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Periodic oscillation in suspension bridge model with a periodic damping term

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Abstract
We study periodic solutions of the suspension bridge model proposed by Lazer and McKenna with a periodic damping term. Under the Dolph-type condition and a small periodic damping term condition, the existence and the uniqueness of a periodic solution have been proved by a constructive method. Two numerical examples are presented to illustrate the effect of the periodic damping term.
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1 Introduction

Many people pay close attention to oscillations in suspension bridges after the collapse of the Tacoma Narrows suspension bridge. In the late 1980s and early 1990s, Lazer and McKenna [1–3] have studied the suspension bridge model

$$U_{tt} + kU_t + cU_{xxxx} + dU^+ = h(t,x), \tag{1}$$

where $U = U(x,t)$, $0 \le x \le L$, $t > 0$ satisfies the boundary conditions

$$U(0,t) = U(L,t) = U_{xx}(0,t) = U_{xx}(L,t) = 0, \tag{2}$$

$c > 0$, $k \ge 0$, $d \ge 0$ are constant, $U^+ = \max\{U,0\}$ and $U^- = \max\{-U,0\}$. System (1)-(2) describes the transverse vibrations of a beam hinged at both ends with length L and external force $h(t,x)$. The term dU^+ takes into account the fact that the cables' restoring force exists only in the situation of stretching. Here kU_t represents the damping term.

In this paper, we consider problem (1) with 2π-periodic damping term $p(t)$

$$U_{tt} + p(t)U_t + cU_{xxxx} + dU^+ = h(t,x), \tag{3}$$

where $h(t,x) = (\sin \pi x/L)f(t)$ is the 2π-periodic external force as same as the assumption in [2]. Looking for a standing-wave solution of (3) and (2), we have

$$U(t,x) = (\sin \pi x/L)u(t),$$

which leads to an equivalent ordinary differential equation

$$u'' + p(t)u' + bu^+ - au^- = f(t), \tag{4}$$

in which $a = c(\pi/L)^4$ and $b = d + c(\pi/L)^4$.

In the past years, the jumping nonlinearity has been discussed by many authors [1–9]. However, to our knowledge, there is no result about periodic damping term. In the early 1990s, Li [10] obtained an ingenious method to discuss the existence and uniqueness of nonlinear two-point boundary value problems with variable coefficient. Recently, the second author of this paper extended this method to the periodic situation [11]. In this paper we refine this method to solve problem (4) and take some numerical simulations to illustrate the effect of periodic damping term.

The rest of this paper is organized as follows. In Section 2, we briefly state the main results. In Section 3, we study the properties of the homogeneous equation by a constructive method. In Section 4, we prove our main results by Leray-Schauder degree theory. In Section 5, we present some numerical experiments. In Section 6, we give the conclusion.

2 Main results

We denote by N a positive integer and $\gamma = \sup_{\mathbb{R}} |p(t)|$. To study the existence of periodic solutions of (4), we need the following assumptions:

(H_1) Dolph-type condition:

$$N^2 < a - \frac{\gamma^2}{4} < b + \frac{(N+1)\pi\gamma}{4} < (N+1)^2.$$

(H_2) Small periodic damping term condition:

$$\sin\frac{\pi\sqrt{4a-\gamma^2}}{4N} < \sqrt{1-\frac{\gamma^2}{4a}}.$$

Theorem 1 *Let* (H_1) *and* (H_2) *hold. Then problem* (4) *has a unique* 2π*-periodic solution.*

The more general form of the suspension bridge model is

$$u'' + p(t)u' + b(t)u^+ - a(t)u^- = f(t). \tag{5}$$

Here $b(t)$ and $a(t)$ are positive 2π-periodic functions satisfying

$$d(t) := b(t) - a(t) \geq 0, \quad t \in [0, 2\pi], \tag{6}$$

where $d(t)$ is the variational coefficient of cables' restoring force. Denote $\alpha_1 = \inf_{\mathbb{R}}(b(t))$, $\beta_1 = \sup_{\mathbb{R}}(b(t))$, $\alpha_2 = \inf_{\mathbb{R}}(a(t))$, $\beta_2 = \sup_{\mathbb{R}}(a(t))$. Then

$$\alpha_1 \leq b(t) \leq \beta_1, \qquad \alpha_2 \leq a(t) \leq \beta_2, \qquad \alpha_1 \geq \alpha_2, \qquad \beta_1 \geq \beta_2.$$

To study the existence of periodic solutions of (5), we make the following assumptions:

(H$_3$) Dolph-type condition:

$$\alpha_2 > N^2 + \frac{\gamma^2}{4}, \qquad \beta_1 < (N+1)^2 - \frac{(N+1)\pi\gamma}{4}.$$

(H$_4$) Small periodic damping term condition:

$$\sin\frac{\pi\sqrt{4\alpha_2-\gamma^2}}{4N} < \sqrt{1-\frac{\gamma^2}{4\alpha_2}}.$$

Theorem 2 *Let* (H$_3$) *and* (H$_4$) *hold. Then problem* (5) *has a unique* 2π*-periodic solution.*

Remark 1 Problem (4) with (H$_1$) and (H$_2$) is a particular case of problem (5) with (H$_3$) and (H$_4$). So we shall only give the proof of Theorem 2.

