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Existence of homoclinic solutions for a class of second order p-Laplacian systems with impulsive effects

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Abstract

This paper is concerned with the existence of homoclinic solutions for a class of second order p-Laplacian systems with impulsive effects. A new result is obtained under more relaxed conditions by using the mountain pass theorem, a weak convergence argument, and a weak version of Lieb's lemma.

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1 Introduction

Consider homoclinic solutions of the following problem:

$$\begin{cases} \frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) - a(t)|u(t)|^{p-2}u(t) + \nabla V(t, u(t)) = 0, & \text{a.e. } t \in (t_j, t_{j+1}), j \in \mathbb{Z}, \\ \Delta(|\dot{u}(t_j)|^{p-2}\dot{u}(t_j)) = |\dot{u}(t_j^+)|^{p-2}\dot{u}(t_j^+) - |\dot{u}(t_j^-)|^{p-2}\dot{u}(t_j^-) = I(u(t_j)), & j \in \mathbb{Z}, \end{cases} \quad (1.1)$$

where $p \in (1, +\infty)$, $V : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of class C^1, $\nabla V(t, u(t)) = \frac{\partial V(t, u(t))}{\partial u}$, $I : \mathbb{R} \to \mathbb{R}$, $a \in C(\mathbb{R}, (0, +\infty))$, and $a(t) \to +\infty$ as $|t| \to +\infty$. $\mathbb{Z}$ denotes the sets of integers, and t_j ($j \in \mathbb{Z}$) are impulsive points. Moreover, there exist a positive integer m and a positive constant T such that $0 < t_0 < t_1 < \cdots < t_{m-1} < T$, $t_{l+km} = t_l + kT$, $\forall k \in \mathbb{Z}$, $l = 0, 1, \ldots, m-1$. $\dot{u}(t_j^+) = \lim_{h \to 0^+} \dot{u}(t_j + h)$ and $\dot{u}(t_j^-) = \lim_{h \to 0^-} \dot{u}(t_j - h)$ represent the right and left limits of $\dot{u}(t)$ at $t = t_j$, respectively.

When $a(t) \equiv 0$ and $p = 2$, problem (1.1) becomes the following problem:

$$\begin{cases} \ddot{u}(t) + \nabla V(t, u(t)) = 0, & \text{a.e. } t \in (t_j, t_{j+1}), j \in \mathbb{Z}, \\ \Delta \dot{u}(t_j) = \dot{u}(t_j^+) - \dot{u}(t_j^-) = I(u(t_j)), & j \in \mathbb{Z}. \end{cases} \quad (1.2)$$

By using the mountain pass theorem, a weak convergence argument, and a weak version of Lieb's methods, Fang and Duan [1] investigated homoclinic solutions of problem (1.2) and obtained the following main result.

Theorem A [1] *Assume that the following conditions hold*:

(V1) *there exists a positive number T such that*

$$\nabla V(t + T, x) = \nabla V(t, x), \qquad V(t + T, x) = V(t, x), \quad \forall (t, x) \in \mathbb{R}^2;$$

(V2) $\lim_{x\to 0} \frac{\nabla V(t,x)}{x} = 0$ *uniformly for $t \in \mathbb{R}$;*

(V3) *there exists a constant $\mu > 2$ such that*

$$\mu V(t,x) \leq \big(\nabla V(t,x),x\big), \quad \forall(t,x) \in \mathbb{R} \times \mathbb{R}\backslash\{0\};$$

(V4) *there exist constants $b_0 > 0$ and $b > 0$ such that*

$$V(t,x) \geq b_0|x|^{\mu}, \quad \forall t \in \mathbb{R}, |x| \geq 1;$$

$$V(t,x) \leq b|x|^{\mu}, \quad \forall t \in \mathbb{R}, |x| \leq 1;$$

(I) *there exists a constant b_1 with $0 < b_1 < \frac{\mu-2}{(\mu+2)mT}$ such that*

$$\big|I(x)\big| \leq b_1|x|, \qquad 2\int_0^x I(t)\,dt - I(x)x \leq 0.$$

Then problem (1.2) *possesses a nontrivial weak homoclinic orbit.*

For $I \neq 0$, problem (1.1) involves impulsive effects. It is well known that impulsive differential equations are used in various fields of science and technology, for example, many biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics, and frequency modulated systems, and so on. For more details of impulsive differential equations, we refer the readers to the books [2, 3].

Recently, the existence and multiplicity of solutions for impulsive differential equations via variational methods have been investigated by some researchers. See for example [4–17] and references therein. However, there are few papers [1, 18–20] concerning homoclinic solutions of impulsive differential equations by variational methods. So it is a novel method to employ variational methods to investigate the existence of homoclinic solutions for impulsive differential equations.

Motivated by the above papers, we will establish a new result for (1.1).

Let

$$E = \left\{ u \in W^{1,p}(\mathbb{R},\mathbb{R}) \Big| \int_{\mathbb{R}} \big[\big|\dot{u}(t)\big|^p + a(t)\big|u(t)\big|^p\big]\,dt < +\infty, \right.$$
$$\left. \big\{u(t_j)\big\}_{j=-\infty}^{+\infty} \in l^2, u(\pm\infty) = 0, u(kT) = 0, k \in \mathbb{Z} \right\},$$

where l^2 denotes the space of sequences whose second powers are summable on $\mathbb{Z}$, that is,

$$\sum_{j\in\mathbb{Z}} |a_j|^2 < +\infty, \quad \forall a = \{a_j\}_{j=-\infty}^{+\infty} \in l^2.$$

The space l^2 is equipped with the following norm:

$$\|a\|_{l^2} = \left(\sum_{j\in\mathbb{Z}} |a_j|^2\right)^{1/2}.$$

Similar to [1], we can check that E is a Banach space with the norm given by

$$\|u\| = \left(\int_{\mathbb{R}} \left[|\dot{u}(t)|^p + a(t)|u(t)|^p \right] dt \right)^{1/p}.$$

It is obvious that

$$E \subset L^{p_1}(\mathbb{R})$$

with the embedding being continuous. Here $L^{p_1}(\mathbb{R})$ $(p_1 \in [p, +\infty])$ denotes the Banach spaces of functions on $\mathbb{R}$ with values in $\mathbb{R}$ under the norm

$$\|u\|_{L^{p_1}} = \left\{ \int_{\mathbb{R}} |u(t)|^{p_1} dt \right\}^{1/p_1}.$$

Here and in subsequence, $(,)$ and $|\cdot|$ denote the inner product and norm in $\mathbb{R}$, respectively. C_i $(i = 1, 2, \ldots)$ denote different positive constants. Now, we state our main result.

