36].

Our goal is to prove the existence of semiclassical solutions of (1.1) by a variational approach. A function $u:\mathbb{R}^N\to\mathbb{R}$ is called a weak solution of (1.1) if $u\in W^{1,p}(\mathbb{R}^N)\cap L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and for all $\varphi\in C_0^\infty(\mathbb{R}^N)$ we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\varepsilon^p\bigl(1+(2\varpi)^{p-1}|u|^{p(2\varpi-1)}\bigr)|\nabla u|^{p-2}\nabla u\nabla\varphi\\&\quad+(2\varpi)^{p-1}(2\varpi-1)\varepsilon^p\int_{\mathbb{R}^N}|\nabla u|^p|u|^{p(2\varpi-1)-2}u\varphi\int_{\mathbb{R}^N}V(x)|u|^{p-2}u\varphi=\int_{\mathbb{R}^N}g(x,u)\varphi,\end{aligned}$$

where $G(x,u)=\int_0^u g(x,s)\,ds=\frac{1}{2p^*}K(x)|u|^{2p^*}+H(x,u)$. We point out that we cannot apply directly a variational method here because of the natural functional corresponding to (1.1) given by

$$I_\varepsilon(u)=\frac{\varepsilon^p}{p}\int_{\mathbb{R}^N}\bigl(1+(2\varpi)^{p-1}|u|^{p(2\varpi-1)}\bigr)|\nabla u|^p+\frac{1}{p}\int_{\mathbb{R}^N}V|u|^p-\int_{\mathbb{R}^N}G(x,u).\tag{1.6}$$

Because the nonhomogeneous term $\Delta_p(|u|^{2\varpi})|u|^{2\varpi-2}u$ prevents us from working directly with the functional I_ε, which is not well defined in $W^{1,p}(\mathbb{R}^N)$ since, for $u\in W^{1,p}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $\int_{\mathbb{R}^N}|u|^{p(2\varpi-1)}|\nabla u|^p=+\infty$ may hold. The other difficulty is the lack of compactness due to the unboundedness of the domain and the appearance of the Sobolev critical exponent $2p^*$. To overcome these difficulties we generalize an argument developed by Liu *et al.* in [28] for $p=2$, $\varpi=1$ (see also [37]). We make the change of variables $v=f^{-1}(u)$, and reformulate the problem into a new one which has an associated functional that is well defined and is of class C^1 on $W^{1,p}(\mathbb{R}^N)$.

Before we end this section, some notations are in order. We use $\int_{\mathbb{R}^N}g(x)$ to denote the integral $\int_{\mathbb{R}^N}g(x)\,dx$, $|u|_s$ denotes the usual $L^s(\mathbb{R}^N)$ norm $(\int_{\mathbb{R}^N}|u|^s\,dx)^{\frac{1}{s}}$. In the whole paper, C denotes a generic constant, which may vary from line to line.

The rest of this paper is organized as follows: in Section 2, we describe the analytic setting where we restate the problems in equivalent form by replacing ε^p with λ^{-1} other than the usual scaling (see [38]), due to the non-autonomy of nonlinearities. In Section 3, we show that the corresponding energy functional satisfies the (PS) condition at the levels less than $\alpha_0\lambda^{1-\frac{N}{p}}$ with some $\alpha_0>0$ independent of λ. Thus in Section 4 we construct minimax levels less than $\sigma\lambda^{1-\frac{N}{p}}$ for all λ large enough. We prove our main results in Section 5.

2 Equivalent variational problems

Let $\lambda=\varepsilon^{-p}$, then (1.1) reads

$$\begin{aligned}&-\Delta_p u+\lambda V(x)|u|^{p-2}u-\Delta_p\bigl(|u|^{2\varpi}\bigr)|u|^{2\varpi-2}u\\&\quad=\lambda K(x)|u|^{2\varpi p^*-2}u+\lambda h(x,u),\quad x\in\mathbb{R}^N,\end{aligned}\tag{2.1}$$

for $\lambda\to\infty$. And we introduce the space

$$E=\left\{u\in W^{1,p}\bigl(\mathbb{R}^N\bigr):\int_{\mathbb{R}^N}V(x)|u|^p<\infty\right\},$$

which is a Banach space with norm

$$\|u\|=\left(\int_{\mathbb{R}^N}|\nabla u|^p+V|u|^p\right)^{1/p}.$$

By (V_1), we know that the embedding $E \hookrightarrow W^{1,p}(\mathbb{R}^N)$ is continuous. Note the norm $\|\cdot\|$ is equivalent to the norm $\|\cdot\|_\lambda$ defined by

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^N} |\nabla u|^p + \lambda V|u|^p\right)^{1/p},$$

for each $\lambda > 0$. It is clear that, for each $s \in [p, p^*]$, there exists $\nu_s > 0$ (independent of λ) such that if $\lambda \geq 1$

$$|u|_s \leq \nu_s\|u\| \leq \nu_s\|u\|_\lambda \quad \text{for all } u \in E. \tag{2.2}$$

Let S be the best Sobolev constant,

$$S|u|_{p^*}^p \leq \int_{\mathbb{R}^N} |\nabla u|^p \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N).$$

We observe that the natural variational functional for (2.1)

$$\begin{aligned} I_\lambda(u) = {}& \frac{1}{p}\int_{\mathbb{R}^N} \bigl(1 + (2\varpi)^{p-1}|u|^{p(2\varpi-1)}\bigr)|\nabla u|^p + \frac{\lambda}{p}\int_{\mathbb{R}^N} V|u|^p \\ & - \frac{\lambda}{2\varpi p^*}\int_{\mathbb{R}^N} K|u|^{2\varpi p^*} - \lambda\int_{\mathbb{R}^N} H(x,u) \end{aligned}$$

is not still well defined in the general function space E. To overcome this difficulty we generalize an argument developed by Liu *et al.* in [28] for $p = 2$, $\varpi = 1$ (see also [37] for $\varpi = 1$). We make the change of variables $v = f^{-1}(u)$, where f is defined by

$$\begin{aligned} & f'(t) = \frac{1}{(1 + (2\varpi)^{p-1}|f(t)|^{p(2\varpi-1)})^{1/p}} \quad \text{on } [0, +\infty), \\ & f(t) = -f(-t) \quad \text{on } (-\infty, 0]. \end{aligned}$$

Thus we collect some properties of f.

Lemma 2.1 *The function $f(t)$ enjoys the following properties*:

(1) *f is uniquely defined C^2 function and invertible.*
(2) $|f'(t)| \leq 1$ *for all* $t \in \mathbb{R}$.
(3) $|f(t)| \leq |t|$ *for all* $t \in \mathbb{R}$.
(4) $\frac{f(t)}{t} \to 1$ *as* $t \to 0$.
(5) $|f(t)| \leq (2\varpi)^{\frac{1}{2p\varpi}}|t|^{\frac{1}{2\varpi}}$ *for all* $t \in \mathbb{R}$.
(6) $\frac{1}{2\varpi}f(t) \leq tf'(t) \leq f(t)$ *for all* $t \geq 0$.
(7) $\frac{f(t)}{t^{\frac{1}{2\varpi}}} \to a > 0$ *as* $t \to +\infty$.
(8) *There exists a positive constant C such that*

$$|f(t)| \geq \begin{cases} C|t|, & |t| \leq 1, \\ C|t|^{\frac{1}{2\varpi}}, & |t| \geq 1. \end{cases}$$

(9) $|f(t)f'(t)| \leq 1$.

Proof Similar to [37]. To prove (1), it is sufficient to remark that the function

$$y(s) = \frac{1}{(1 + (2\varpi)^{p-1}|s|^{p(2\varpi-1)})^{1/p}}$$

has a bound derivative. The point (2) is immediate by the definition of f. Inequality (3) is a consequence of (2) and the fact that $f(t)$ is an odd and concave function for $t > 0$. Next, we prove (4). As a consequence of the mean value theorem for integrals, we see that

$$f(t) = \int_0^t \frac{1}{(1 + (2\varpi)^{p-1}|f(s)|^{p(2\varpi-1)})^{1/p}}\,ds = \frac{t}{(1 + (2\varpi)^{p-1}|f(\xi)|^{p(2\varpi-1)})^{1/p}}, \quad \xi \in (0,t).$$

Since $f(0) = 0$, we get

$$\lim_{t\to 0}\frac{f(t)}{t} = \lim_{\xi\to 0}\frac{1}{(1 + (2\varpi)^{p-1}|f(\xi)|^{p(2\varpi-1)})^{1/p}} = 1.$$

To show item (5), we integrate $f'(t)(1 + (2\varpi)^{p-1}|f(t)|^{p(2\varpi-1)})^{1/p} = 1$ and we obtain

$$\int_0^t f'(s)\bigl(1 + (2\varpi)^{p-1}\bigl|f(s)\bigr|^{p(2\varpi-1)}\bigr)^{1/p}\,ds = t.$$

Using the change of variables $y = f(s)$, we get

$$t = \int_0^{f(t)} \bigl(1 + (2\varpi)^{p-1}|y|^{p(2\varpi-1)}\bigr)^{1/p}\,dy \geq (2\varpi)^{-\frac{1}{p}}\bigl|f(t)\bigr|^{2\varpi},$$

thus (5) is proved for $t \geq 0$. For $t < 0$, we use the fact $f(t)$ is odd. The first inequality in (6) is equivalent to $2\varpi t \geq f(t)(1 + (2\varpi)^{p-1}|f(t)|^{p(2\varpi-1)})^{1/p}$. To show the inequality, we study the function $G : \mathbb{R}^+ \to \mathbb{R}$, defined by $G(t) = 2\varpi t - f(t)(1 + (2\varpi)^{p-1}|f(t)|^{p(2\varpi-1)})^{1/p}$. Since $G(0) = 0$ and using the definition of f, we obtain, for all $t > 0$,

$$G'(t) = (2\varpi - 1)\bigl|f'(t)\bigr|^p > 0 \quad \text{if } \varpi \geq 1,$$

and the first inequality in (6) is proved. The second inequality in (6) is obtained in a similar way.