3 Homogeneous equation

The following lemmas will be used in this section.

Lemma 3 (see [12]) *Let* $x \in C^1([0,h],\mathbb{R})$, $h > 0$, *with*

$$x(0) = x(h) = 0, \qquad x(t) > 0, \quad t \in (0,h).$$

Then

$$\int_0^h |x(t)x'(t)|\,dt \le \frac{h}{4}\int_0^h x'^2(t)\,dt,$$

and the constant $\frac{h}{4}$ *is optimal.*

Lemma 4 (see [12]) *Let* $x \in C^1([a,b],\mathbb{R})$ $a,b \in \mathbb{R}$, $a < b$, *with the boundary value conditions* $x(a) = x(b) = 0$. *Then*

$$\int_a^b x^2(t)\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b x'^2(t)\,dt.$$

Consider the periodic boundary value problem

$$\begin{cases} u'' + p(t)u' + b(t)u^+ - a(t)u^- = 0, \\ u(0) = u(2\pi), \qquad u'(0) = u'(2\pi). \end{cases} \tag{7}$$

We will prove the following proposition by similar methods to [11].

Proposition 5 *Suppose that* $p(t)$, $b(t)$, $a(t)$ *are* L^2*-integrable* 2π*-periodic functions satisfying* (6), (H$_3$) *and* (H$_4$), *then* (7) *has only the trivial* 2π*-periodic solution* $u(t) \equiv 0$.

Proof We assume that (7) has a nonzero 2π-periodic solution $u(t)$. A contradiction will be proved in six steps.

Step 1. We will prove that $u(t)$ has at least one zero in $(0,2\pi)$. Otherwise, we may we assume $u(t) > 0$, $t \in (0,2\pi)$. Then we have $u'' + p(t)u' + b(t)u^+ = 0$ in $(0,2\pi)$. Consider the following equivalent equation:

$$\left(e^{\int_{t_0}^{t} p(s)\,ds}u'\right)' + e^{\int_{t_0}^{t} p(s)\,ds}\left(b(t)u^+ - a(t)u^-\right) = 0,$$

where $t_0 \in [0, 2\pi]$ is undetermined. By Rolle's theorem, there exists a $t_0' \in (0, 2\pi)$ with $u'(t_0') = 0 = u'(t_0' + 2\pi)$. Then

$$0 = \int_{t_0'}^{t_0'+2\pi} \left(e^{\int_{t_0}^{t} p(s)\,ds}u'\right)'\,dt = -\int_{t_0'}^{t_0'+2\pi} e^{\int_{t_0}^{t} p(s)\,ds} b(t)u^+\,dt < 0,$$

which leads to a contradiction.

Without loss of generality, we assume $u(0) = u(2\pi) = 0$, $u'(0) = u'(2\pi) = A > 0$.

Step 2. We construct two auxiliary equations. Consider the initial value problem

$$u'' + p(t)u' + b(t)u^+ = 0, \qquad u(0) = 0, \qquad u'(0) = A.$$

The first auxiliary equation is

$$\varphi'' - \gamma\varphi' + \alpha_1\varphi = 0, \qquad \varphi(0) = 0, \qquad \varphi'(0) = A. \tag{8}$$

Obviously,

$$\varphi(t) = \frac{2A}{\sqrt{4\alpha_1 - \gamma^2}} e^{\frac{\gamma t}{2}} \sin\frac{\sqrt{4\alpha_1 - \gamma^2}}{2}t$$

is the solution of (8) and

$$\varphi'(t) = 2A\sqrt{\frac{\alpha_1}{4\alpha_1 - \gamma^2}} e^{\frac{\gamma t}{2}} \sin\left(\frac{\sqrt{4\alpha_1 - \gamma^2}}{2}t + \theta\right),$$

where $\theta \in (0, \frac{\pi}{2}]$ with $\sin\theta = \sqrt{\frac{4\alpha_1 - \gamma^2}{4\alpha_1}}$. Since

$$N < \frac{\sqrt{4\alpha_1 - \gamma^2}}{2} < \sqrt{\alpha_1} \le \sqrt{\beta_1} < N + 1 \tag{9}$$

holds under the assumptions of (H_3), there is a $t_0 \in (0, \pi)$ such that

$$\frac{\sqrt{4\alpha_1 - \gamma^2}}{2}t_0 + \theta = \pi, \quad \text{i.e.}, \frac{\pi}{2} \le \frac{\sqrt{4\alpha_1 - \gamma^2}}{2}t_0 < \pi. \tag{10}$$