Theorem 1.1 *Suppose that a, I, and V satisfy* (V1) *and the following conditions*:

(A) $a \in C(\mathbb{R}, (0, +\infty))$, $a(t + T) = a(t)$, *and* $a(t) \to +\infty$ *as* $|t| \to +\infty$;

(V2)′ $V(t,x) = V_1(t,x) - V_2(t,x)$, $V_1, V_2 \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, *and there exists a constant* $R > 0$ *such that*

$$\left| \nabla V(t,x) \right| = o\left(|x|^{p-1}\right) \quad as\ x \to 0\ uniformly\ in\ t \in \mathbb{R};$$

(V3)′ *there is a constant* $\mu > p$ *such that*

$$0 < \mu V_1(t,x) \le \left(\nabla V_1(t,x), x \right), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R} \backslash \{0\};$$

(V5) $V_2(t, 0) = 0$ *and there exists a constant* $\varrho \in (p, \mu)$ *such that*

$$\left(\nabla V_2(t,x), x \right) \le \varrho V_2(t,x), \quad \forall (t,x) \in \mathbb{R} \times \mathbb{R};$$

(I)′ $I \in C(\mathbb{R}, \mathbb{R})$, *and there exists a constant c with* $0 < c < \frac{(\mu - p)}{(\mu + p)mT^{p/q}}$ *such that*

$$\left| I(x) \right| \le c|x|^{p-1}, \qquad p \int_0^x I(t)\,dt - I(x)x \le 0, \quad \forall x, y \in \mathbb{R}.$$

Then problem (1.1) *has a nontrivial homoclinic solution.*

2 Preliminaries

Lemma 2.1 [21] *Let E be a real Banach space and $\varphi \in C^1(E, \mathbb{R})$, $e \in E$, $r > 0$ be such that $\|e\| > r$ and*

$$b := \inf_{\|y\| = r} \varphi(y) > \varphi(0) \ge \varphi(e).$$

Let

$$\Phi = \big\{ h \in C\big([0,1],E\big) \,\big|\, h(0) = 0, h(1) = e \big\}, \qquad d = \inf_{h \in \Phi} \max_{s \in [0,1]} \varphi\big(h(s)\big).$$

Then, for each $\varepsilon > 0$, $\delta > 0$, there exists $y \in E$ such that

(i) $c - 2\varepsilon \le \varphi(y) \le c + 2\varepsilon$;

(ii) $\mathrm{dist}(y, E) \le 2\delta$;

(iii) $\|\varphi'(y)\| \le \frac{8\varepsilon}{\delta}$.

Lemma 2.2 *Assume that* (V3)′ *and* (V5) *hold. Then for every* $(t, x) \in \mathbb{R} \times \mathbb{R}$,

(i) $s^{-\mu} V_1(t, sx)$ *is nondecreasing on* $(0, +\infty)$;

(ii) $s^{-\varrho} V_2(t, sx)$ *is nonincreasing on* $(0, +\infty)$.

The proof of Lemma 2.2 is routine and we omit it. Similar to [21–23], we have the following lemma.

Lemma 2.3 *For any $u \in E$, the following inequalities hold*:

$$\|u\|_\infty := \sup_{t \in \mathbb{R}} \big|u(t)\big| \le \big(\max\{(p-1)/2, 1/2a_0^{p-1}\}\big)^{1/p} \|u\| := C_1 \|u\|,$$

$$\sum_{j=-\infty}^{+\infty} \big|u(t_j)\big|^p \le m T^{p/q} \|\dot{u}\|_{L^p}^p,$$

where $C_1 = (\max\{(p-1)/2, 1/2a_2^{p-1}\})^{1/p}$, $a_0 = \min_{t \in \mathbb{R}}\{a(t)\}$ from (A).

The following lemma comes from [1], which is similar to a weak version of Lieb's lemma [24].

Lemma 2.4 [1] *If $\{u_n\}$ is bounded in E and u_n does not converge to 0 in measure, then there exist a sequence $\{x_{n_k}\} \subset \mathbb{Z}$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k}(\cdot + x_{n_k} T) \rightharpoonup u \ne 0$ in E.*

The functional φ corresponding to (1.1) on E is given by

$$\varphi(u) = \int_{\mathbb{R}} \frac{1}{p}\big[\big|\dot{u}(t)\big|^p + a(t)\big|u(t)\big|^p\big]\,dt - \int_{\mathbb{R}} V\big(t, u(t)\big)\,dt$$

$$+ \sum_{j=-\infty}^{\infty} \int_0^{u(t_j)} I(s)\,ds, \quad u \in E. \tag{2.1}$$

Lemma 2.5 *If* (V1), (A), (I)′, *and* (V2)′ *hold, then $\varphi \in C^1(E, \mathbb{R})$ and*

$$\big\langle \varphi'(u), v \big\rangle = \int_{\mathbb{R}} \big[\big|\dot{u}(t)\big|^{p-2}\big(\dot{u}(t), \dot{v}(t)\big) + a(t)\big|u(t)\big|^{p-2}\big(u(t), v(t)\big)\big]\,dt$$

$$- \int_{\mathbb{R}} \big(\nabla V\big(t, u(t)\big), v(t)\big)\,dt + \sum_{j=-\infty}^{\infty} \big(I\big(u(t_j)\big), v(t_j)\big). \tag{2.2}$$

Furthermore, the critical points of φ in E are classical solutions of (1.1) *with $u(\pm\infty) = 0$.*

Proof Firstly, we show that $\varphi : E \to \mathbb{R}$. By (V2)$'$, for any given $\varepsilon_0 > 0$, there exists $\gamma_0 > 0$ such that

$$\left| \nabla V(t,x) \right| \le p\varepsilon_0 a_0 |x|^{p-1}, \quad t \in \mathbb{R}, |x| \le \gamma_0. \tag{2.3}$$

Then, by $V(t,0) = 0$ and (2.3), we have

$$\left| V(t,x) \right| = \left| \int_0^1 \left(\nabla V(t,sx), x \right) ds \right| \le \varepsilon_0 a_0 |x|^p, \quad \forall t \in \mathbb{R}, x \in \mathbb{R}. \tag{2.4}$$

