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Infinitely many weak solutions for a fractional Schrödinger equation

Wei Dong[1*], Jiafa Xu[2] and Zhongli Wei[2,3]

*Correspondence: wdongau@aliyun.com
[1] Department of Mathematics, Hebei University of Engineering, Handan, Hebei 056038, China
Full list of author information is available at the end of the article

Abstract

In this paper we are concerned with the fractional Schrödinger equation $(-\Delta)^{\alpha}u + V(x)u = f(x,u)$, $x \in \mathbb{R}^N$, where $0 < \alpha < 1$, $N > 2\alpha$, $(-\Delta)^{\alpha}$ stands for the fractional Laplacian of order α, V is a positive continuous potential, and f is a continuous subcritical nonlinearity. We obtain the existence of infinitely many weak solutions for the above problem by the fountain theorem in critical point theory.

Keywords: fractional Laplacian; subcritical nonlinearity; fountain theorem; weak solution

1 Introduction

In this paper we consider the following fractional Schrödinger equation:

$$(-\Delta)^{\alpha}u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $0 < \alpha < 1$, $N > 2\alpha$, $(-\Delta)^{\alpha}$ stands for the fractional Laplacian of order α, and the potential $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying

(V) $0 < \inf_{x\in\mathbb{R}^N} V(x) = V_0 < \liminf_{|x|\to\infty} V(x) = V_\infty < \infty$.

The nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function, satisfying the subcritical condition.

(H1) There exist $d_1 > 0$, $d_2 > 0$ and $p \in (2, 2_\alpha^*)$ such that

$$\left|f(x,s)\right| \le d_1|s| + d_2|s|^{p-1}, \quad \forall (x,s) \in \mathbb{R}^N \times \mathbb{R},$$

where $2_\alpha^* = \frac{2N}{N-2\alpha}$ is the fractional critical exponent.

Recently, there have appeared plenty of works on the fractional Schrödinger equations; for example, see [1–11] and the references therein. In [1], Shang and Zhang considered the critical fractional Schrödinger equation

$$\varepsilon^{2\alpha}(-\Delta)^{\alpha}u + V(x)u = |u|^{2_\alpha^*-2}u + \lambda f(u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where ε and λ are positive parameters, V and f satisfy (V) and (H1), respectively. They obtained the result that (1.2) has a nonnegative ground state solution and investigated the relation between the number of solutions and the topology of the set where V attains its minimum for all sufficiently large λ and small ε. In [2], Shang *et al.* considered the existence of nontrivial solutions for (1.2) with $f(u) = |u|^{q-2}u$, where $2 < q < 2^*_\alpha$.

In [8], Hua and Yu studied the critical fractional Laplacian equation

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = |u|^{2^*_\alpha - 2}u + \mu u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $0 < \alpha < 2$, $\Omega \subset \mathbb{R}^N$, $N > (1 + \sqrt{2})\alpha$ is a bounded domain. They obtained the result that the problem (1.3) possesses a nontrivial ground state solution for any $\mu > 0$.

In [7], Secchi investigated the existence of radially symmetric solutions for (1.1) replacing $f(t,u)$ by $g(u)$, where g satisfies the following conditions.

(g1) $g : \mathbb{R} \to \mathbb{R}$ is of class $C^{1,\gamma}$ for some $\gamma > \max\{0, 1 - 2\alpha\}$, and odd,

(g2) $-\infty < \liminf_{t\to 0^+} \frac{g(t)}{t} \leq \limsup_{t\to 0^+} \frac{g(t)}{t} = -m < 0$,

(g3) $-\infty < \limsup_{t\to +\infty} \frac{g(t)}{t^{2^*_\alpha - 1}} \leq 0$,

(g4) for some $\xi > 0$ such that $G(\xi) = \int_0^\xi g(t)\,\mathrm{d}t > 0$.

Inspired by the mentioned papers, we first establish a compact embedding lemma via a fractional Gagliardo-Nirenberg inequality. Then by virtue of the fountain theorem in critical point theory, we get two existence results of infinitely many weak solutions for (1.1).

2 Preliminary results

In this section we offer some preliminary results which enable us to obtain the main existence theorems. First, we collect some useful facts of the fractional order Sobolev spaces.