Thus, we have

$$\sin\frac{\sqrt{4\alpha_1 - \gamma^2}}{2}t_0 = \sin\theta = \sqrt{\frac{4\alpha_1 - \gamma^2}{4\alpha_1}} \ge \sqrt{\frac{4\alpha_2 - \gamma^2}{4\alpha_2}}.$$

Since $\frac{\pi}{2} < \frac{\pi\sqrt{4\alpha_1 - \gamma^2}}{4N} < \pi$ and $\sin t$ is decreasing in $[\frac{\pi}{2}, \pi)$, we have

$$\sin\frac{\pi\sqrt{4\alpha_2 - \gamma^2}}{4N} \ge \sin\frac{\pi\sqrt{4\alpha_1 - \gamma^2}}{4N}.$$

By (H_4), we have

$$\sin\frac{\sqrt{4\alpha_1-\gamma^2}}{2}t_0=\sqrt{\frac{4\alpha_1-\gamma^2}{4\alpha_1}}>\sin\frac{\pi\sqrt{4\alpha_1-\gamma^2}}{4N},\tag{11}$$

and $0<t_0<\frac{\pi}{2N}$. Therefore

$$\varphi'(t)>0,\qquad \varphi(t)>0,\quad \text{for } t\in(0,t_0);\qquad \varphi(t_0)>0,\qquad \varphi'(t_0)=0.\tag{12}$$

We also consider the initial value problem

$$\psi''+\gamma\psi'+\alpha_1\psi=0,\qquad \psi(t_0)=\varphi(t_0),\qquad \psi'(t_0)=0.\tag{13}$$

Clearly,

$$\psi(t)=2\sqrt{\frac{\alpha_1}{4\alpha_1-\gamma^2}}\varphi(t_0)e^{\frac{-\gamma(t-t_0)}{2}}\sin\left(\frac{\sqrt{4\alpha_1-\gamma^2}}{2}(t-t_0)+\theta\right)$$

is the solution of (13), where θ is the same as the previous one, and

$$\psi'(t)=-\frac{2\alpha_1}{\sqrt{4\alpha_1-\gamma^2}}\varphi(t_0)e^{\frac{-\gamma(t-t_0)}{2}}\sin\frac{\sqrt{4\alpha_1-\gamma^2}}{2}(t-t_0).$$

Hence there exists a $t_1\in(0,2\pi)$ with $t_1-t_0\in(0,\pi)$, such that

$$\frac{\sqrt{4\alpha_1-\gamma^2}}{2}(t_1-t_0)+\theta=\pi.\tag{14}$$

Then $\psi(t_1)=0$. From (10) and (14), it follows that

$$\frac{\sqrt{4\alpha_1-\gamma^2}}{4}t_1=\pi-\theta,\quad \textit{i.e.}, \frac{\pi}{2}\le\frac{\sqrt{4\alpha_1-\gamma^2}}{4}t_1<\pi.\tag{15}$$

By (11) and (15), we have

$$\sin\frac{\sqrt{4\alpha_1-\gamma^2}}{4}t_1=\sin\theta=\sqrt{\frac{4\alpha_1-\gamma^2}{4\alpha_1}}>\sin\frac{\pi\sqrt{4\alpha_1-\gamma^2}}{4N}.$$

Since $\sin t$ is decreasing on $[\frac{\pi}{2},\pi)$, we have $0<t_1<\frac{\pi}{N}$, and

$$\psi'(t)<0,\qquad \psi(t)>0,\quad \text{for } t\in(t_0,t_1);\qquad \psi(t_1)=0,\qquad \psi'(t_1)<0.$$

Step 3. We will prove that $u(t)$ has a zero point in $(0,t_1]$. Assume, on the contrary, $u(t)>0$ for $t\in(0,t_1]$.

Let $y=\varphi'(t)u(t)-\varphi(t)u'(t)$ on $[0,t_0]$. Since $\varphi'(t)\ge0$, $\varphi(t)>0$, $u(t)>0$ on $t\in(0,t_0]$, we have

$$\begin{aligned}y'&=\varphi''(t)u(t)+\varphi'(t)u'(t)-\varphi'(t)u'(t)-\varphi(t)u''(t)\\&=\big(\gamma\varphi'(t)-\alpha_1\varphi(t)\big)u(t)-\varphi(t)\big(-p(t)u'(t)-b(t)u(t)\big)\end{aligned}$$

$$
\begin{aligned}
&= \bigl(\gamma + p(t)\bigr)\varphi'(t)u(t) + \bigl(-p(t)\bigr)\bigl(\varphi'(t)u(t) - \varphi(t)u'(t)\bigr) + \bigl(b(t) - \alpha_1\bigr)\varphi(t)u(t) \\
&\geq -p(t)y,
\end{aligned}
$$

which implies

$$
\bigl(ye^{\int_0^t p(s)\,ds}\bigr)' \geq 0, \quad t \in (0, t_0].
$$

Notice that $y(0) = \varphi'(0)u(0) - \varphi(0)u'(0) = 0$, we have

$$
y = \varphi'(t)u(t) - \varphi(t)u'(t) \geq 0, \quad t \in (0, t_0]. \tag{16}
$$

By (12), we know that $u'(t_0) \leq 0$.