From (2.4), we have

$$
\begin{aligned}
\left| \int_{\mathbb{R}} V\left(t, u(t)\right) dt \right| &\le \int_{\mathbb{R}} \left| V\left(t, u(t)\right) \right| dt \\
&\le \int_{\mathbb{R}} \varepsilon_0 a_0 \left| u(t) \right|^p dt \\
&\le \int_{\mathbb{R}} \varepsilon_0 a(t) \left| u(t) \right|^p dt \le \varepsilon_0 \|u\|^p, \quad u \in E.
\end{aligned} \tag{2.5}
$$

From (I)$'$ and Lemma 2.3, we have

$$
\begin{aligned}
\sum_{j=-\infty}^{\infty} \left| \int_0^{u(t_j)} I(s)\, ds \right| &\le \sum_{j=-\infty}^{\infty} \int_{\min\{0,u(t_j)\}}^{\max\{0,u(t_j)\}} \left| I(s) \right| ds \\
&\le \frac{1}{p} \sum_{j=-\infty}^{\infty} c \left| u(t_j) \right|^p \le \frac{cmT^{p/q}}{p} \|\dot{u}\|_{L^p}^p \le \frac{cmT^{p/q}}{p} \|u\|^p.
\end{aligned} \tag{2.6}
$$

It follows from (2.1), (2.5), and (2.6) that $\varphi : E \to \mathbb{R}$. Next we prove that $\varphi \in C^1(E, \mathbb{R})$. Rewrite φ as follows:

$$\varphi(u) = \varphi_1(u) - \varphi_2(u) + \varphi_3(u),$$

where

$$\varphi_1(u) := \int_{\mathbb{R}} \frac{1}{p} \left[\left| \dot{u}(t) \right|^p + a(t) \left| u(t) \right|^p \right] dt,$$

$$\varphi_2(u) := \int_{\mathbb{R}} V\left(t, u(t)\right) dt, \qquad \varphi_3(u) := \sum_{j=-\infty}^{\infty} \int_0^{u(t_j)} I(s)\, ds.$$

It is easy to check that $\varphi_1, \varphi_3 \in C^1(E, \mathbb{R})$ and

$$\left\langle \varphi_1'(u), v \right\rangle = \int_{\mathbb{R}} \left[\left| \dot{u}(t) \right|^{p-2} \left(\dot{u}(t), \dot{v}(t) \right) + a(t) \left| u(t) \right|^{p-2} \left(u(t), v(t) \right) \right] dt,$$

$$\left\langle \varphi_3'(u), v \right\rangle = \sum_{j=-\infty}^{\infty} \left(I\left(u(t_j)\right), v(t_j) \right), \quad \forall u, v \in E.$$

Next we prove that $\varphi_2 \in C^1(E, \mathbb{R})$ and

$$\left\langle \varphi_2'(u), v \right\rangle = \int_{\mathbb{R}} \left(\nabla V\left(t, u(t)\right), v(t) \right) dt, \quad \forall u, v \in E.$$

By (2.3), we have

$$
\begin{aligned}
\left|\left(\nabla V\bigl(t, u_n(t)\bigr), v(t)\right)\right| &\leq \varepsilon_0 p a_0 \bigl|u_n(t)\bigr|^{p-1}\bigl|v(t)\bigr| \\
&\leq 2^{p-2}\varepsilon_0 p a(t)\bigl(\bigl|u_n(t) - u(t)\bigr|^{p-1} + \bigl|u(t)\bigr|^{p-1}\bigr)\bigl|v(t)\bigr| \\
&:= g_n(t), \quad t \in \mathbb{R}.
\end{aligned}
\tag{2.7}
$$

Let $u_n \to u$ in E, so $u_n(t) \to u(t)$ for almost every $t \in \mathbb{R}$, we have

$$
\lim_{n \to +\infty} g_n(t) = 2^{p-2} p \varepsilon_0 a(t)\bigl|u(t)\bigr|^{p-1}\bigl|v(t)\bigr| := g(t)
\tag{2.8}
$$

and

$$
\begin{aligned}
&\lim_{n \to +\infty} \int_{\mathbb{R}} g_n(t)\, dt \\
&= \lim_{n \to +\infty} \int_{\mathbb{R}} \Bigl[2^{p-2} p \varepsilon_0 a(t)\bigl(\bigl|u_n(t) - u(t)\bigr|^{p-1} + \bigl|u(t)\bigr|^{p-1}\bigr)\bigl|v(t)\bigr|\Bigr]\, dt \\
&= \lim_{n \to +\infty} \int_{\mathbb{R}} 2^{p-2} p \varepsilon_0 a(t)\bigl|u_n(t) - u(t)\bigr|^{p-1}\, dt + \int_{\mathbb{R}} \Bigl[2^{p-2} p \varepsilon_0 a(t)\bigl|u(t)\bigr|^{p-1}\bigl|v(t)\bigr|\Bigr]\, dt \\
&= \int_{\mathbb{R}} \Bigl[2^{p-2} p \varepsilon_0 a(t)\bigl|u(t)\bigr|^{p-1}\bigl|v(t)\bigr|\Bigr]\, dt \\
&:= \int_{\mathbb{R}} g(t)\, dt < +\infty.
\end{aligned}
\tag{2.9}
$$

Then by (2.7), (2.8), (2.9), and Lebesgue's dominated convergence theorem, we have

$$
\lim_{n \to +\infty} \int_{\mathbb{R}} \left(\nabla V\bigl(t, u_n(t)\bigr), v(t)\right) dt = \int_{\mathbb{R}} \left(\nabla V\bigl(t, u(t)\bigr), v(t)\right) dt.
\tag{2.10}
$$

Therefore, for any $u, v \in E$ and for any function $\theta : \mathbb{R} \to (0,1)$, from (2.10), we have

$$
\begin{aligned}
\left\langle \varphi_2'(u), v \right\rangle &= \lim_{h \to 0^+} \frac{\varphi_2(u + hv) - \varphi_2(u)}{h} \\
&= \lim_{h \to 0^+} \frac{1}{h} \int_{\mathbb{R}} \Bigl[V\bigl(t, u(t) + hv(t)\bigr) - V\bigl(t, u(t)\bigr)\Bigr]\, dt \\
&= \lim_{h \to 0^+} \int_{\mathbb{R}} \left(\nabla V\bigl(t, u(t) + \theta(t) h v(t)\bigr), v(t)\right) dt \\
&= \int_{\mathbb{R}} \left(\nabla V\bigl(t, u(t)\bigr), v(t)\right) dt, \quad \forall u, v \in E.
\end{aligned}
\tag{2.11}
$$