Now by point (4) it follows that $\lim_{t\to 0}\frac{f(t)}{t^{\frac{1}{2\varpi}}} = 0$ and the inequality (6) implies that for all $t > 0$

$$\frac{d}{dt}\left(\frac{f(t)}{t^{\frac{1}{2\varpi}}}\right) = t^{-(1+\frac{1}{2\varpi})}\left[tf'(t) - \frac{1}{2\varpi}f(t)\right] \geq 0.$$

Thus $\frac{f(t)}{t^{\frac{1}{2\varpi}}}$ is a nondecreasing function for $t > 0$ and this together with estimate (5) shows item (7). Point (8) is an immediate consequence of (4) and (7). Point (9) is obtained from the definition of f. □

After the change of variables, $I_\lambda(u)$ can be reduced to the following functional:

$$J_\lambda(v) = \frac{1}{p}\int_{\mathbb{R}^N}\bigl[|\nabla v|^p + \lambda V(x)\bigl|f(v)\bigr|^p\bigr] - \frac{\lambda}{2\varpi p^*}\int_{\mathbb{R}^N} K(x)\bigl|f(v)\bigr|^{2\varpi p^*} - \lambda\int_{\mathbb{R}^N} H\bigl(x,f(v)\bigr),$$

which is C^1 on the usual Sobolev space $W^{1,p}(\mathbb{R}^N)$. Moreover, the critical points of J_λ are the weak solutions of the following equation:

$$-\Delta_p v = \lambda f'(v)\big[K(x)\big|f(v)\big|^{2\varpi p^*-2}f(v) + h\big(x,f(v)\big) - V(x)\big|f(v)\big|^{p-2}f(v)\big] \quad \text{in } \mathbb{R}^N. \tag{2.3}$$

Now we can restate Theorem 1.1 and Theorem 1.2 as follows.

Theorem 2.2 *Let* (V_1)-(V_2), (K), *and* (h_1)-(h_3) *hold. Then for any* $\sigma > 0$ *there is* $\Lambda_\sigma > 0$ *such that if* $\lambda \ge \Lambda_\sigma$ *then problem* (2.3) *has at least one positive solution* v_λ *satisfying*

(i)

$$\frac{\mu-p}{p}\int_{\mathbb{R}^N} H\big(x,f(v_\lambda)\big) + \frac{1}{2\varpi N}\int_{\mathbb{R}^N} K(x)\big|f(v_\lambda)\big|^{2\varpi p^*} \le \sigma\lambda^{-\frac{N}{p}}$$

and

(ii)

$$\frac{\mu-p}{p\mu}\int_{\mathbb{R}^N}\big[|\nabla v_\lambda|^p + \lambda V(x)\big|f(v_\lambda)\big|^p\big] \le \sigma\lambda^{1-\frac{N}{p}}.$$

Moreover, $f(v_\lambda) \to 0$ *in* $W^{1,p}(\mathbb{R}^N)$ *as* $\lambda \to \infty$.

Theorem 2.3 *Let* (V_1)-(V_2), (K), *and* (h_1)-(h_3) *hold, and* $h(x,-u) = -h(x,u)$. *Then for any* $m \in \mathbb{N}$ *and* $\sigma > 0$ *there is* $\Lambda_\sigma^m > 0$ *such that if* $\lambda \ge \Lambda_\sigma^m$, *problem* (2.3) *has at least* m *pairs of solutions* $v_{\lambda,i}$, $-v_{\lambda,i}$, $i = 1,2,\ldots,m$, *which satisfy the estimates* (i) *and* (ii) *in Theorem* 2.2. *Moreover,* $f(v_{\lambda,i}) \to 0$ *in* $W^{1,p}(\mathbb{R}^N)$ *as* $\lambda \to \infty$.

Remark 2.4 To prove the existence of positive solutions, we may consider in E

$$J_\lambda^+(v) = \frac{1}{p}\int_{\mathbb{R}^N}\big[|\nabla v|^p + \lambda V(x)\big|f(v)\big|^p\big] - \frac{\lambda}{2\varpi p^*}\int_{\mathbb{R}^N} K(x)\big|f(v^+)\big|^{2\varpi p^*} - \lambda\int_{\mathbb{R}^N} H\big(x,f(v^+)\big),$$

where $v^\pm = \pm\max\{\pm v, 0\}$, then $J_\lambda^+ \in C^1(E,\mathbb{R})$ and critical points of J_λ^+ are positive solutions for (2.3).

3 Behaviors of (PS) sequences

Let E be a real Banach space and $J_\lambda : E \to \mathbb{R}$ be a function of class C^1. We say that $\{v_n\} \subset E$ is a $(\mathrm{PS})_c$ sequence if $J_\lambda(v_n) \to c$ and $J'_\lambda(v_n) \to 0$. J_λ is said to satisfy the $(\mathrm{PS})_c$ condition if any $(\mathrm{PS})_c$ sequence contains a convergent subsequence.

The main result of the section is the following compactness result.

Lemma 3.1 *Assume that* (V_1)-(V_2), (K), *and* (h_1)-(h_3) *are satisfied. Let* $\{v_n\}$ *be a* $(PS)_c$ *sequence for* J_λ. *Then* $c \ge 0$ *and* $\{v_n\}$ *is bounded in* E.

Proof Let $\{v_n\}$ be a $(\mathrm{PS})_c$ sequence for J_λ, we have

$$J_\lambda(v_n) - \frac{1}{\mu}J'_\lambda(v_n)v_n = c + o(1) + \varepsilon_n\|v_n\|_\lambda, \tag{3.1}$$

where $\varepsilon_n \to 0$ as $n \to \infty$.

By (h_3) and Lemma 2.1(6), we deduce

$$\begin{aligned}
&J_\lambda(v_n) - \frac{1}{\mu}J'_\lambda(v_n)v_n \\
&\quad= \frac{1}{p}\int_{\mathbb{R}^N}\left[|\nabla v_n|^p + \lambda V(x)\left|f(v_n)\right|^p\right] \\
&\qquad - \frac{1}{\mu}\int_{\mathbb{R}^N}\left[|\nabla v_n|^p + \lambda V(x)\left|f(v_n)\right|^{p-2}f(v_n)f'(v_n)v_n\right] \\
&\qquad + \lambda\int_{\mathbb{R}^N}\left[\frac{1}{\mu}h\big(x,f(v_n)\big)f'(v_n)v_n - H\big(x,f(v_n)\big)\right] \\
&\qquad + \lambda\int_{\mathbb{R}^N}\left[\frac{1}{\mu}K(x)\left|f(v_n)\right|^{2\varpi p^*-2}f(v_n)f'(v_n)v_n - \frac{1}{2\varpi p^*}K(x)\left|f(v_n)\right|^{2\varpi p^*}\right] \\
&\quad\geq \left(\frac{1}{p}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}\left[|\nabla v_n|^p + \lambda V(x)\left|f(v_n)\right|^p\right] \\
&\qquad + \lambda\int_{\mathbb{R}^N}\left[\frac{1}{2\varpi\mu}h\big(x,f(v_n)\big)f(v_n) - H\big(x,f(v_n)\big)\right] \\
&\qquad + \lambda\left(\frac{1}{2\varpi\mu}-\frac{1}{2\varpi p^*}\right)\int_{\mathbb{R}^N}K(x)\left|f(v_n)\right|^{2\varpi p^*} \\
&\quad\geq \left(\frac{1}{p}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}\left[|\nabla v_n|^p + \lambda V(x)\left|f(v_n)\right|^p\right].
\end{aligned} \tag{3.2}$$

Hence combining (3.1) and (3.2), for n large enough,

$$\left(\frac{1}{p}-\frac{1}{\mu}\right)\int_{\mathbb{R}^N}\left[|\nabla v_n|^p + \lambda V(x)\left|f(v_n)\right|^p\right] \leq c + o(1) + \varepsilon_n\|v_n\|_\lambda,$$

which implies that there exists $C > 0$ such that

$$\int_{\mathbb{R}^N}\left[|\nabla v_n|^p + \lambda V(x)\left|f(v_n)\right|^p\right] < C. \tag{3.3}$$

Taking the limit in (3.2), we can obtain $c \geq 0$.