Let $z = \psi'(t)u(t) - \psi(t)u'(t)$ on $[t_0, t_1]$. Since $\psi(t_0) > 0$, $\psi'(t_0) = 0$, we have

$$
z(t_0) = \psi'(t_0)u(t_0) - \psi(t_0)u'(t_0) \geq 0.
$$

Since $\psi'(t) \leq 0$, $\psi(t) \geq 0$, $u(t) > 0$ on $[t_0, t_1]$, we have

$$
\begin{aligned}
z' &= \psi''(t)u(t) + \psi'(t)u'(t) - \psi'(t)u'(t) - \psi(t)u''(t) \\
&= \bigl(-\gamma\psi'(t) - \alpha_1\psi(t)\bigr)u(t) - \psi(t)\bigl(-p(t)u'(t) - b(t)u(t)\bigr) \\
&= \bigl(-\gamma + p(t)\bigr)\psi'(t)u(t) + \bigl(-p(t)\bigr)\bigl(\psi'(t)u(t) - \psi(t)u'(t)\bigr) + \bigl(b(t) - \alpha_1\bigr)\psi(t)u(t) \\
&\geq -p(t)z,
\end{aligned}
$$

which implies

$$
\bigl(ze^{\int_0^t p(s)\,ds}\bigr)' \geq 0, \quad t \in [t_0, t_1], \tag{17}
$$

i.e.,

$$
z(t) \geq 0, \quad t \in [t_0, t_1].
$$

But $\psi(t_1) = 0$ and $\psi'(t_1) < 0$ imply that

$$
z(t_1) = \psi'(t_1)u(t_1) - \psi(t_1)u'(t_1) < 0,
$$

which is a contradiction to $u(t) > 0$ on $(0, t_1]$. Therefore $u(t)$ has at least one zero in $(0, t_1]$ with $t_1 < \frac{\pi}{N}$.

Step 4. We will prove that $u(t)$ has at least $2N + 2$ zeros on $[0, 2\pi]$. Let $u(t^1)$ be the first zero point in $(0, t_1]$ such that $u(t^1) = 0$, $u'(t^1) = B < 0$. We claim that there must exist a zero point in $(t^1, 2\pi]$. Otherwise, we consider $u'' + p(t)u' - a(t)u^- = 0$. With a similar argument to Step 3, we have a t_2 such that there must be a zero in $(t^1, t_2]$ and $t_2 - t^1 < \frac{\pi}{N}$. Step by step, we find that $u(t)$ has at least $2N + 2$ zeros on $[0, 2\pi]$.

Step 5. We will prove that $u(t)$ has at least $2N + 3$ zeros on $[0, 2\pi]$. On the contrary, we assume $u(t)$ has exactly $2N + 2$ zeros on $[0, 2\pi]$. We write them as

$$
0 = t^0 < t^1 < \cdots < t^{2N+1} = 2\pi.
$$

Obviously,

$$u'(t^i) \neq 0, \quad i = 0, 1, \ldots, 2N + 1.$$

Without loss of generality, we may assume $u'(t^0) > 0$. Since

$$u'(t^i)u'(t^{i+1}) < 0, \quad i = 0, 1, \ldots, 2N,$$

we obtain $u'(t^{2N+1}) < 0$, which contradicts $u'(t^{2N+1}) = u'(t^0) > 0$. Therefore $u(t)$ has at least $2N + 3$ zeros on $[0, 2\pi]$.

Step 6. Since $u(t)$ has at least $2N + 3$ zeros on $[0, 2\pi]$, there are two zeros ξ_1 and ξ_2 with $0 < \xi_2 - \xi_1 \leq \frac{\pi}{N+1}$. Integrating u'^2 from ξ_1 to ξ_2, we have

$$\begin{aligned}
\int_{\xi_1}^{\xi_2} u'^2(t)dt &= -\int_{\xi_1}^{\xi_2} u(t)u''(t)\,dt \\
&= \int_{\xi_1}^{\xi_2} p(t)u(t)u'(t)\,dt + \int_{\xi_1}^{\xi_2} b(t)u(t)u^+(t) - a(t)u(t)u^-(t)\,dt \\
&= \int_{\xi_1}^{\xi_2} p(t)u(t)u'(t)\,dt + \int_{\xi_1}^{\xi_2} a(t)u^2(t)\,dt + \int_{\xi_1}^{\xi_2} d(t)u(t)u^+(t)\,dt \\
&\leq \int_{\xi_1}^{\xi_2} \left|p(t)u(t)u'(t)\right|dt + \int_{\xi_1}^{\xi_2} a(t)u^2(t)\,dt + \int_{\xi_1}^{\xi_2} d(t)u^2(t)\,dt \\
&\leq \gamma \int_{\xi_1}^{\xi_2} \left|u(t)u'(t)\right|dt + \beta_1 \int_{\xi_1}^{\xi_2} u^2(t)\,dt.
\end{aligned}$$