Now, we prove that $\varphi_2 \in C^1(E, \mathbb{R})$. From (2.11), $u_n \to u$ in E and $V \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, we have

$$
\begin{aligned}
&\lim_{n \to +\infty} \left|\left\langle \varphi_2'(u_n) - \varphi_2'(u), v \right\rangle\right| \\
&= \lim_{n \to +\infty} \left|\int_{\mathbb{R}} \left(\nabla V\bigl(t, u_n(t)\bigr) - \nabla V\bigl(t, u(t)\bigr), v(t)\right) dt\right| \\
&\leq \lim_{n \to +\infty} \int_{\mathbb{R}} \bigl|\nabla V\bigl(t, u_n(t)\bigr) - \nabla V\bigl(t, u(t)\bigr)\bigr|\bigl|v(t)\bigr|\, dt \\
&= \int_{\mathbb{R}} \lim_{n \to +\infty} \bigl|\nabla V\bigl(t, u_n(t)\bigr) - \nabla V\bigl(t, u(t)\bigr)\bigr|\bigl|v(t)\bigr|\, dt = 0, \quad \forall v \in E.
\end{aligned}
$$

This shows that $\varphi_2 \in C^1(E, \mathbb{R})$. Therefore $\varphi \in C^1(E, \mathbb{R})$ and (2.2) holds. Finally, we prove that the critical points of φ in E are classical solutions of (1.1) with $u(\pm\infty) = 0$. Let $u \in E$ be a critical point of φ, then for any $v \in C_0^\infty(\mathbb{R})$, we have

$$
\begin{aligned}
0 &= \langle \varphi'(u), v \rangle \\
&= \int_{\mathbb{R}} \left[|\dot{u}(t)|^{p-2} (\dot{u}(t), \dot{v}(t)) + a(t) |u(t)|^{p-2} (u(t), v(t)) \right] dt \\
&\quad - \int_{\mathbb{R}} (\nabla V(t, u(t)), v(t)) \, dt + \sum_{j=-\infty}^{\infty} (I(u(t_j)), v(t_j)).
\end{aligned}
\tag{2.12}
$$

Let $v \in C_0^\infty(\mathbb{R})$ such that $v(t) = 0$ for any $t \in (-\infty, t_j] \cup [t_{j+1}, +\infty)$, $\forall j \in \mathbb{Z}$, and $v \in C_0^\infty([t_j, t_{j+1}])$. Hence, we have

$$
\begin{aligned}
0 &= \int_{t_j}^{t_{j+1}} \left[|\dot{u}(t)|^{p-2} (\dot{u}(t), \dot{v}(t)) + a(t) |u(t)|^{p-2} (u(t), v(t)) \right] dt \\
&\quad - \int_{t_j}^{t_{j+1}} (\nabla V(t, u(t)), v(t)) \, dt,
\end{aligned}
$$

which implies that

$$
\frac{d}{dt} \left(|\dot{u}(t)|^{p-2} \dot{u}(t) \right) - a(t) |u(t)|^{p-2} u(t) + \nabla V(t, u(t)) = 0, \quad \text{a.e. } t \in (t_j, t_{j+1}).
$$

The proof is complete. □

3 Proof of Theorem 1.1

Proof of Theorem 1.1 Firstly, we prove that under the assumptions of Theorem 1.1, there exist $e \in E$ and $r > 0$ such that $\|e\| > r$ and

$$
b := \inf_{\|y\|=r} \varphi(y) > \varphi(0) \geq \varphi(e).
$$

It is easy to see that $\varphi(0) = 0$. From (V2)′, there exists $\delta \in (0, 1)$ such that

$$
|\nabla V(t, x)| \leq \frac{a_0}{2} |x|^{p-1} \quad \text{for } t \in \mathbb{R}, |x| \leq \delta.
\tag{3.1}
$$

By $V(t, 0) = 0$ and (3.1), we have

$$
|V(t, x)| \leq \frac{a_0}{2p} |x|^p \quad \text{for } t \in \mathbb{R}, |x| \leq \delta.
\tag{3.2}
$$

Let $\|u\| = \frac{\delta}{C_1} := r$, it follows from Lemma 2.3 that $|u(t)| \leq \delta$. From (2.6) and (3.2), we have

$$
\begin{aligned}
\varphi(u) &= \frac{1}{p} \|u\|^p + \sum_{j=-\infty}^{\infty} \int_0^{u(t_j)} I(t) \, dt - \int_{\mathbb{R}} V(t, u(t)) \, dt \\
&\geq \frac{1}{p} \|u\|^p - \frac{cmT^{p/q}}{p} \|\dot{u}\|_{L^p}^p - \frac{1}{2p} \int_{\mathbb{R}} a_0 |u(t)|^p \, dt
\end{aligned}
$$

$$\geq \frac{1}{p}\|u\|^p - \frac{cmT^{p/q}}{p}\|\dot{u}\|_{L^p}^p - \frac{1}{2p}\int_{\mathbb{R}} a(t)|u(t)|^p \, dt$$

$$\geq \min\left\{\frac{1}{p} - \frac{cmT^{p/q}}{p}, \frac{1}{2p}\right\}\|u\|^p := b. \tag{3.3}$$

Since $0 < c < \frac{(\mu - p)}{(\mu + p)mT^{p/q}}$, we know that $b > 0$. From Lemma 2.2(ii), we have for any $u \in E$

$$\int_{-3}^{3} V_2(t, u(t)) \, dt = \int_{\{t \in [-3,3]: |u(t)| > 1\}} V_2(t, u(t)) \, dt + \int_{\{t \in [-3,3]: |u(t)| \leq 1\}} V_2(t, u(t)) \, dt$$

$$\leq \int_{\{t \in [-3,3]: |u(t)| > 1\}} V_2\left(t, \frac{u(t)}{|u(t)|}\right)|u(t)|^\varrho \, dt + \int_{-3}^{3} \max_{|x| \leq 1} V_2(t, x) \, dt$$

$$\leq \|u\|_\infty^\varrho \int_{-3}^{3} \max_{|x|=1} V_2(t, x) \, dt + \int_{-3}^{3} \max_{|x| \leq 1} V_2(t, x) \, dt$$