For any $0 < \alpha < 1$, the fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ is defined by

$$H^\alpha(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N+2\alpha}{2}}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\},$$

endowed with the norm

$$\|u\|_{H^\alpha(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} |u|^2\,\mathrm{d}x + \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y \right)^{\frac{1}{2}},$$

where $[u]_{H^\alpha(\mathbb{R}^N)} = (\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2\alpha}}\,\mathrm{d}x\,\mathrm{d}y)^{1/2}$ is the so-called Gagliardo semi-norm of u. Let $\mathscr{S}$ be the Schwartz space of rapidly decaying C^∞ functions in $\mathbb{R}^N$, for any $u \in \mathscr{S}$ and $\alpha \in (0,1)$, and let $(-\Delta)^\alpha$ be defined as

$$(-\Delta)^\alpha u(x) = k_{N,\alpha}\,\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}}\,\mathrm{d}y = k_{N,\alpha} \lim_{\varepsilon \to 0} \int_{\mathscr{C}B_\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2\alpha}}\,\mathrm{d}y. \tag{2.1}$$

The symbol P.V. stands for the Cauchy principal value, and $k_{N,\alpha}$ is a dimensional constant that depends on N and α, precisely given by $k_{N,\alpha} = (\int_{\mathbb{R}^N} \frac{1-\cos\zeta_1}{|\zeta|^{N+2\alpha}}\,\mathrm{d}\zeta)^{-1}$.

Indeed, the fractional Laplacian $(-\Delta)^\alpha$ can be viewed as a pseudo-differential operator of symbol $|\xi|^{2\alpha}$, as stated in the following.

Lemma 2.1 (see [12]) *Let $\alpha \in (0,1)$ and $(-\Delta)^\alpha : \mathscr{S} \to L^2(\mathbb{R}^N)$ be the fractional Laplacian operator defined by* (2.1). *Then for any $u \in \mathscr{S}$,*

$$(-\Delta)^\alpha u(x) = \mathscr{F}^{-1}\big(|\xi|^{2\alpha}(\mathscr{F}u)\big)(x), \quad \forall \xi \in \mathbb{R}^N,$$

where $\mathscr{F}$ is the Fourier transform, i.e.,

$$\mathscr{F}(\phi)(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp\{-2\pi i\xi \cdot x\}\phi(x)\,\mathrm{d}x.$$

Now we can see that an alternative definition of the fractional Sobolev space $H^\alpha(\mathbb{R}^N)$ via the Fourier transform is as follows:

$$H^\alpha(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \big(1 + |\xi|^{2\alpha}\big)|\mathscr{F}u|^2\,\mathrm{d}\xi < +\infty\right\}.$$

It can be proved that

$$2k_{N,\alpha}^{-1} \int_{\mathbb{R}^N} |\xi|^{2\alpha}|\mathscr{F}u|^2\,\mathrm{d}\xi = 2k_{N,\alpha}^{-1} \big\|(-\Delta)^{\frac{\alpha}{2}}u\big\|^2_{L^2(\mathbb{R}^N)} = [u]^2_{H^\alpha(\mathbb{R}^N)}. \tag{2.2}$$

As a result, the norms on $H^\alpha(\mathbb{R}^N)$,

$$\begin{aligned} &u \mapsto \|u\|_{H^\alpha(\mathbb{R}^N)}, \\ &u \mapsto \big(\|u\|^2_{L^2(\mathbb{R}^N)} + \big\|(-\Delta)^{\frac{\alpha}{2}}u\big\|^2_{L^2(\mathbb{R}^N)}\big)^{\frac{1}{2}}, \\ &u \mapsto \left(\|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} |\xi|^{2\alpha}|\mathscr{F}u|^2\,\mathrm{d}\xi\right)^{\frac{1}{2}}, \end{aligned} \tag{2.3}$$

are all equivalent.

In this paper, in view of the presence of potential $V(x)$, we consider its subspace

$$E = \left\{u \in H^\alpha(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2\,\mathrm{d}x < \infty\right\}.$$

We define the norm in E by

$$\|u\|_E = \left(\int_{\mathbb{R}^N} \big(|\xi|^{2\alpha}\hat{u}^2 + \hat{u}^2\big)\,\mathrm{d}\xi + \int_{\mathbb{R}^N} V(x)u^2\,\mathrm{d}x\right)^{\frac{1}{2}},$$

where $\hat{u} = \mathscr{F}(u)$. Moreover, by [6], E is a Hilbert space with the inner product

$$\langle u,v\rangle_E = \int_{\mathbb{R}^N} \big(|\xi|^{2\alpha}\hat{u}(\xi)\hat{v}(\xi) + \hat{u}(\xi)\hat{v}(\xi)\big)\,\mathrm{d}\xi + \int_{\mathbb{R}^N} V(x)u(x)v(x)\,\mathrm{d}x, \quad \forall u,v \in E.$$