In the following, we need to show $\{v_n\}$ is bounded in E. From (3.3), we need to prove that $\int_{\mathbb{R}^N} V(x)|v_n|^p$ is bounded.

By (V_2),

$$\int_{\{|v_n|>1\}} V(x)|v_n|^p \leq M\int_{\{|v_n|>1\}}|v_n|^{p^*} \leq MS^{-\frac{p^*}{p}}\left(\int_{\mathbb{R}^N}|\nabla v_n|^p\right)^{\frac{p^*}{p}}$$

and using Lemma 2.1(8),

$$\int_{\{|v_n|\leq 1\}} V(x)|v_n|^p \leq \frac{1}{C^2}\int_{\{|v_n|\leq 1\}} V(x)\left|f(v_n)\right|^p \leq \frac{1}{C^2}\int_{\mathbb{R}^N} V(x)\left|f(v_n)\right|^p.$$

These estimates imply that $\{v_n\}$ is bounded in E. □

From Lemma 3.1, we know that every $(PS)_c$ sequence is bounded, hence, without loss of generality, we may assume $v_n \rightharpoonup v$ in E and $L^p(\mathbb{R}^N)$, $v_n \to v$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $s \in [p,p^*)$, and $v_n(x) \to v(x)$ a.e. for $x \in \mathbb{R}^N$. Obviously, v is a critical point of J_λ.

Lemma 3.2 *Assume that* (V_1)-(V_2), (K), *and* (h_1)-(h_3) *are satisfied. Let* $s \in [p, 2\varpi p^*)$ *and* $\{v_n\}$ *be a bounded* $(PS)_c$ *sequence. Then there is a subsequence* $\{v_{nj}\}$ *such that, for each* $\varepsilon > 0$, *there exists* $r_\varepsilon > 0$

$$\limsup_{j\to\infty} \int_{B_j\setminus B_r} \left|f(v_{nj})\right|^s \le \varepsilon$$

for all $r \ge r_\varepsilon$, *where* $B_k = \{x \in \mathbb{R}^N, |x| \le k\}$.

Proof For $s \in [2\varpi p, 2\varpi p^*)$. Noting that $v_n \to v$ in $L^{\frac{s}{2\varpi}}_{\mathrm{loc}}$ as $n \to \infty$, we have, for each $j \in \mathbb{N}$,

$$\int_{B_j} |v_n|^{\frac{s}{2\varpi}} \to \int_{B_j} |v|^{\frac{s}{2\varpi}} \quad \text{as } n \to \infty,$$

and there exists $\hat{n}_j \in \mathbb{N}$ such that

$$\int_{B_j} \left(|v_n|^{\frac{s}{2\varpi}} - |v|^{\frac{s}{2\varpi}}\right) < \frac{1}{j} \quad \text{as } n = \hat{n}_j + i, i = 1, 2, \ldots.$$

Without loss of generality, we can assume $\hat{n}_{j+1} \ge \hat{n}_j$. In particular, for $n_j = \hat{n}_j + j$, we deduce

$$\int_{B_j} \left(|v_{nj}|^{\frac{s}{2\varpi}} - |v|^{\frac{s}{2\varpi}}\right) < \frac{1}{j}.$$

Observe that there exists an r_ε such that $r \ge r_\varepsilon$, and the following relation is satisfied:

$$\int_{\mathbb{R}^N\setminus B_r} |v|^{\frac{s}{2\varpi}} < \varepsilon. \tag{3.4}$$

We have

$$\begin{aligned}\int_{B_j\setminus B_r} |v_{nj}|^{\frac{s}{2\varpi}} &= \int_{B_j} \left(|v_{nj}|^{\frac{s}{2\varpi}} - |v|^{\frac{s}{2\varpi}}\right) + \int_{B_j\setminus B_r} |v|^{\frac{s}{2\varpi}} + \int_{B_r} \left(|v|^{\frac{s}{2\varpi}} - |v_{nj}|^{\frac{s}{2\varpi}}\right) \\ &\le \frac{1}{j} + \int_{\mathbb{R}^N\setminus B_r} |v|^{\frac{s}{2\varpi}} + \int_{B_r} \left(|v|^{\frac{s}{2\varpi}} - |v_{nj}|^{\frac{s}{2\varpi}}\right) \le \varepsilon \quad \text{as } j \to \infty.\end{aligned}$$

From Lemma 2.1(5), we know

$$\limsup_{j\to\infty} \int_{B_j\setminus B_r} \left|f(v_{nj})\right|^s \le C \limsup_{j\to\infty} \int_{B_j\setminus B_r} |v_{nj}|^{\frac{s}{2\varpi}} \le \varepsilon$$

for all $r \ge r_\varepsilon$.

For $s \in [p, 2\varpi p)$, we only need Lemma 2.1. □

Remark 3.3 From the proof of Lemma 3.2, we can find the same subsequence $\{v_{nj}\}$ such that the result of Lemma 3.2 holds for both $s = p$ and $s = q$.

Let $\eta : [0, \infty) \to [0, 1]$ be a smooth function satisfying $\eta(t) = 1$ if $t \le 1$, $\eta(t) = 0$ if $t \ge p$. Define $\tilde{v}_j = \eta(\frac{p|x|}{j})v(x)$. Clearly,

$$\|\tilde{v}_j - v\|_\lambda \to 0 \quad \text{as } j \to \infty. \tag{3.5}$$

Lemma 3.4 *Assume that* (V_1)-(V_2), (K), *and* (h_1)-(h_3) *are satisfied. Let* $\{v_{nj}\}$ *be defined as in Lemma* 3.2, *then we have*

$$\lim_{j\to\infty}\int_{\mathbb{R}^N}\bigl[h\bigl(x,f(v_{nj})\bigr)f'(v_{nj})-h\bigl(x,f(v_{nj}-\tilde{v}_j)\bigr)f'(v_{nj}-\tilde{v}_j)-h\bigl(x,f(\tilde{v}_j)\bigr)f'(\tilde{v}_j)\bigr]\varphi=0$$

uniformly in $\varphi\in E$ *with* $\|\varphi\|_\lambda\le 1$.

Proof From (3.5) and local compactness of the Sobolev embedding, for any $r>0$,

$$\lim_{j\to\infty}\left|\int_{B_r}\bigl[h\bigl(x,f(v_{nj})\bigr)f'(v_{nj})-h\bigl(x,f(v_{nj}-\tilde{v}_j)\bigr)f'(v_{nj}-\tilde{v}_j)-h\bigl(x,f(\tilde{v}_j)\bigr)f'(\tilde{v}_j)\bigr]\varphi\right|=0$$

uniformly in $\|\varphi\|_\lambda\le 1$.

Let $s=p,q$. By (2.2)

$$|\varphi|_s\le\nu_s\|\varphi\|_\lambda\le\nu_s,$$

and, for any $\varepsilon>0$, it follows from (3.4) that

$$\limsup_{j\to\infty}\int_{B_j\setminus B_r}|\tilde{v}_j|^s\le\int_{\mathbb{R}^N\setminus B_r}|v|^s<\varepsilon,$$

for all $r\ge r_\varepsilon$. By (h_1), (h_2), and Lemma 2.1(2), (5), and (6), we have, for all $v\in E$,

$$\begin{aligned}\bigl|h\bigl(x,f(v)\bigr)f'(v)\bigr||\varphi| &\le c_0\bigl(\bigl|f(v)\bigr|^{p-1}+\bigl|f(v)\bigr|^{2\varpi q-1}\bigr)\bigl|f'(v)\bigr||\varphi| \\ &\le C\left(\bigl|f(v)\bigr|^{p-1}+\frac{|f(v)|^{2\varpi q}}{|v|}\right)|\varphi| \\ &\le C\bigl(\bigl|f(v)\bigr|^{p-1}+|v|^{q-1}\bigr)|\varphi| \\ &\le C\bigl(|v|^{p-1}+|v|^{q-1}\bigr)|\varphi|. \end{aligned}\tag{3.6}$$