$$\leq C_1^\varrho \|u\|^\varrho \int_{-3}^{3} \max_{|x|=1} V_2(t, x) \, dt + \int_{-3}^{3} \max_{|x| \leq 1} V_2(t, x) \, dt$$

$$= C_2\|u\|^\varrho + C_3, \tag{3.4}$$

where $C_2 = C_1^\varrho \int_{-3}^{3} \max_{|x|=1} V_2(t, x) \, dt$, $C_3 = \int_{-3}^{3} \max_{|x| \leq 1} V_2(t, x) \, dt$. Take $\omega \in E$ such that

$$|\omega(t)| = \begin{cases} 1 & \text{for } |t| \leq 1, \\ 0 & \text{for } |t| \geq 3, \end{cases} \tag{3.5}$$

and $|\omega(t)| \leq 1$ for $|t| \in (1, 3]$. For $s > 1$, from Lemma 2.2(i) and (3.5), we get

$$\int_{-1}^{1} V_1(t, s\omega(t)) \, dt \geq s^\mu \int_{-1}^{1} V_1(t, \omega(t)) \, dt = C_4 s^\mu, \tag{3.6}$$

where $C_4 = \int_{-1}^{1} V_1(t, \omega(t)) \, dt > 0$. From (2.1), (2.6), (3.4), (3.5), (3.6), we get for $s > 1$

$$\varphi(s\omega) = \frac{s^p}{p}\|\omega\|^p + \sum_{j=-\infty}^{\infty} \int_{0}^{s\omega(t)} I(t) \, dt + \int_{\mathbb{R}} \left[V_2(t, s\omega(t)) - V_1(t, s\omega(t))\right] dt$$

$$\leq \frac{s^p}{p}\|\omega\|^p + \frac{cmT^{p/q}s^p}{p}\|\dot{\omega}\|_{L^p}^p + \int_{-3}^{3} V_2(t, s\omega(t)) \, dt - \int_{-1}^{1} V_1(t, s\omega(t)) \, dt$$

$$\leq \left(\frac{1 + cmT^{p/q}}{p}\right)s^p\|\omega\|^p + C_2 s^\varrho \|\omega\|^\varrho + C_3 - C_4 s^\mu. \tag{3.7}$$

Since $\mu > \varrho > p$ and $C_4 > 0$, it follows from (3.7) that there exists $s_1 > 1$ such that $\|s_1\omega\| > r$ and $\varphi(s_1\omega) \leq 0$. Set $e = s_1\omega(t)$, then $e \in E$, $\|e\| = \|s_1\omega\| > r$ and $\varphi(e) = \varphi(s_1\omega) \leq 0$.

Secondly, we prove that under the assumptions of Theorem 1.1, there exists a bounded sequence $\{u_n\}$ in E such that

$$\varphi(u_n) \to d, \qquad \varphi'(u_n) \to 0, \qquad \text{dist}(u_n, E) \to 0, \quad n \to \infty, \tag{3.8}$$

where $\Phi = \{h \in C([0,1], E) | h(0) = 0, h(1) = e\}$, $d = \inf_{h \in \Phi} \max_{s \in [0,1]} \varphi(h(s))$. Furthermore, $\{u_n\}$ does not converge to 0 in measure.

From (I)′ and Lemma 2.3, we have

$$\sum_{j=-\infty}^{\infty} \left(I\big(u(t_j)\big), u(t_j)\right) \le \sum_{j=-\infty}^{\infty} \left|I\big(u(t_j)\big)\right|\left|u(t_j)\right| \le \sum_{j=-\infty}^{\infty} c\left|u(t_j)\right|^p \le cmT^{p/q}\|\dot{u}\|_{L^p}^p. \tag{3.9}$$

From (2.1), (2.2), (2.6), (3.9), (V3)′, and (V5), we have

$$
\begin{aligned}
pd + \frac{pd}{\mu}\|u_n\| &\ge p\varphi(u_n) - \frac{p}{\mu}\big\langle \varphi'(u_n), u_n\big\rangle \\
&= \frac{\mu - p}{\mu}\|u_n\|^p + p\sum_{j=-\infty}^{\infty}\int_0^{u(t_j)} I(s)\,ds - \frac{p}{\mu}\sum_{j=-\infty}^{\infty}\big(I\big(u_n(t_j)\big), u_n(t_j)\big) \\
&\quad - p\int_{\mathbb{R}}\left[V_1\big(t, u_n(t)\big) - \frac{1}{\mu}\big(\nabla V_1\big(t, u_n(t)\big), u_n(t)\big)\right]dt \\
&\quad + p\int_{\mathbb{R}}\left[V_2\big(t, u_n(t)\big) - \frac{1}{\mu}\big(\nabla V_2\big(t, u_n(t)\big), u_n(t)\big)\right]dt \\
&\ge \frac{\mu - p}{\mu}\|u_n\|^p - cmT^{p/q}\|\dot{u}_n\|_{L^p}^p - \frac{pcmT^{p/q}}{\mu}\|\dot{u}_n\|_{L^p}^p \\
&\ge \left(\frac{\mu - p}{\mu} - \frac{(\mu + p)cmT^{p/q}}{\mu}\right)\|u_n\|^p.
\end{aligned}
$$

Since $0 < c < \frac{(\mu - p)}{(\mu + p)pT^{p/q}}$, the above inequalities implies that there exists a constant $C_5 > 0$ such that

$$\|u_n\| \le C_5, \quad n \in \mathbb{N}. \tag{3.10}$$

By (V2)′, we have

$$\frac{1}{p}\big(\nabla V(t, x), x\big) - V(t, x) = o\big(|x|^p\big) \quad \text{as } x \to 0, t \in \mathbb{R}, \tag{3.11}$$

which implies that

$$C_6 := \sup_{|x| \le C_1 C_5} \frac{\frac{1}{p}(\nabla V(t, x), x) - V(t, x)}{|x|^p} < \infty. \tag{3.12}$$

For any $\varepsilon_1 > 0$, there exists $\delta_1 > 0$ such that

$$\frac{1}{p}\big(\nabla V(t, x), x\big) - V(t, x) \le \varepsilon_1 a_0 |x|^p, \quad |x| \le \delta_1, t \in \mathbb{R}. \tag{3.13}$$