Note that by (2.2) and (2.3), together with the condition (V), we know that $\|\cdot\|_E$ is equivalent to the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} \big(\big|(-\Delta)^{\frac{\alpha}{2}}u\big|^2 + V(x)u^2\big)\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{2.4}$$

The corresponding inner product is

$$\langle u,v\rangle = \int_{\mathbb{R}^N} \big((-\Delta)^{\frac{\alpha}{2}} u(x)(-\Delta)^{\frac{\alpha}{2}} v(x) + V(x)u(x)v(x)\big)\,\mathrm{d}x.$$

Throughout out this paper, we will use the norm $\|\cdot\|$ in E.

Definition 2.2 We say that $u \in E$ is a weak solution of (1.1), if

$$\int_{\mathbb{R}^N} \big((-\Delta)^{\frac{\alpha}{2}} u(-\Delta)^{\frac{\alpha}{2}} \phi + V(x)u\phi\big)\,\mathrm{d}x = \int_{\mathbb{R}^N} f(x,u)\phi\,\mathrm{d}x, \quad \forall \phi \in E.$$

Lemma 2.3 (see [7] and [12]) *E is continuously embedded into $L^p(\mathbb{R}^N)$ for $p \in [2,2_\alpha^*]$ and compactly embedded into $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ for $p \in [2,2_\alpha^*)$.*

Lemma 2.4 *E is compactly embedded into $L^p(\mathbb{R}^N)$ for $p \in [2,2_\alpha^*)$ with $2_\alpha^* = \frac{2N}{N-2\alpha}$.*

Proof By [4], we know E is compactly embedded into $L^2(\mathbb{R}^N)$, *i.e.*, if there exists a sequence $\{u_n\} \subset E$ and $u_0 \in E$ such that $u_n \rightharpoonup u_0$ weakly in E, passing to a subsequence if necessary, we have $u_n \to u_0$ strongly in $L^2(\mathbb{R}^N)$. Therefore, we only consider $p \in (2,2_\alpha^*)$. In order to do this, we need the following fractional Gagliardo-Nirenberg inequality, see [13, Corollary 2.3]. Let $1 \le p, p_2 < \infty$, $0 < s < p < \infty$, $0 < \alpha < N$ and $1 < p_1 < N/\alpha$. Then

$$\|u(x)\|_{L^p(\mathbb{R}^N)} \le \eta^{\frac{s}{p}} \|(-\Delta)^{\alpha/2} u(x)\|^{\frac{s}{p}}_{L^{p_1}(\mathbb{R}^N)} \|u(x)\|^{1-\frac{s}{p}}_{L^{p_2}(\mathbb{R}^N)} \tag{2.5}$$

with

$$s\left(\frac{1}{p_1} - \frac{\alpha}{N}\right) + \frac{p-s}{p_2} = 1 \quad \text{and} \quad \eta = 2^{-\alpha}\pi^{-\alpha/2}\frac{\Gamma((N-\alpha)/2)}{\Gamma((N+\alpha)/2)}\left(\frac{\Gamma(N)}{\Gamma(N/2)}\right)^{\alpha/N}.$$

Note that the dimension $N > 2\alpha$, we can take $p_1 = p_2 = 2$, and then $s(\frac{1}{2} - \frac{\alpha}{N}) + \frac{p-s}{2} = 1$ whence $s = \frac{(p-2)N}{2\alpha} \in (0,p)$ as $p \in (2,2_s^*)$. Consequently, from (2.5) we have

$$\|u\|_{L^p(\mathbb{R}^N)} \le \eta^{\frac{s}{p}} \|(-\Delta)^{\alpha/2} u\|^{\frac{s}{p}}_{L^2(\mathbb{R}^N)} \|u\|^{1-\frac{s}{p}}_{L^2(\mathbb{R}^N)}.$$

Furthermore, note that (2.4); we see that

$$\|u\|_{L^p(\mathbb{R}^N)} \le \eta^{\frac{s}{p}} \|u\|^{\frac{s}{p}} \|u\|^{1-\frac{s}{p}}_{L^2(\mathbb{R}^N)}. \tag{2.6}$$