Therefore, using Lemma 3.2 and Remark 3.3,

$$\begin{aligned}&\limsup_{j\to\infty}\left|\int_{\mathbb{R}^N}\bigl[h\bigl(x,f(v_{nj})\bigr)f'(v_{nj})-h\bigl(x,f(\tilde{v}_j)\bigr)f'(\tilde{v}_j)-h\bigl(x,f(v_{nj}-\tilde{v}_j)\bigr)f'(v_{nj}-\tilde{v}_j)\bigr]\varphi\right| \\ &\quad=\limsup_{j\to\infty}\left|\int_{B_j\setminus B_r}\bigl[h\bigl(x,f(v_{nj})\bigr)f'(v_{nj})-h\bigl(x,f(\tilde{v}_j)\bigr)f'(\tilde{v}_j)\right. \\ &\qquad\left.-h\bigl(x,f(v_{nj}-\tilde{v}_j)\bigr)f'(v_{nj}-\tilde{v}_j)\bigr]\varphi\right| \\ &\quad\le C\limsup_{j\to\infty}\int_{B_j\setminus B_r}\bigl(\bigl|f(v_{nj})\bigr|^{p-1}+\bigl|f(\tilde{v}_j)\bigr|^{p-1}+\bigl|f(v_{nj}-\tilde{v}_j)\bigr|^{p-1}\bigr)|\varphi| \\ &\qquad+C\limsup_{j\to\infty}\int_{B_j\setminus B_r}\bigl(|v_{nj}|^{q-1}+|\tilde{v}_j|^{q-1}+|v_{nj}-\tilde{v}_j|^{q-1}\bigr)|\varphi| \\ &\quad\le C\limsup_{j\to\infty}\bigl(\bigl|f(v_{nj})\bigr|^{p-1}_{L^p(B_j\setminus B_r)}+\bigl|f(\tilde{v}_j)\bigr|^{p-1}_{L^p(B_j\setminus B_r)}\bigr)|\varphi|_p \\ &\qquad+C\limsup_{j\to\infty}\bigl(|v_{nj}|^{p-1}_{L^p(B_j\setminus B_r)}+|\tilde{v}_j|^{p-1}_{L^p(B_j\setminus B_r)}\bigr)|\varphi|_p\end{aligned}$$

$$
\begin{aligned}
&+ C \limsup_{j\to\infty}\bigl(|v_{nj}|^{q-1}_{L^q(B_j\setminus B_r)} + |\tilde{v}_j|^{q-1}_{L^q(B_j\setminus B_r)}\bigr)|\varphi|_q \\
&\le C\bigl(\varepsilon^{\frac{p-1}{p}} + \varepsilon^{\frac{q-1}{q}}\bigr),
\end{aligned}
$$

which implies the conclusion as required. □

Lemma 3.5 *Assume that* (V_1)-(V_2), (K), *and* (h_1)-(h_3) *are satisfied. Let* $\{v_{nj}\}$ *be the defined in Lemma* 3.2, *then we have, as* $j \to \infty$,

(i) $J_\lambda(v_{nj} - \tilde{v}_j) \to c - J_\lambda(v)$;

(ii) $J'_\lambda(v_{nj} - \tilde{v}_j) \to 0$.

Proof

$$
\begin{aligned}
J_\lambda(v_{nj} - \tilde{v}_j) = {} & J_\lambda(v_{nj}) - J_\lambda(\tilde{v}_j) \\
& - \frac{1}{p}\int_{\mathbb{R}^N}\bigl[|\nabla v_{nj}|^p - \bigl|\nabla(v_{nj} - \tilde{v}_j)\bigr|^p - |\nabla \tilde{v}_j|^p\bigr] \\
& - \frac{\lambda}{p}\int_{\mathbb{R}^N} V(x)\bigl[\bigl|f(v_{nj})\bigr|^p - \bigl|f(v_{nj} - \tilde{v}_j)\bigr|^p - \bigl|f(\tilde{v}_j)\bigr|^p\bigr] \\
& + \frac{\lambda}{2\varpi p^*}\int_{\mathbb{R}^N} K(x)\bigl[\bigl|f(v_{nj})\bigr|^{2\varpi p^*} - \bigl|f(v_{nj} - \tilde{v}_j)\bigr|^{2\varpi p^*} - \bigl|f(\tilde{v}_j)\bigr|^{2\varpi p^*}\bigr] \\
& + \lambda\int_{\mathbb{R}^N}\bigl[H\bigl(x, f(v_{nj})\bigr) - H\bigl(x, f(v_{nj} - \tilde{v}_j)\bigr) - H\bigl(x, f(\tilde{v}_j)\bigr)\bigr].
\end{aligned}
$$

By (h_1)-(h_3) and Lemma 2.1, similar to the proof of Lemma 3.4, it is not difficult to check that

$$
\lim_{j\to\infty}\int_{\mathbb{R}^N}\bigl[H\bigl(x, f(v_{nj})\bigr) - H\bigl(x, f(v_{nj} - \tilde{v}_j)\bigr) - H\bigl(x, f(\tilde{v}_j)\bigr)\bigr] = 0.
$$

By (3.5) and the Brezis-Lieb lemma, we can deduce that

$$
\lim_{j\to\infty}\int_{\mathbb{R}^N}\bigl[|\nabla v_{nj}|^p - \bigl|\nabla(v_{nj} - \tilde{v}_j)\bigr|^p - |\nabla \tilde{v}_j|^p\bigr] = 0.
$$

Recalling that, for any fixed $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that, for all $a, b \in \mathbb{R}$,

$$
\bigl||a + b|^s - |a|^s\bigr| \le \varepsilon|a|^s + C_\varepsilon|b|^s, \quad 1 < s < \infty,
$$

therefore,

$$
\bigl|f(v_{nj})\bigr|^p - \bigl|f(v_{nj} - \tilde{v}_j)\bigr|^p \le \varepsilon\bigl|f(v_{nj} - \tilde{v}_j)\bigr|^p + C_\varepsilon\bigl|f'(v_{nj} - \theta_j\tilde{v}_j)\tilde{v}_j\bigr|^p, \quad 0 < \theta_j < 1.
$$

Using Lemma 2.1(3), we obtain

$$
\begin{aligned}
\Gamma_j^\varepsilon &= \bigl(\bigl|f(v_{nj})\bigr|^p - \bigl|f(v_{nj} - \tilde{v}_j)\bigr|^p - \bigl|f(\tilde{v}_j)\bigr|^p - \varepsilon\bigl|f(v_{nj} - \tilde{v}_j)\bigr|^p\bigr)^+ \\
&\le \bigl(\bigl|f(\tilde{v}_j)\bigr|^p + C_\varepsilon\bigl|f'(v_{nj} - \theta_j\tilde{v}_j)\tilde{v}_j\bigr|^p\bigr) \\
&\le C|v|^p.
\end{aligned}
$$

Applying the Lebesgue dominated convergence theorem, we know that $\int_{\mathbb{R}^N} \Gamma_j^\varepsilon \to 0$ as $j \to \infty$. Since $V(x)$ is bounded and

$$\left| \left|f(v_{nj})\right|^p - \left|f(v_{nj} - \tilde{v}_j)\right|^p - \left|f(\tilde{v}_j)\right|^p \right| \le \Gamma_j^\varepsilon + \varepsilon \left|f(v_{nj} - \tilde{v}_j)\right|^p,$$

we deduce that

$$\lim_{j\to\infty} \int_{\mathbb{R}^N} V(x)\left[\left|f(v_{nj})\right|^p - \left|f(v_{nj} - \tilde{v}_j)\right|^p - \left|f(\tilde{v}_j)\right|^p\right] = 0.$$

Similarly, we can obtain

$$\lim_{j\to\infty} \int_{\mathbb{R}^N} K(x)\left[\left|f(v_{nj})\right|^{2\varpi p^*} - \left|f(v_{nj} - \tilde{v}_j)\right|^{2\varpi p^*} - \left|f(\tilde{v}_j)\right|^{2\varpi p^*}\right] = 0.$$

These, together with the facts $J_\lambda(v_{nj}) \to c$ and $J_\lambda(\tilde{v}_j) \to J_\lambda(v)$ as $j \to \infty$, give conclusion (i).

To verify conclusion (ii), observe that, for any $\varphi \in E$,

$$\begin{aligned}
J_\lambda'(v_{nj} - \tilde{v}_j)\varphi &= J_\lambda'(v_{nj})\varphi - J_\lambda'(\tilde{v}_j)\varphi \\
&\quad - \int_{\mathbb{R}^N} \left[|\nabla v_{nj}|^{p-2}\nabla v_{nj} - \left|\nabla(v_{nj} - \tilde{v}_j)\right|^{p-2}\nabla(v_{nj} - \tilde{v}_j) - |\nabla\tilde{v}_j|^{p-2}\nabla\tilde{v}_j\right]\nabla\varphi \\
&\quad - \lambda \int_{\mathbb{R}^N} V(x)\left[\left|f(v_{nj})\right|^{p-2} f(v_{nj})f'(v_{nj}) - \left|f(\tilde{v}_j)\right|^{p-2} f(\tilde{v}_j)f'(\tilde{v}_j)\right. \\
&\quad \left. - \left|f(v_{nj} - \tilde{v}_j)\right|^{p-2} f(v_{nj} - \tilde{v}_j)f'(v_{nj} - \tilde{v}_j)\right]\varphi \\
&\quad + \lambda \int_{\mathbb{R}^N} K(x)\left[\left|f(v_{nj})\right|^{2\varpi p^*-2} f(v_{nj})f'(v_{nj}) - \left|f(\tilde{v}_j)\right|^{2\varpi p^*-2} f(\tilde{v}_j)f'(\tilde{v}_j)\right. \\
&\quad \left. - \left|f(v_{nj} - \tilde{v}_j)\right|^{2\varpi p^*-2} f(v_{nj} - \tilde{v}_j)f'(v_{nj} - \tilde{v}_j)\right]\varphi \\
&\quad + \lambda \int_{\mathbb{R}^N} \left[h\big(x, f(v_{nj})\big)f'(v_{nj}) - h\big(x, f(\tilde{v}_j)\big)f'(\tilde{v}_j)\right. \\
&\quad \left. - h\big(x, f(v_{nj} - \tilde{v}_j)\big)f'(v_{nj} - \tilde{v}_j)\right]\varphi.
\end{aligned}$$