It follows from (3.10), (3.12), and (3.13) that

$$
\begin{aligned}
&\int_{\mathbb{R}}\left[\frac{1}{p}\big(\nabla V\big(t, u_n(t)\big), u_n(t)\big) - V\big(t, u_n(t)\big)\right]dt \\
&= \left[\int_{\{\mathbb{R}:|u_n(t)|>\delta_1\}} + \int_{\{\mathbb{R}:|u_n(t)|\le\delta_1\}}\right]\left[\frac{1}{p}\big(\nabla V\big(t, u_n(t)\big), u_n(t)\big) - V\big(t, u_n(t)\big)\right]dt \\
&\le \int_{\{\mathbb{R}:|u_n(t)|>\delta_1\}} C_6\left|u_n(t)\right|^p dt + \int_{\{\mathbb{R}:|u_n(t)|\le\delta_1\}} \varepsilon_1 a_0\left|u_n(t)\right|^p dt
\end{aligned}
$$

$$\leq \int_{\{\mathbb{R}:|u_n(t)|>\delta_1\}} C_6 |u_n(t)|^p \, dt + \int_{\{\mathbb{R}:|u_n(t)|\leq\delta_1\}} \varepsilon_1 a(t) |u_n(t)|^p \, dt$$

$$\leq \text{meas}\{|u_n(t)| > \delta_1\} C_6 \|u_n\|_\infty^p + \varepsilon_1 \|u_n\|^p$$

$$\leq \text{meas}\{|u_n(t)| > \delta_1\} C_6 C_1^p \|u_n\|^p + \varepsilon_1 \|u_n\|^p$$

$$\leq \text{meas}\{|u_n(t)| > \delta_1\} C_5^p C_6 C_1^p + \varepsilon_1 C_5^p. \tag{3.14}$$

If $\{u_n\}$ converges to 0 in measure, from (I)$'$ and (3.14), we have

$$0 < d = \varphi(u_n) - \frac{1}{p}\langle \varphi'(u_n), u_n \rangle + o(1)$$

$$= \int_\mathbb{R} \left[\frac{1}{p} (\nabla V(t, u_n(t)), u_n(t)) - V(t, u_n(t)) \right] dt$$

$$+ \sum_{j=-\infty}^\infty \int_0^{u_n(t_j)} I(s) \, ds - \frac{1}{p} \sum_{j=-\infty}^\infty (I(u_n(t_j)), u_n(t_j)) + o(1)$$

$$\leq \text{meas}\{|u_n(t)| > \delta_1\} C_5^p C_6 C_1^p + \varepsilon_1 C_5^p$$

$$+ \frac{1}{p} \sum_{j=-\infty}^\infty \left[p \int_0^{u_n(t_j)} I(s) \, ds - (I(u_n(t_j)), u_n(t_j)) \right] + o(1)$$

$$= o(1). \tag{3.15}$$

This is a contradiction. Hence, (3.8) holds and $\{u_n\}$ does not converge to 0 in measure.

Finally, from (3.10), we know that $u_n \rightharpoonup u$ in E, what we need to do is to prove that $\varphi'(u) = 0$. By (3.8), $\{u_n\}$ does not converge to 0 in measure and Lemma 2.4, there exists a sequence $\{x_{n_k}\}$ in $\mathbb{Z}$ such that $\omega_k := u_{n_k}(\cdot + x_{n_k}T) \rightharpoonup u \neq 0$ in E. For any fixed $k \in \mathbb{N}$, set $s = t + x_{n_k}T$ and $v_k(s) := v(s - x_{n_k}T)$. Then $s_j := t_j + x_{n_k}T$ ($j \in \mathbb{Z}$) are impulsive points and

$$\|v_k\| = \left\{ \int_\mathbb{R} \left[|\dot{v}_k(s)|^p + a(s)|v_k(s)|^p \right] ds \right\}^{1/p}$$

$$= \left\{ \int_\mathbb{R} \left[|\dot{v}(s)|^p + a(s)|v(s)|^p \right] ds \right\}^{1/p} = \|v\|. \tag{3.16}$$

For any $v \in C_0^\infty(\mathbb{R})$ with $v(kT) = 0$, by (V1) and (A), we have

$$\langle \varphi'(\omega_k), v \rangle = \int_\mathbb{R} \left[|\dot{\omega}_k(t)|^{p-2} (\dot{\omega}_k(t), \dot{v}(t)) + a(t)|\omega_k(t)|^{p-2} (\omega_k(t), v(t)) \right] dt$$

$$- \int_\mathbb{R} (\nabla V(t, \omega_k(t)), v(t)) \, dt + \sum_{j=-\infty}^\infty (I(\omega_k(t_j)), v(t_j))$$

$$= \int_\mathbb{R} |\dot{u}_{n_k}(t + x_{n_k}T)|^{p-2} (\dot{u}_{n_k}(t + x_{n_k}T), \dot{v}(t)) \, dt$$

$$+ \int_\mathbb{R} a(s - x_{n_k}T) |\dot{u}_{n_k}(t + x_{n_k}T)|^{p-2} (u_{n_k}(t + x_{n_k}T), v(t)) \, dt$$

$$- \int_\mathbb{R} (\nabla V(t, u_{n_k}(t + x_{n_k}T)), v(t)) \, dt + \sum_{j=-\infty}^\infty (I(u_{n_k}(t + x_{n_k}T)), v(t_j))$$

$$= \int_\mathbb{R} \left[|\dot{u}_{n_k}(s)|^{p-2} (\dot{u}_{n_k}(s), \dot{v}(s - x_{n_k}T)) \right.$$

$$+ a(s)\big|\dot{u}_{n_k}(s)\big|^{p-2}\big(u_{n_k}(s), v(s - x_{n_k}T)\big)\big]\,dt$$

$$- \int_{\mathbb{R}} \big(\nabla V\big(s - x_{n_k}T, u_{n_k}(s)\big), v(s - x_{n_k}T)\big)\,dt$$

$$+ \sum_{j=-\infty}^{\infty} \big(I\big(u_{n_k}(s_j)\big), v(s_j - x_{n_k}T)\big)$$

$$= \int_{\mathbb{R}} \Big[\big|\dot{u}_{n_k}(s)\big|^{p-2}\big(\dot{u}_{n_k}(s), \dot{v}_k(s)\big) + a(s)\big|\dot{u}_{n_k}(s)\big|^{p-2}\big(u_{n_k}(s), v_k(s)\big)\Big]\,dt$$

$$- \int_{\mathbb{R}} \big(\nabla V\big(s, u_{n_k}(s)\big), v_k(s)\big)\,dt + \sum_{j=-\infty}^{\infty} \big(I\big(u_{n_k}(s_j)\big), v_k(s_j)\big)$$

$$= \big\langle \varphi'(u_{n_k}), v_k \big\rangle. \tag{3.17}$$

By (3.17), we have

$$\big|\big\langle \varphi'(\omega_k), v\big\rangle\big| = \big|\big\langle \varphi'(u_{n_k}), v_k\big\rangle\big| \le \big\|\varphi'(u_{n_k})\big\|_{E^*} \cdot \|v_k\| = \big\|\varphi'(u_{n_k})\big\|_{E^*} \cdot \|v\|, \tag{3.18}$$

where E^* is the dual space of E. Equation (3.18) implies that

$$\big\langle \varphi'(\omega_k), v\big\rangle \to 0 \quad \text{as } k \to \infty. \tag{3.19}$$