Assume that there are k zeros in (ξ_1, ξ_2) denoted by τ_k, $k \in \mathbb{N}$. By Lemma 3,

$$\begin{aligned}
\int_{\xi_1}^{\xi_2} \left|u(t)u'(t)\right|dt &= \int_{\xi_1}^{\tau_1} + \int_{\tau_1}^{\tau_2} + \cdots + \int_{\tau_k}^{\xi_2} \left|u(t)u'(t)\right|dt \\
&\leq \frac{1}{4}\max\{\tau_1 - \xi_1, \tau_2 - \tau_1, \ldots, \xi_2 - \tau_k\} \\
&\quad \times \left(\int_{\xi_1}^{\tau_1} + \int_{\tau_1}^{\tau_2} + \cdots + \int_{\tau_k}^{\xi_2} u'^2(t)\,dt\right) \\
&\leq \frac{\xi_2 - \xi_1}{4} \int_{\xi_1}^{\xi_2} u'^2(t)\,dt.
\end{aligned}$$

By Lemma 4, we have

$$\int_{\xi_1}^{\xi_2} u^2(t)\,dt \leq \frac{(\xi_2 - \xi_1)^2}{\pi^2} \int_{\xi_1}^{\xi_2} u'^2(t)\,dt.$$

Since $\beta_1 + \frac{(N+1)\pi\gamma}{4} < (N+1)^2$, we have

$$\frac{\gamma}{4}(\xi_2 - \xi_1) + \frac{\beta_1}{\pi^2}(\xi_2 - \xi_1)^2 \leq \frac{\pi\gamma}{4(N+1)} + \frac{\beta_1}{(N+1)^2} < 1.$$

Hence

$$\int_{\xi_1}^{\xi_2} u'^2(t)\,dt = 0,$$

which implies $u'(t) = 0$ for $t \in [\xi_1, \xi_2]$. Due to the uniqueness of the solution for the initial value problem, we get $u(t) \equiv 0$ for $t \in [0, 2\pi]$, a contradiction. □

4 Non-homogeneous equation

In this section, we will give the complete proof of Theorem 2.

4.1 Uniqueness

Proof Let u_1 and u_2 be two 2π-periodic solutions of (5). Denote $v = u_1 - u_2$, then v is a solution of the following problem:

$$v'' + p(t)v' + a(t)v + d(t)\big(u_1^+ - u_2^+\big) = 0, \tag{18}$$

where $d(t)$ is defined in (6). Since

$$-v^- = \big(u_1 - u_2 - |u_1 - u_2|\big)/2 < u_1^+ - u_2^+ < \big(|u_1 - u_2| + u_1 - u_2\big)/2 = v^+,$$

there exists a $\theta \in (0,1)$ such that

$$u_1^+ - u_2^+ = (1-\theta)v^+ + \theta\big(-v^-\big).$$

Then (18) equals

$$v'' + p(t)v' + \big((1-\theta)b(t) + \theta a(t)\big)v^+ - \big(\theta b(t) + (1-\theta)a(t)\big)v^- = 0.$$

If $0 < \theta \leq \frac{1}{2}$,

$$\big((1-\theta)b(t) + \theta a(t)\big) - \big(\theta b(t) + (1-\theta)a(t)\big) = (1-2\theta)\big(b(t) - a(t)\big) > 0.$$

Equation (18) satisfies (H_3) and (H_4). Otherwise, $-v$ as a solution satisfies (H_3) and (H_4). By Proposition 5, we have $v \equiv 0$. □

4.2 Boundedness

We consider the homotopy equation

$$u'' + \alpha_2 u = \lambda\big(-p(t)u' - b(t)u^+ + a(t)u^- + \alpha_2 u + f(t)\big) \equiv \lambda F\big(t, u, u'\big), \tag{19}$$

where $\lambda \in [0,1]$. Denote by $\|\cdot\|$ the usual normal in $C^1[0, 2\pi]$, *i.e.*, $\|u\| = |u| + |u'|$. We assert there exists $B > 0$ such that every possible periodic solution $u(t)$ of (19) satisfies $\|u\| \leq B$. If not, there exists $\lambda_k \to \lambda_0$ and the solution $u_k(t)$ with $\|u_k\| \to \infty$ $(k \to \infty)$. Let $y_k = \frac{u_k}{\|u_k\|}$, we have $y_k^+ = \frac{u_k^+}{\|u_k\|}$ and $y_k^- = \frac{u_k^-}{\|u_k\|}$. Obviously, $\|y_k\| = 1$ $(k = 1, 2, \ldots)$. It satisfies the following problem:

$$y_k'' + \alpha_2 y_k = \lambda_k\left(-b(t)y_k^+ + a(t)y_k^- - p(t)y_k' + \alpha_2 y_k + \frac{f(t)}{\|u_k\|}\right), \tag{20}$$

in which we have

$$\frac{f(t)}{\|u_k\|} \to 0 \quad (k \to \infty).$$

Since

$$y_k(t) = y_k(0) + \int_0^t y_k'(s)\,ds,$$
$$y_k'(t) = y_k'(0) + \int_0^t -\alpha_2 y_k + \lambda_k\left(-b(s)y_k^+ + a(s)y_k^- - p(s)y_k' + \alpha_2 y_k + \frac{f(s)}{\|u_k\|}\right) ds,$$

$\{y_k\}$, $\{y_k'\}$ are uniformly bounded and equicontinuous. By the Ascoli lemma, there exists a continuous function $w(t)$, $v(t)$, and a subsequence of $\{k\}_{k=1}^\infty$ (denote it again by $\{k\}_{k=1}^\infty$) such that

$$\lim_{k\to\infty} y_k(t) = w(t), \qquad \lim_{k\to\infty} y_k'(t) = v(t) \quad \text{uniformly on } [0,2\pi].$$

As a consequence (20) weakly converges to the following equation in $L^2[0,2\pi]$:

$$w''(t) + \lambda_0 p(t)w' + \big(\lambda_0 b(t) + \alpha_2 - \alpha_2\lambda_0\big)w^+ - \big(\alpha_2 + \lambda_0 a(t) - \alpha_2\lambda_0\big)w^- = 0, \tag{21}$$

which satisfy (H_3) and (H_4). By Proposition 5, we have $w(t) \equiv 0$ for $t \in [0,2\pi]$, which contradicts $\|w\| = 1$. Thus, the possible periodic solution is bounded.

4.3 Existence

Proof Assume $\Phi(t)$ is the fundamental solution matrix of $u'' + \alpha_2 u = 0$ with $\Phi(0) = I$. Obviously, it is nonresonant by (H_3). Equation (19) can be transformed into the integral equation

$$\begin{pmatrix} u \\ u' \end{pmatrix}(t) = \Phi(t)\left(\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} + \int_0^t \Phi^{-1}(s)\begin{pmatrix} 0 \\ \lambda F(s,u(s),u'(s)) \end{pmatrix} ds\right). \tag{22}$$

Because $u(t)$ is a 2π-periodic solution of (22), then

$$\big(I - \Phi(2\pi)\big)\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} = \Phi(2\pi)\int_0^{2\pi} \Phi^{-1}(s)\begin{pmatrix} 0 \\ \lambda F(s,u(s),u'(s)) \end{pmatrix} ds. \tag{23}$$

Obviously, $(I - \Phi(2\pi))$ is invertible,

$$\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} = \big(I - \Phi(2\pi)\big)^{-1}\Phi(2\pi)\int_0^{2\pi} \Phi^{-1}(s)\begin{pmatrix} 0 \\ \lambda F(s,u(s),u'(s)) \end{pmatrix} ds. \tag{24}$$

We substitute (24) into (22)

$$\begin{aligned}\begin{pmatrix} u \\ u' \end{pmatrix}(t) &= \Phi(t)\big(I - \Phi(2\pi)\big)^{-1}\Phi(2\pi)\int_0^{2\pi} \Phi^{-1}(s)\begin{pmatrix} 0 \\ \lambda F(s,u(s),u'(s)) \end{pmatrix} ds \\ &\quad + \Phi(t)\int_0^t \Phi^{-1}(s)\begin{pmatrix} 0 \\ \lambda F(s,u(s),u'(s)) \end{pmatrix} ds. \end{aligned}\tag{25}$$

Define an operator

$$P_\lambda : C^1[0,2\pi] \times C[0,2\pi] \to C^1[0,2\pi] \times C[0,2\pi],$$

such that

$$P_\lambda \left[\begin{pmatrix} u \\ u' \end{pmatrix} \right](t) \equiv \Phi(t)\bigl(I - \Phi(2\pi)\bigr)^{-1}\Phi(2\pi) \int_0^{2\pi} \Phi^{-1}(s) \begin{pmatrix} 0 \\ \lambda F(s,u,u') \end{pmatrix} ds$$
$$+ \Phi(t) \int_0^{t} \Phi^{-1}(s) \begin{pmatrix} 0 \\ \lambda F(s,u,u') \end{pmatrix} ds. \tag{26}$$