By (3.5) and Lemma 3.2 in [39], we can check that

$$\lim_{j\to\infty} \left(\int_{\mathbb{R}^N} \left||\nabla v_{nj}|^{p-2}\nabla v_{nj} - \left|\nabla(v_{nj} - \tilde{v}_j)\right|^{p-2}\nabla(v_{nj} - \tilde{v}_j) - |\nabla\tilde{v}_j|^{p-2}\nabla\tilde{v}_j\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} = 0.$$

Hence we have

$$\lim_{j\to\infty} \int_{\mathbb{R}^N} \left[|\nabla v_{nj}|^{p-2}\nabla v_{nj} - \left|\nabla(v_{nj} - \tilde{v}_j)\right|^{p-2}\nabla(v_{nj} - \tilde{v}_j) - |\nabla\tilde{v}_j|^{p-2}\nabla\tilde{v}_j\right]\nabla\varphi = 0.$$

By Lemma 2.1(6) and (5), we have

$$\left|\left|f(v)\right|^{2\varpi p^*-2} f(v)f'(v)\right| \le \frac{|f(v)|^{2\varpi p^*}}{|v|} \le C|v|^{p^*-1}.$$

Then by the Rellich imbedding theorem and the continuity of the Nemytskii operator, we obtain

$$\lim_{j\to\infty}\int_{\mathbb{R}^N} K(x)\big[\big|f(v_{nj})\big|^{2\varpi p^*-2}f(v_{nj})f'(v_{nj}) - \big|f(\tilde{v}_j)\big|^{2\varpi p^*-2}f(\tilde{v}_j)f'(\tilde{v}_j) - \big|f(v_{nj}-\tilde{v}_j)\big|^{2\varpi p^*-2}f(v_{nj}-\tilde{v}_j)f'(v_{nj}-\tilde{v}_j)\big]\varphi = 0$$

uniformly in $\|\varphi\|_\lambda \le 1$. Moreover, since $V(x)$ is bounded, using the same arguments as in Lemma 3.4 and (3.6), we obtain

$$\lim_{j\to\infty}\int_{\mathbb{R}^N} V(x)\big[\big|f(v_{nj})\big|^{p-2}f(v_{nj})f'(v_{nj}) - \big|f(\tilde{v}_j)\big|^{p-2}f(\tilde{v}_j)f'(\tilde{v}_j) - \big|f(v_{nj}-\tilde{v}_j)\big|^{p-2}f(v_{nj}-\tilde{v}_j)f'(v_{nj}-\tilde{v}_j)\big]\varphi = 0$$

and

$$\lim_{j\to\infty}\int_{\mathbb{R}^N}\big[h\big(x,f(v_{nj})\big)f'(v_{nj}) - h\big(x,f(\tilde{v}_j)\big)f'(\tilde{v}_j) - h\big(x,f(v_{nj}-\tilde{v}_j)\big)f'(v_{nj}-\tilde{v}_j)\big]\varphi = 0,$$

uniformly in $\|\varphi\|_\lambda \le 1$, proving (ii). □

Lemma 3.6 *Assume that* (V_1)-(V_2), (K), *and* (h_1)-(h_3) *are satisfied. Then there exists a constant* α_0 *independent of* λ *such that, for any* $(PS)_c$ *sequence* $\{v_n\}$ *for* J_λ *with* $v_n \rightharpoonup v$, *either* $v_n \to v$ *for a subsequence or*

$$c - J_\lambda(v) \ge \alpha_0 \lambda^{1-\frac{N}{p}}.$$

Proof Taking

$$v_j^1 = v_{nj} - \tilde{v}_j,$$

then $v_{nj} - v = v_j^1 + (\tilde{v}_j - v)$, by (3.5), $v_{nj} \to v$ if and only if $v_j^1 \to 0$. Assume that $\{v_n\}$ has no convergent subsequence. Then $\liminf_{n\to\infty}\|v_n - v\|_\lambda > 0$. By Lemma 3.5, one also has a subsequence that $J_\lambda(v_j^1) \to c - J_\lambda(v) > 0$ and $J'_\lambda(v_j^1) \to 0$.

Denote

$$V_b(x) = \max\big\{V(x), b\big\},$$

where b is the positive constant from assumption of (V_1). Since the V^b has a finite measure and $v_j^1 \to 0$ in L^p_{loc}, we see that

$$\int_{\mathbb{R}^N} V(x)\big|f\big(v_j^1\big)\big|^p = \int_{\mathbb{R}^N} V_b\big|f\big(v_j^1\big)\big|^p + o(1). \tag{3.7}$$

From (h_1)-(h_2), we deduce for any fixed $\varepsilon > 0$ that there exists C_ε such that

$$h\big(x,f(v)\big)f(v) \le \varepsilon\big|f(v)\big|^p + C_\varepsilon\big|f(v)\big|^{2\varpi p^*},$$

thus by (K), we can find a constant $C_{\frac{b}{2\varpi}}$ such that

$$h(x,f(v))f(v)+K(x)\left|f(v)\right|^{2\varpi p^*}\le \frac{b}{2\varpi}\left|f(v)\right|^p+C_{\frac{b}{2\varpi}}\left|f(v)\right|^{2\varpi p^*}\quad \text{for all } (x,v). \tag{3.8}$$

From Lemma 2.1(5) and (6), (3.7), and (3.8), we know

$$\begin{aligned}
&\frac{S}{2\varpi}\left|f\left(v_j^1\right)\right|_{2\varpi p^*}^{2\varpi p}\\
&\quad\le S\left|v_j^1\right|_{p^*}^p\le \int_{\mathbb{R}^N}\left[\left|\nabla v_j^1\right|^p+\lambda V(x)\left|f\left(v_j^1\right)\right|^p\right]-\lambda\int_{\mathbb{R}^N}V(x)\left|f\left(v_j^1\right)\right|^p\\
&\quad\le 2\varpi\int_{\mathbb{R}^N}\left[\left|\nabla v_j^1\right|^p+\lambda V(x)\left|f\left(v_j^1\right)\right|^{p-2}f\left(v_j^1\right)f'\left(v_j^1\right)v_j^1\right]-\lambda\int_{\mathbb{R}^N}V(x)\left|f\left(v_j^1\right)\right|^p\\
&\quad\le 2\varpi\lambda\int_{\mathbb{R}^N}h\left(x,f\left(v_j^1\right)\right)f'\left(v_j^1\right)v_j^1+2\varpi\lambda\int_{\mathbb{R}^N}K(x)\left|f\left(v_j^1\right)\right|^{2\varpi p^*-2}f\left(v_j^1\right)f'\left(v_j^1\right)v_j^1\\
&\qquad-\lambda\int_{\mathbb{R}^N}V(x)\left|f\left(v_j^1\right)\right|^p+o(1)\\
&\quad\le 2\varpi\lambda\int_{\mathbb{R}^N}\left[h\left(x,f\left(v_j^1\right)\right)f\left(v_j^1\right)+K(x)\left|f\left(v_j^1\right)\right|^{2\varpi p^*}\right]\\
&\qquad-\lambda\int_{\mathbb{R}^N}V_b(x)\left|f\left(v_j^1\right)\right|^p+o(1)\\
&\quad\le 2\varpi\lambda\int_{\mathbb{R}^N}\left[h\left(x,f\left(v_j^1\right)\right)f\left(v_j^1\right)+K(x)\left|f\left(v_j^1\right)\right|^{2\varpi p^*}\right]-\lambda b\int_{\mathbb{R}^N}\left|f\left(v_j^1\right)\right|^p+o(1)\\
&\quad\le 2\varpi\lambda C_{\frac{b}{2\varpi}}\left|f\left(v_j^1\right)\right|_{2\varpi p^*}^{2\varpi p^*}+o(1).
\end{aligned}\tag{3.9}$$

We have

$$J_\lambda\left(v_j^1\right)-\frac{1}{p}J'_\lambda\left(v_j^1\right)v_j^1\ge\frac{\lambda}{2\varpi N}\int_{\mathbb{R}^N}K(x)\left|f\left(v_j^1\right)\right|^{2\varpi p^*}\ge\frac{\lambda K_{\min}}{2\varpi N}\int_{\mathbb{R}^N}\left|f\left(v_j^1\right)\right|^{2\varpi p^*},$$

where $K_{\min}=\inf K(x)>0$. It is easy to see that

$$\left|f\left(v_j^1\right)\right|_{2\varpi p^*}^{2\varpi p^*}\le\frac{2\varpi N(c-J_\lambda(v))}{\lambda K_{\min}}+o(1). \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$\begin{aligned}
\frac{S}{4\varpi^2}&\le\lambda C_{\frac{b}{2\varpi}}\left|f\left(v_j^1\right)\right|_{2\varpi p^*}^{2\varpi p^*-2\varpi p}+o(1)\\
&\le\lambda C_{\frac{b}{2\varpi}}\left(\frac{2\varpi N(c-J_\lambda(v))}{\lambda K_{\min}}\right)^{p/N}+o(1)\\
&=\lambda^{1-\frac{p}{N}}C_{\frac{b}{2\varpi}}\left(\frac{2\varpi N}{K_{\min}}\right)^{p/N}\left(c-J_\lambda(v)\right)^{\frac{p}{N}}+o(1),
\end{aligned}$$

or, equivalently,

$$\alpha_0\lambda^{1-\frac{N}{p}}\le c-J_\lambda(v)+o(1),$$

where

$$\alpha_0 = \left(\frac{S}{4\varpi^2}\right)^{\frac{p}{N}} C_{\frac{b}{2\varpi}}^{-\frac{p}{N}} \frac{K_{\min}}{2\varpi N}.$$

The proof is complete. □

From Lemma 3.6, we have the following conclusions.