For any $a > 0$, let $\chi_a(t) = 1$ for $t \in [-a, a]$ and $\chi_a(t) = 0$ for $t \in (-\infty, -a) \cup (a, \infty)$. Then from (2.2), we have

$$\big\langle \varphi'(u_n) - \varphi'(u), \chi_a(u_n - u)\big\rangle$$

$$= \int_{-a}^{a} \big(\big|\dot{u}_n(t)\big|^{p-2}\dot{u}_n(t) - \big|\dot{u}(t)\big|^{p-2}\dot{u}(t), \dot{u}_n(t) - \dot{u}(t)\big)\,dt$$

$$+ \int_{-a}^{a} a(t)\big(\big|u_n(t)\big|^{p-2}u_n(t) - \big|u(t)\big|^{p-2}u(t), u_n(t) - u(t)\big)\,dt$$

$$- \int_{-a}^{a} \big(\nabla V\big(t, u_n(t)\big) - \nabla V\big(t, u(t)\big), u_n(t) - u(t)\big)\,dt$$

$$+ \sum_{t_j \in [-a,a]} \big(I\big(t, u_n(t_j)\big) - I\big(t, u(t_j)\big), u_n(t_j) - u(t_j)\big)$$

$$\ge \int_{-a}^{a} \big|\dot{u}_n(t) - \dot{u}(t)\big|^p\,dt + \int_{-a}^{a} a(t)\big|u_n(t) - u(t)\big|^p\,dt$$

$$- \int_{-a}^{a} \big(\nabla V\big(t, u_n(t)\big) - \nabla V\big(t, u(t)\big), u_n(t) - u(t)\big)\,dt$$

$$+ \sum_{t_j \in [-a,a]} \big(I\big(t, u_n(t_j)\big) - I\big(t, u(t_j)\big), u_n(t_j) - u(t_j)\big). \tag{3.20}$$

Since $\varphi'(u_n) \to 0$ as $n \to +\infty$ and $u_n \rightharpoonup u$ in E, it follows from (3.8) that

$$\big\langle \varphi'(u_n) - \varphi'(u), \chi_a(u_n - u)\big\rangle \to 0 \quad \text{as } n \to \infty, \tag{3.21}$$

$$\int_{-a}^{a} \big(\nabla V\big(t, u_n(t)\big) - \nabla V\big(t, u(t)\big), u_n(t) - u(t)\big)\,dt \to 0 \quad \text{as } n \to \infty \tag{3.22}$$

and

$$\sum_{t_j \in [-a,a]} \left(I\big(t, u_n(t_j)\big) - I\big(t, u(t_j)\big), u_n(t_j) - u(t_j) \right) \to 0 \quad \text{as } n \to \infty. \tag{3.23}$$

It follows from (3.20), (3.21), (3.22), and (3.23) that

$$\int_{-a}^{a} |\dot{u}_n(t) - \dot{u}(t)|^p \, dt \to 0, \qquad \int_{-a}^{a} a(t) |u_n(t) - u(t)|^p \, dt \to 0 \quad \text{as } n \to \infty. \tag{3.24}$$

Assume that for some $A > 0$, $\text{supp}(v) \subset [-A, A]$. Since

$$\lim_{n \to \infty} \dot{u}_n(t) = \dot{u}(t), \qquad \lim_{n \to \infty} u_n(t) = \dot{u}(t), \quad \forall \text{a.e. } t \in \mathbb{R}, \tag{3.25}$$

$$\left| \left(|\dot{u}_n(t)|^{p-2} \dot{u}_n(t), \dot{v}(t) \right) \right| \le \frac{p-1}{p} |\dot{u}_n(t)|^p + \frac{1}{p} |\dot{v}(t)|^p, \quad \forall t \in \mathbb{R}, n = 1, 2, \ldots, \tag{3.26}$$

$$a(t) \left| \left(|u_n(t)|^{p-2} u_n(t), v(t) \right) \right|$$
$$\le a(t) \left[\frac{p-1}{p} |u_n(t)|^p + \frac{1}{p} |v(t)|^p \right], \quad \forall t \in \mathbb{R}, n = 1, 2, \ldots, \tag{3.27}$$

$$\lim_{n \to \infty} \int_{-A}^{A} \left[\frac{p-1}{p} |\dot{u}_n(t)|^p + \frac{1}{p} |\dot{v}(t)|^p \right] dt$$
$$= \frac{p-1}{p} \lim_{n \to \infty} \|\dot{u}_n\|_{L^p[-A,A]}^p + \frac{1}{p} \|\dot{v}\|_{L^p[-A,A]}^p$$
$$= \frac{p-1}{p} \|\dot{u}\|_{L^p[-A,A]}^p + \frac{1}{p} \|\dot{v}\|_{L^p[-A,A]}^p$$
$$= \int_{-A}^{A} \left[\frac{p-1}{p} |\dot{u}(t)|^p + \frac{1}{p} |\dot{v}(t)|^p \right] dt < +\infty \tag{3.28}$$

and

$$\lim_{n \to \infty} \int_{-A}^{A} a(t) \left[\frac{p-1}{p} |u_n(t)|^p + \frac{1}{p} |v(t)|^p \right] dt$$
$$= \frac{p-1}{p} \lim_{n \to \infty} \int_{-A}^{A} a(t) |u_n(t)|^p \, dt + \frac{1}{p} \int_{-A}^{A} a(t) |v(t)|^p \, dt$$
$$= \frac{p-1}{p} \int_{-A}^{A} a(t) |u(t)|^p \, dt + \frac{1}{p} \int_{-A}^{A} a(t) |v(t)|^p \, dt < +\infty, \tag{3.29}$$

then it follows from (3.25), (3.26), (3.27), (3.28), (3.29), and the Lebesgue dominated convergence theorem that

$$\int_{-A}^{A} \left(|\dot{u}_n(t)|^{p-2} \dot{u}_n(t), \dot{v}(t) \right) dt \to \int_{-A}^{A} \left(|\dot{u}(t)|^{p-2} \dot{u}(t), \dot{v}(t) \right) dt \quad \text{as } n \to \infty \tag{3.30}$$

and

$$\int_{-A}^{A} a(t) \left(|u_n(t)|^{p-2} u_n(t), v(t) \right) dt \to \int_{-A}^{A} a(t) \left(|u(t)|^{p-2} u(t), v(t) \right) dt \quad \text{as } n \to \infty. \tag{3.31}$$