Since the right-hand side of P_λ is continuous (non-smooth), it is easy to see that P_λ is a completely continuous operator in $C^1[0,2\pi] \times C[0,2\pi]$. Denote

$$\Omega = \bigl\{ u \in C^1[0,2\pi], \|u\| \le B + 1 \bigr\}$$

and

$$h_\lambda(u) = \begin{pmatrix} u \\ u' \end{pmatrix} - P_\lambda \left[\begin{pmatrix} u \\ u' \end{pmatrix} \right].$$

Because $0 \notin h_\lambda(\partial\Omega)$ for $\lambda \in [0,1]$, by Leray-Schauder degree theory, we have

$$\deg\left(\begin{pmatrix} u \\ u' \end{pmatrix} - P_1 \left[\begin{pmatrix} u \\ u' \end{pmatrix} \right], \Omega, 0 \right) = \deg\bigl(h_1(u), \Omega, 0\bigr)$$
$$= \deg\bigl(h_0(u), \Omega, 0\bigr) = 1 \ne 0.$$

So we conclude that P_1 has at least one fixed point in Ω, that is, (5) has a unique solution. □

5 Numerical experiment

5.1 Example 1

Let us consider

$$u'' + 0.1 \sin t u' + \bigl(0.5\cos(t) + 8\bigr)u^+ - \bigl(0.5\cos(t) + 6\bigr)u^- = 0.2\sin(t). \tag{27}$$

By Theorem 2, there is a unique 2π-periodic solution.

In Figure 1, we make a 10-fold Newton iteration to get an approximate solution of (27), displayed by a blue line. It is obvious that the solution here is locally stable and unique. The error here is about 10^{-10}. The red line in Figure 1 is the approximate solution of (27) without periodic damping term. Our simulation illustrates that the effect of the small periodic damping term is limited.

In Figure 2, we consider the effect of the cables' restoring force $d(t)$. If there is no cables' restoring force, we have the following system:

$$u'' + 0.1 \sin t u' + \bigl(0.5\cos(t) + 6\bigr)u = 0.2\sin(t). \tag{28}$$

The blue line in Figure 2 is the approximate solution of (27) and the red line in Figure 2 is the approximate solution of (28). The latter one is a particular case that can be tackled by [11].

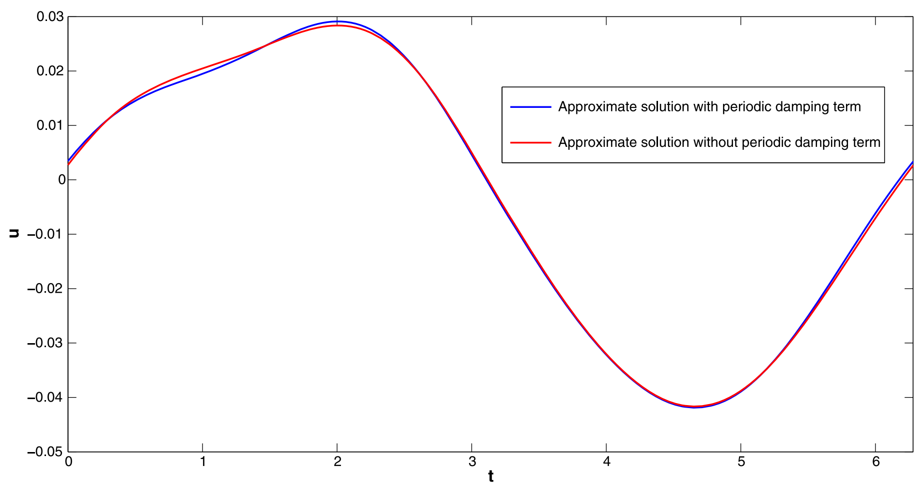

Figure 1 The approximate solution of (27) with and without periodic damping term.

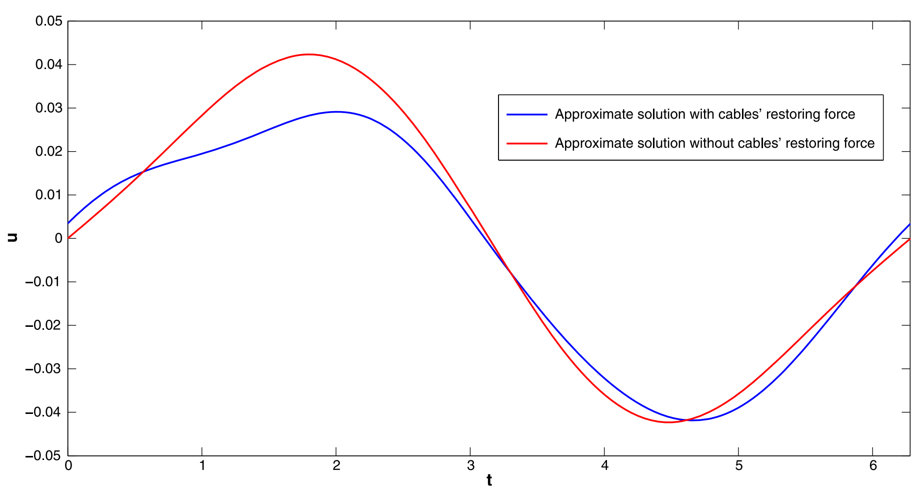

Figure 2 The approximate solution of (27) and (28).