Lemma 3.7 *Assume that* (V_1)-(V_2), (K), *and* (h_1)-(h_3) *are satisfied. Then* J_λ *satisfies the* $(PS)_c$ *condition for all* $c < \alpha_0 \lambda^{1-\frac{N}{p}}$.

Lemma 3.8 *Assume that* (V_1)-(V_2), (K), *and* (h_1)-(h_3) *are satisfied. Then* J_λ^+ *satisfies the* $(PS)_c$ *condition for all* $c < \alpha_0 \lambda^{1-\frac{N}{p}}$.

4 The mountain pass geometry

Lemma 4.1 *Let* E *be a real Banach space and* $J : E \to \mathbb{R}$ *be a functional of class of* C^1. *Assume that* $\widetilde{E}$ *is a closed subset of* E *which disconnects (arcwise)* E *into distinct connected components* E_1 *and* E_2. *Suppose further that* $J(0) = 0$ *and*

(i) $0 \in E_1$ *and there exists* $\alpha > 0$ *such that* $J|_{\widetilde{E}} \geq \alpha > 0$;

(ii) *there exists* $e \in E_2$ *such that* $J(e) < 0$.

Then J *possesses a* $(PS)_c$ *sequence with* $c \geq \alpha > 0$ *given by*

$$c = \inf_{\gamma \in \Lambda} \max_{0 \leq t \leq 1} J(\gamma(t)),$$

where $\Lambda = \{\gamma \in C([0,1], E) : \gamma(0) = 0, J(\gamma(1)) < 0\}$.

From now on, we consider $\lambda \geq 1$, and the following lemma implies that J_λ possesses the mountain pass geometry.

Lemma 4.2 *Assume that* (V_1)-(V_2), (K), *and* (h_1)-(h_3) *are satisfied. For each* λ *there is a closed subset* $\widetilde{E}_\lambda$ *of* E *which disconnects (arcwise)* E *into distinct connected components* E_1 *and* E_2. *Then* J_λ *satisfies*:

(i) $0 \in E_1$ *and there exists* $\alpha_\lambda > 0$ *such that* $J_\lambda|_{\widetilde{E}_\lambda} \geq \alpha_\lambda > 0$.

(ii) *For any finite-dimensional subspace* $F \subset E$,

$$J_\lambda(v) \to -\infty \quad \text{as } v \in F \text{ and } \|v\|_\lambda \to \infty.$$

(iii) *For any* $\sigma > 0$ *there exists* $\Lambda_\sigma > 0$ *such that, for each* $\lambda \geq \Lambda_\sigma$, *there is* $\bar{e}_\lambda \in E_2$ *such that* $J_\lambda(\bar{e}_\lambda) < 0$ *and*

$$\max_{t \in [0,1]} J_\lambda(t\bar{e}_\lambda) \leq \sigma \lambda^{1-\frac{N}{p}}.$$

Proof (i) First note that, for each λ, $J_\lambda(0) = 0$. Now, for every $\rho > 0$, define

$$\widetilde{E}_{\lambda,\rho} = \left\{ v \in E : \int_{\mathbb{R}^N} \left[|\nabla v|^p + \lambda V(x) \left| f(v) \right|^p \right] = \rho^p \right\}.$$

Since $\int_{\mathbb{R}^N}[|\nabla v|^p + \lambda V(x)|f(v)|^p]$ is continuous, then $\widetilde{E}_{\lambda,\rho}$ is a closed subset which disconnects the space E. From (h_1)-(h_2), for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$\int_{\mathbb{R}^N} H(x,f(v)) \le \delta \int_{\mathbb{R}^N} |f(v)|^p + C_\delta \int_{\mathbb{R}^N} |f(v)|^{2\varpi q}. \tag{4.1}$$

From Lemma 2.1(3), we know $|f(v)|, |f(v)|^p \in E$, and since the embedding from E to $L^s(\mathbb{R}^N)$, $p \le s \le p^*$, is continuous, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |f(v)|^p &\le \nu_p^p \int_{\mathbb{R}^N} \left[|\nabla f(v)|^p + \lambda V(x)|f(v)|^p\right] \\ &\le \nu_p^p \int_{\mathbb{R}^N} \left[|\nabla v|^p + \lambda V(x)|f(v)|^p\right] \le \nu_p^p \rho^p.\end{aligned} \tag{4.2}$$

Taking $0 < \tau < 1$ such that $q = \frac{p}{2\varpi}\tau + p^*(1-\tau)$, using the Hölder inequality and the Sobolev embedding theorem, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} |f(v)|^{2\varpi q} &\le \left(\int_{\mathbb{R}^N} |f(v)|^p\right)^\tau \left(\int_{\mathbb{R}^N} |f(v)|^{2\varpi p^*}\right)^{1-\tau} \\ &\le (2\varpi)^{\frac{p^*(1-\tau)}{p}} \left(\int_{\mathbb{R}^N} |f(v)|^p\right)^\tau \left(\int_{\mathbb{R}^N} |v|^{p^*}\right)^{1-\tau} \\ &\le (2\varpi)^{\frac{p^*(1-\tau)}{p}} \nu_p^{p\tau} \rho^{p\tau} S^{\frac{p^*(\tau-1)}{p}} \left(\int_{\mathbb{R}^N} |\nabla v|^p\right)^{\frac{p^*(1-\tau)}{p}} \\ &\le (2\varpi)^{\frac{p^*(1-\tau)}{p}} \nu_p^{p\tau} \rho^{p\tau+p^*(1-\tau)} S^{\frac{p^*(\tau-1)}{p}}.\end{aligned} \tag{4.3}$$

Furthermore, since $K(x)$ is bounded, by Lemma 2.1(5) and the Sobolev embedding theorem, we get

$$\begin{aligned}\int_{\mathbb{R}^N} K(x)|f(v)|^{2\varpi p^*} &\le (2\varpi)^{\frac{p^*}{p}} |K|_\infty \int_{\mathbb{R}^N} |v|^{p^*} \\ &\le (2\varpi)^{\frac{p^*}{p}} S^{-\frac{p^*}{p}} |K|_\infty \left(\int_{\mathbb{R}^N} |\nabla v|^p\right)^{\frac{p^*}{p}} \\ &\le (2\varpi)^{\frac{p^*}{p}} S^{-\frac{p^*}{p}} |K|_\infty \rho^{p^*}.\end{aligned} \tag{4.4}$$

By (4.1)-(4.4), we know that

$$J_\lambda(v) \ge \left(\frac{1}{p} - \lambda\delta\nu_p^p\right)\rho^p - \lambda C_\delta (2\varpi)^{\frac{p^*(1-\tau)}{p}} \nu_p^{p\tau} S^{\frac{p^*(\tau-1)}{p}} \rho^{p\tau+p^*(1-\tau)} - \lambda \frac{(2\varpi)^{\frac{p^*}{p}}}{2\varpi p^*} S^{-\frac{p^*}{p}} |K|_\infty \rho^{p^*}$$

for every $v \in \widetilde{E}_{\lambda,\rho}$. Since $p\tau + p^*(1-\tau) > p$, we conclude that there are $\alpha_\lambda > 0$ and ρ_λ such that $J_\lambda|_{\widetilde{E}_\lambda := \widetilde{E}_{\lambda,\rho_\lambda}} \ge \alpha_\lambda > 0$.