For any $v \in E$ and $\varepsilon > 0$, take J_0 sufficiently large such that

$$\left(\sum_{j=J_0+1}^{+\infty} \left|v(t_j)\right|^p\right)^{1/p} \le \varepsilon, \qquad \left(\sum_{j=-\infty}^{-J_0-1} \left|v(t_j)\right|^p\right)^{1/p} \le \varepsilon. \tag{3.32}$$

Since $u_n \rightharpoonup u$ in E, $u_n \rightharpoonup u$ in $H^1([t_{-J_0}, t_{J_0}])$, therefore u_n uniformly converges to u in $[t_{-J_0}, t_{J_0}]$. By the continuity of I, there exists $N > 0$ such that, for $n > N$, we have

$$\left|\sum_{j=-J_0}^{J_0} \left[I\left(u_n(t_j)\right) - I\left(u(t_j)\right)\right]v(t_j)\right| \le \varepsilon. \tag{3.33}$$

From (I)$'$, we have

$$\left(\sum_{j=J_0+1}^{+\infty} \left[I\left(u_n(t_j)\right) - I\left(u(t_j)\right)\right]^{p/(p-1)}\right)^{(p-1)/p}$$

$$\le \left(\sum_{j=J_0+1}^{+\infty} \left[c\left(\left|u_n(t_j)\right|^{p-1} + \left|u(t_j)\right|^{p-1}\right)\right]^{p/(p-1)}\right)^{(p-1)/p}$$

$$\le 2^{1/p} c \left(\sum_{j=J_0+1}^{+\infty} \left(\left|u_n(t_j)\right|^p + \left|u(t_j)\right|^p\right)\right)^{(p-1)/p}$$

$$\le 2^{1/p} c \left(mT^{p/q} \|\dot{u}_n\|_{L^p}^p + mT^{p/q} \|\dot{u}\|_{L^p}^p\right)^{1/q}$$

$$\le 2cm^{1/q} T^{p/q^2} \max\left\{\sup_n \|\dot{u}_n\|_{L^p}^{p/q}, \|\dot{u}\|_{L^p}^{p/q}\right\}, \tag{3.34}$$

where $q = \frac{p}{p-1}$. Similarly, we have

$$\left(\sum_{j=-J_0-1}^{-\infty} \left[I\left(u_n(t_j)\right) - I\left(u(t_j)\right)\right]^{p/(p-1)}\right)^{(p-1)/p}$$

$$\le 2cm^{1/q} T^{p/q^2} \max\left\{\sup_n \|\dot{u}_n\|_{L^p}^{p/q}, \|\dot{u}\|_{L^p}^{p/q}\right\}. \tag{3.35}$$

It follows from (3.33), (3.34), (3.35), and the Cauchy-Schwarz inequality that

$$\left|\sum_{j=-\infty}^{+\infty} \left[I\left(u_n(t_j)\right) - I\left(u(t_j)\right)\right]v(t_j)\right|$$

$$\le \left|\sum_{j=J_0+1}^{+\infty} \left[I\left(u_n(t_j)\right) - I\left(u(t_j)\right)\right]v(t_j)\right|$$

$$+ \left|\sum_{j=-J_0}^{J_0} \left[I\left(u_n(t_j)\right) - I\left(u(t_j)\right)\right]v(t_j)\right|$$

$$+ \left|\sum_{j=-\infty}^{-J_0-1} \left[I\left(u_n(t_j)\right) - I\left(u(t_j)\right)\right]v(t_j)\right|$$

$$\leq \left(\sum_{j=J_0+1}^{+\infty} \left| I\big(u_n(t_j)\big) - I\big(u(t_j)\big) \right|^q \right)^{1/q} \left(\sum_{j=J_0+1}^{+\infty} \left| v(t_j) \right|^p \right)^{1/p}$$

$$+ \left| \sum_{j=-J_0}^{J_0} \left[I\big(u_n(t_j)\big) - I\big(u(t_j)\big) \right] v(t_j) \right|$$

$$+ \left(\sum_{j=-J_0-1}^{-\infty} \left| I\big(u_n(t_j)\big) - I\big(u(t_j)\big) \right|^q \right)^{1/q} \left(\sum_{j=-J_0-1}^{-\infty} \left| v(t_j) \right|^p \right)^{1/p}$$

$$\leq \left(1 + 4cm^{1/q} T^{p/q^2} \max\left\{ \sup_n \|\dot{u}_n\|_{L^p}^{p/q}, \|\dot{u}\|_{L^p}^{p/q} \right\} \right) \varepsilon. \tag{3.36}$$

It follows from (3.36) that

$$\lim_{n\to\infty} \sum_{j=-\infty}^{+\infty} \big(I\big(u_n(t_j)\big), v(t_j) \big) = \sum_{j=-\infty}^{+\infty} \big(I\big(u(t_j)\big), v(t_j) \big). \tag{3.37}$$

Notice that

$$\int_{-A}^{A} \big(\nabla V\big(t, u_n(t)\big), v(t) \big) \, dt \to \int_{-A}^{A} \big(\nabla V\big(t, u(t)\big), v(t) \big) \, dt \quad \text{as } n \to \infty. \tag{3.38}$$

Hence, from (3.30), (3.31), (3.37), and (3.38), we have

$$\big\langle \varphi'(u), v \big\rangle = \lim_{n\to\infty} \big\langle \varphi'(u_n), v \big\rangle = 0.$$

Therefore, $\varphi'(u) = 0$ and u is a nontrivial homoclinic solution of φ. □

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