5.2 Example 2

Let us consider

$$u'' + 0.5 \sin t u' + 15u^{+} - 10u^{-} = 1 + 2\cos(t). \tag{29}$$

By Theorem 1, there is a unique 2π-periodic solution.

The blue line in Figure 3 is the approximate solution of (29). The red line in Figure 3 is the approximate solution of (29) without periodic damping term. Our simulation illustrates that the effect of the small periodic damping term is limited. The method we applied is a 10-fold Newton iteration and the error here is about 10^{-10}.

In Figure 4, we consider the effect of the cables' restoring force $d(t)$. If there is no cables' restoring force, we have the following system:

$$u'' + 0.5 \sin t u' + 10u = 1 + 2\cos(t). \tag{30}$$

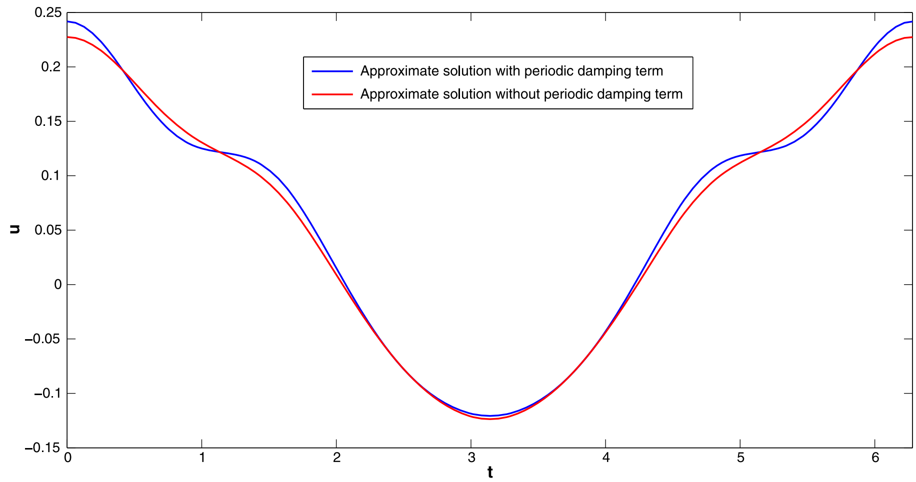

Figure 3 **The approximate solution of (29) with and without periodic damping term.**

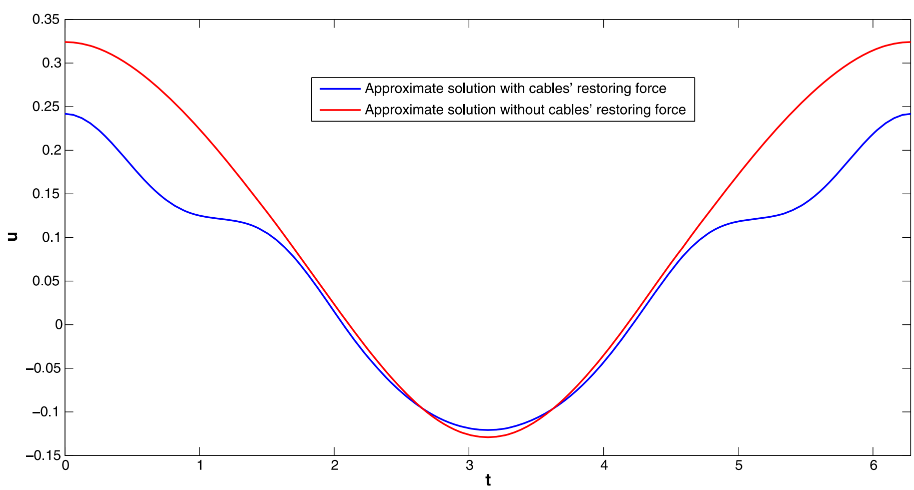

Figure 4 **The approximate solution of (29) and (30).**

The blue line in Figure 4 is the approximate solution of (29) and the red line in Figure 4 is the approximate solution of (30).

6 Conclusions

Periodic solutions of the suspension bridge model with a periodic damping term have been studied. After transforming this system into an equivalent ordinary differential equation, we get the existence and the uniqueness of a periodic solution by the Dolph-type condition and a small periodic damping term condition. Our constructive method is very adaptable to this kind of non-smooth problem. Two numerical examples have been presented to simulate our main results. By the numerical experiment, we know that the effect of the small periodic damping term is limited. Furthermore, we compare the approximate solution of our system to the suspension bridge model without the cables' restoring force, the latter one is a particular case of [11].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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