(ii) Observe that, by (h_3), $|H(x,f(v))| \ge \widetilde{c}_0(|f(v)|^{2\varpi} + |f(v)|)^l$. Define the functional $\Phi_\lambda \in C^1(E,\mathbb{R})$ by

$$\Phi_\lambda(v) = \frac{1}{p} \int_{\mathbb{R}^N} \left[|\nabla v|^p + \lambda V(x)|f(v)|^p\right] - \lambda\widetilde{c}_0 \int_{\mathbb{R}^N} \left(|f(v)|^{2\varpi} + |f(v)|\right)^l.$$

Then

$$J_\lambda(v) \le \Phi_\lambda(v) \quad \text{for all } v \in E.$$

For any finite-dimensional subspace $F \subset E$, we only need to prove

$$\Phi_\lambda(v) \to -\infty \quad \text{as } v \in F, \|v\|_\lambda \to \infty.$$

In fact, by Lemma 2.1(8), we get

$$\left|f(v)\right|^{2\varpi} + \left|f(v)\right| \ge C|v|.$$

Thus

$$\Phi_\lambda(v) \le \frac{1}{p}\int_{\mathbb{R}^N}\left[|\nabla v|^p + \lambda V(x)\left|f(v)\right|^p\right] - \lambda\widetilde{c}_0 C^l \int_{\mathbb{R}^N} |v|^l.$$

Since all norms in a finite-dimensional space are equivalent and $l > p$, one easily obtains the desired conclusion.

(iii) From Lemma 4.1 and Lemma 4.2(i)-(ii), if J_λ satisfies the $(\mathrm{PS})_c$ condition for all $c > 0$, then Theorem 2.2 follows from a variant mountain pass theorem. However, in general we do not know if J_λ satisfies the $(\mathrm{PS})_c$ condition. By Lemma 3.7 for λ large and c_λ small enough, J_λ satisfies the $(\mathrm{PS})_{c_\lambda}$ condition. Thus we will find a special finite-dimensional subspace by which we construct sufficiently small minimax levels for J_λ when λ is large enough.

Recall that

$$\inf\left\{\int_{\mathbb{R}^N} |\nabla\varphi|^p : \varphi \in C_0^\infty(\mathbb{R}^N), |\varphi|_l = 1\right\} = 0, \quad p < l < p^*.$$

For any $\delta > 0$, we can choose $\varphi_\delta \in C_0^\infty$ with $|\varphi_\delta|_l = 1$ and $\operatorname{supp}\varphi_\delta \subset B_{r_\delta}(0)$ such that $|\nabla\varphi_\delta|_p^p < \delta$. Set

$$e_\lambda(x) := \varphi_\delta\left(\lambda^{\frac{1}{p}} x\right), \tag{4.5}$$

then $\operatorname{supp} e_\lambda \subset B_{\lambda^{-\frac{1}{p}} r_\delta}(0)$. Remark that, for $t \ge 0$,

$$\begin{aligned}
J_\lambda(te_\lambda) &\le \Phi_\lambda(te_\lambda) \\
&= \frac{t^p}{p}\int_{\mathbb{R}^N}\left(|\nabla e_\lambda|^p + \lambda V(x)\left|f(te_\lambda)\right|^p\right) - \lambda\widetilde{c}_0\int_{\mathbb{R}^N}\left(\left|f(te_\lambda)\right|^{2\varpi} + \left|f(te_\lambda)\right|\right)^l \\
&\le \frac{t^p}{p}\int_{\mathbb{R}^N}\left(|\nabla e_\lambda|^p + \lambda V(x)|e_\lambda|^p\right) - \lambda\widetilde{c}_0 C^l t^l \int_{\mathbb{R}^N} |e_\lambda|^l \\
&\le \lambda^{1-\frac{N}{p}}\left(\frac{t^p}{p}\int_{\mathbb{R}^N}\left(|\nabla\varphi_\delta|^p + V\left(\lambda^{-\frac{1}{p}}x\right)|\varphi_\delta|^p\right) - \widetilde{c}_0 C^l t^l \int_{\mathbb{R}^N} |\varphi_\delta|^l\right) \\
&= \lambda^{1-\frac{N}{p}}\Psi_\lambda(t\varphi_\delta),
\end{aligned}$$

where $\Psi_\lambda \in C^1(E,\mathbb{R})$ is defined by

$$\Psi_\lambda(v) = \frac{1}{p}\int_{\mathbb{R}^N}\bigl(|\nabla v|^p + V\bigl(\lambda^{-\frac{1}{p}}x\bigr)|v|^p\bigr) - \widetilde{c}_0 C^l \int_{\mathbb{R}^N}|v|^l.$$

It is easy to show that

$$\max_{t\geq 0}\Psi_\lambda(t\varphi_\delta) = \frac{l-p}{lp(l\widetilde{c}_0 C^l)^{\frac{p}{l-p}}}\left(\int_{\mathbb{R}^N}|\nabla\varphi_\delta|^p + V\bigl(\lambda^{-\frac{1}{p}}x\bigr)|\varphi_\delta|^p\right)^{\frac{l}{l-p}}.$$

Since $V(0) = 0$ and $\operatorname{supp}\varphi_\delta \subset B_{r_\delta}(0)$, there is $\hat{\Lambda}_\delta > 0$ such that

$$V\bigl(\lambda^{-\frac{1}{p}}x\bigr) \leq \frac{\delta}{|\varphi_\delta|_p^p} \quad \text{for all } |x| \leq r_\delta \text{ and } \lambda \geq \hat{\Lambda}_\delta.$$

Thus

$$\max_{t\geq 0}\Psi_\lambda(t\varphi_\delta) \leq \frac{l-p}{lp(l\widetilde{c}_0 C^l)^{\frac{p}{l-p}}}(2\delta)^{\frac{l}{l-p}}.$$

Therefore, for all $\lambda \geq \hat{\Lambda}_\delta$,

$$\max_{t\geq 0}\Phi_\lambda(te_\lambda) \leq \frac{l-p}{lp(l\widetilde{c}_0 C^l)^{\frac{p}{l-p}}}(2\delta)^{\frac{l}{l-p}}\lambda^{1-\frac{N}{p}}.$$

Choosing $\delta > 0$ such that

$$\frac{l-p}{lp(l\widetilde{c}_0 C^l)^{\frac{p}{l-p}}}(2\delta)^{\frac{l}{l-p}} \leq \sigma$$

and taking $\Lambda_\sigma = \hat{\Lambda}_\delta$, from (ii), we can choose $\bar{t}$ large enough and define $\bar{e}_\lambda = \bar{t}e_\lambda$; then we get

$$J_\lambda(\bar{e}_\lambda) < 0 \quad \text{and} \quad \max_{0\leq t\leq 1} J_\lambda(t\bar{e}_\lambda) \leq \sigma\lambda^{1-\frac{N}{p}}. \qquad \square$$

Remark 4.3 For any $\delta > 0$, one can choose nonnegative $\varphi_\delta \in C_0 \cap W^{1,p}(\mathbb{R}^N)$ such that the function e_λ defined by (4.5) is nonnegative. In fact, if $\{\varphi_j\}$ is a sequence in C_0^∞ with $|\varphi_j|_l = 1$ and $|\nabla\varphi_j|_p^p \to 0$, then by Kato's inequality, the absolute value sequence $|\varphi_j| \in C_0 \cap W^{1,p}(\mathbb{R}^N)$ with $|\varphi_j|_l = 1$ and $|\nabla(|\varphi_j|)|_p^p \leq |\nabla\varphi_j|_p^p \to 0$, where C_0 denotes the set of all continuous functions in $\mathbb{R}^N$ with compact supports. Therefore, Lemma 4.2 is still true with the function $\bar{e}_\lambda \geq 0$.

As a consequence of Lemma 4.2 and Remark 4.3, we have the following conclusions.

Corollary 4.4 *Assume that* (V_1)-(V_2), (K), *and* (h_1)-(h_3) *are satisfied. For any* $\sigma > 0$ *there exists* $\Lambda_\sigma > 0$ *such that, for each* $\lambda \geq \Lambda_\sigma$, *there is* $\alpha_\lambda > 0$ *and a* $(PS)_{c_\lambda}$ *sequence* $\{v_n\}$ *satisfying*

$$J_\lambda(v_n) \to c_\lambda, \qquad J'_\lambda(v_n) \to 0 \quad \text{as } n \to \infty,$$

where $0 < \alpha_\lambda \leq c_\lambda \leq \sigma\lambda^{1-\frac{N}{p}}$.

Corollary 4.5 *Assume that* (V_1)-(V_2), (K), *and* (h_1)-(h_3) *are satisfied. For any* $\sigma > 0$ *there exists* $\Lambda_\sigma > 0$ *such that, for each* $\lambda \geq \Lambda_\sigma$, *there is* $\alpha_\lambda > 0$ *and a* $(PS)_{c_\lambda}$ *sequence* $\{v_n\}$ *satisfying*

$$J_\lambda^+(v_n) \to c_\lambda, \qquad J_\lambda'^+(v_n) \to 0 \quad \textit{as } n \to \infty,$$

where $0 < \alpha_\lambda \leq c_\lambda \leq \sigma\lambda^{1-\frac{N}{p}}$.

5 Proof of the main results

In section, we prove the existence and multiplicity results.

Proof of Theorem 2.2 In virtue of Corollary 4.4, for any $0 < \sigma < \alpha_0$, there exists $\lambda \geq \Lambda_\sigma$, there is $\alpha_\lambda > 0$ and a $(\mathrm{PS})_{c_\lambda}$ sequence $\{v_n\}$ satisfying

$$J_\lambda(v_n) \to c_\lambda, \qquad J_\lambda'(v_n) \to 0 \quad \text{as } n \to \infty,$$

where $0 < \alpha_\lambda \leq c_\lambda \leq \sigma\lambda^{1-\frac{N}{p}}$. Lemma 3.7 implies that J_λ satisfies the $(\mathrm{PS})_{c_\lambda}$ condition, thus there is $v_\lambda \in E$ such that $J_\lambda(v_\lambda) = c_\lambda$ and $J_\lambda'(v_\lambda) = 0$, then v_λ is a positive solution of (2.1). Moreover, it is well known that a mountain pass solution is a state solution of (2.1).

Since v_λ is a critical point of J_λ, for $\nu \in [p, p^*]$,

$$\begin{aligned}
\sigma\lambda^{1-\frac{N}{p}} &\geq J_\lambda(v_\lambda) - \frac{1}{\nu} J_\lambda'(v_\lambda)v_\lambda \\
&\geq \left(\frac{1}{p} - \frac{1}{\nu}\right) \int_{\mathbb{R}^N} \left[|\nabla v_\lambda|^p + \lambda V(x)\left|f(v_\lambda)\right|^p\right] \\
&\quad + \lambda\left(\frac{\mu}{\nu} - 1\right) \int_{\mathbb{R}^N} H\bigl(x, f(v_\lambda)\bigr) \\
&\quad + \lambda \int_{\mathbb{R}^N} \left(\frac{1}{2\varpi\nu} - \frac{1}{2\varpi p^*}\right) K(x)\left|f(v_\lambda)\right|^{2\varpi p^*},
\end{aligned}$$

where μ is the constant in (h_3). Taking $\nu = p$ yields

$$\frac{\mu - p}{p} \int_{\mathbb{R}^N} H\bigl(x, f(v_\lambda)\bigr) + \frac{1}{2\varpi N} \int_{\mathbb{R}^N} K(x)\left|f(v_\lambda)\right|^{2\varpi p^*} \leq \sigma\lambda^{-\frac{N}{p}}$$

and taking $\nu = \mu$ gives

$$\frac{\mu - p}{p\mu} \int_{\mathbb{R}^N} \left[|\nabla v_\lambda|^p + \lambda V(x)\left|f(v_\lambda)\right|^p\right] \leq \sigma\lambda^{1-\frac{N}{p}}.$$

Then

$$\int_{\mathbb{R}^N} \left[\left|\nabla f(v_\lambda)\right|^p + \lambda V(x)\left|f(v_\lambda)\right|^p\right] \leq \int_{\mathbb{R}^N} \left[|\nabla v_\lambda|^p + \lambda V(x)\left|f(v_\lambda)\right|^p\right] \leq \sigma\lambda^{1-\frac{N}{p}},$$

which means $f(v_\lambda) \to 0$ in $W^{1,p}(\mathbb{R}^N)$ as $\lambda \to \infty$. The proof is completed. □

Remark 5.1 By the same arguments as applied to J_λ^+, we can obtain the existence of positive solutions for (2.3).

In order to obtain the multiplicity of critical points, we will apply the index theory defined by the Krasnoselski genus. Denote the set of all symmetric (in the sense that $-A = A$) and closed subsets of E by Σ. For each $A \in \Sigma$, let $\operatorname{gen}(A)$ be the Krasnoselski genus and

$$i(A) = \min_{h\in\Sigma} \operatorname{gen}\bigl(h(A) \cap \widetilde{E}_\lambda\bigr),$$

where Σ is the set of all odd homeomorphisms $h \in C(E, E)$ and $\widetilde{E}_\lambda$ is the closed symmetric set

$$\widetilde{E}_\lambda = \left\{ v \in E : \int_{\mathbb{R}^N} \bigl[|\nabla v|^p + \lambda V(x)\bigl|f(v)\bigr|^p\bigr] = \rho^p \right\}$$

such that $J_\lambda|_{\widetilde{E}_\lambda} \geq \alpha_\lambda > 0$. Then i is a version of Benci's pseudoindex [40]. Let

$$c_{\lambda j} = \inf_{i(A)\geq j} \sup_{v\in A} J_\lambda(v), \quad 1 \leq j \leq m. \tag{5.1}$$

If $c_{\lambda j}$ is finite and J_λ satisfies the $(\mathrm{PS})_{c_{\lambda j}}$ condition, then we know all $c_{\lambda j}$ are critical values for J_λ.

Proof of Theorem 2.2 Consider the functional J_λ, from (h_1)-(h_3), we know, for each λ, there is a closed subset $\widetilde{E}_\lambda$ of E and $\alpha_\lambda > 0$ such that $J_\lambda|_{\widetilde{E}_\lambda} \geq \alpha_\lambda > 0$.

In the same way as we have done in Lemma 4.2, for any $m \in \mathbb{N}$ and $\delta > 0$, we can choose m functions $\varphi_\delta^j \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\varphi_\delta^i \cap \operatorname{supp}\varphi_\delta^k = \varnothing$ if $i \neq k$, $|\varphi_\delta^j|_l = 1$ and $|\nabla\varphi_\delta^j|_p^p < \delta$. Let $r_\delta^m > 0$ be such that $\operatorname{supp}\varphi_\delta^j \subset B_{r_\delta^m}(0)$, $1 \leq j \leq m$. Set

$$e_\lambda^j(x) := \varphi_\delta^j\bigl(\lambda^{\frac{1}{p}} x\bigr), \quad 1 \leq j \leq m$$

and

$$H_\lambda^m(x) := \operatorname{spann}\bigl\{e_\lambda^1, \ldots, e_\lambda^m\bigr\}.$$

Then $i(H_\lambda^m) = \dim H_\lambda^m = m$. Observe that, for each $v = \sum_{j=1}^m t_j e_\lambda^j \in H_\lambda^m$,

$$J_\lambda(v) = \sum_{j=1}^m J_\lambda\bigl(t_j e_\lambda^j\bigr)$$

and as before

$$J_\lambda\bigl(t_j e_\lambda^j\bigr) \leq \lambda^{1-\frac{N}{p}} \Psi_\lambda\bigl(|t_j|\varphi_\delta^j\bigr).$$

Set

$$\beta_\delta = \max\bigl\{\bigl|\varphi_\delta^j\bigr|_p^p : 1 \leq j \leq m\bigr\}$$

and choose $\hat{\Lambda}_\delta^m$ such that

$$V\bigl(\lambda^{-\frac{1}{p}} x\bigr) \leq \frac{\delta}{\beta_\delta} \quad \text{for all } |x| \leq r_\delta^m \text{ and } \lambda \geq \hat{\Lambda}_\delta^m.$$

Thus it is easily to obtain

$$\sup_{v\in H_\lambda^m} J_\lambda(v) \le \frac{m(l-p)}{lp(l\widetilde{c}_0 C^l)^{\frac{p}{l-p}}}(2\delta)^{\frac{l}{l-p}}\lambda^{1-\frac{N}{p}}$$

for all $\lambda \ge \hat{\Lambda}_\delta^m$. Choose $\delta > 0$ such that

$$\frac{m(l-p)}{lp(l\widetilde{c}_0 C^l)^{\frac{p}{l-p}}}(2\delta)^{\frac{l}{l-p}} \le \sigma.$$

Thus, for any $m \in \mathbb{N}$ and $\sigma \in (0,\alpha_0)$, there exists $\hat{\Lambda}_\delta^m$ such that $\lambda \ge \hat{\Lambda}_\delta^m$, we can choose an m-dimensional subspace H_λ^m with $\max J_\lambda(H_\lambda^m) \le \sigma\lambda^{1-\frac{N}{p}}$.

Since $J_\lambda|_{\widetilde{E}_\lambda} \ge \alpha_\lambda > 0$ and $\max J_\lambda(H_\lambda^m) \le \sigma\lambda^{1-\frac{N}{p}}$, we deduce

$$\alpha_\lambda \le c_{\lambda 1} \le c_{\lambda 2} \le \cdots \le c_{\lambda m} \le \sup_{v\in H_\lambda^m} J_\lambda(v) \le \sigma\lambda^{1-\frac{N}{p}},$$

where $c_{\lambda j}$ defined by (5.1).

It follows from Lemma 3.7, J_λ satisfies the $(\mathrm{PS})_c$ condition if $c < \alpha_0\lambda^{1-\frac{N}{p}}$. Then all $c_{\lambda j}$ are critical values and J_λ has at least m pairs of nontrivial critical points satisfying

$$\alpha_\lambda \le J_\lambda\bigl(v_\lambda^j\bigr) \le \sigma\lambda^{1-\frac{N}{p}}, \quad 1 \le j \le m.$$

Therefore, (2.3) has at least m pairs of solutions and $u_j = f(v_{\lambda j})$ must solve problem (2.1). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The work presented here was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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