Then by (2.6) and $E \hookrightarrow\hookrightarrow L^2(\mathbb{R}^N)$, we find

$$\begin{aligned} \|u_n - u_0\|_{L^p(\mathbb{R}^N)} &\le \eta^{\frac{s}{p}} \|u_n - u_0\|^{\frac{s}{p}} \|u_n - u_0\|^{1-\frac{s}{p}}_{L^2(\mathbb{R}^N)} \\ &\le \eta^{\frac{s}{p}} \big(\|u_n\|^{\frac{s}{p}} + \|u_0\|^{\frac{s}{p}}\big) \|u_n - u_0\|^{1-\frac{s}{p}}_{L^2(\mathbb{R}^N)} \to 0. \end{aligned}$$

Therefore, E is compactly embedded into $L^p(\mathbb{R}^N)$ for $p \in [2,2_\alpha^*)$ with $2_\alpha^* = \frac{2N}{N-2\alpha}$, as required. This completes the proof. □

The functional associated with (1.1) is defined by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(\left|(-\Delta)^{\frac{\alpha}{2}}u\right|^2 + V(x)u^2\right)\mathrm{d}x - \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x, \quad \forall u \in E, \tag{2.7}$$

where $F(x,u) = \int_0^u f(x,s)\,\mathrm{d}s$.

Now, we list our assumptions on f and F.

(H2) $\lim_{|s|\to\infty}\frac{F(x,s)}{|s|^2} = +\infty$ uniformly for $x \in \mathbb{R}^N$.

(H3) There exist $d_3 > 0$ and $\varphi_1 > 0$ such that $\int_{\mathbb{R}^N}\varphi_1(x)\,\mathrm{d}x < +\infty$ and

$$tf(x,t) - 2F(x,t) \le sf(x,s) - 2F(x,s) + d_3\varphi_1, \quad \forall 0 < t < s \text{ or } s < t < 0, x \in \mathbb{R}^N.$$

(H4) $F(x,s) \ge 0$ for $(x,s) \in \mathbb{R}^N \times \mathbb{R}$.

(H5) There exist $\beta > 2$, $r_0 > 0$ such that

$$-\beta F(x,s) + sf(x,s) \ge 0, \quad |s| \ge r_0, \text{uniformly for } x \in \mathbb{R}^N.$$

(H6) $F(x,s) = F(x,-s)$ for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}$.

Remark 2.5 (1) Let $F(x,s) = s^2\ln(|s|+1)$, for all $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$. Then (H1), (H2), (H4), and (H6) hold. Moreover, we easily have $f(x,s) = 2s\ln(|s|+1) + s^3[|s|(|s|+1)]^{-1}$ and $sf(x,s) - 2F(x,s) = |s|^3(|s|+1)^{-1}$, so (H3) is satisfied.

However, we can see that $F(x,s)$ does not satisfy the Ambrosetti-Rabinowitz condition (see [6, (f_4)]):

(AR) there is a constant $\mu > 2$ such that

$$0 < \mu F(x,s) \le sf(x,s) \quad \text{for all } x \in \mathbb{R}^N \text{ and } s \in \mathbb{R}\backslash\{0\}.$$

Indeed, $sf(x,s) - \mu F(x,s) = s^2[(2-\mu)\ln(|s|+1) + |s|(|s|+1)^{-1}] \ge 0$ is impossible for all $x \in \mathbb{R}^N$ and $s \in \mathbb{R}\backslash\{0\}$.

(2) Let $\beta > 2$ and $F(x,s) = |s|^\beta\ln(|s|+1)$, for all $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$. Then (H1), (H2), (H4), and (H6) hold. Moreover, from $-\beta F(x,s) + sf(x,s) = |s|^{\beta+1}(|s|+1)^{-1}$, and (H5) holds.

Note that from Theorem 4 in [14] we have (H4) and (H5) imply (H2).

Lemma 2.6 (see [15, Lemma 1]) *Let* (V) *and* (H1) *hold. Then $J \in C^1(E,\mathbb{R})$ and its derivative*

$$\left(J'(u),\phi\right) = \int_{\mathbb{R}^N}\left((-\Delta)^{\frac{\alpha}{2}}u(-\Delta)^{\frac{\alpha}{2}}\phi + V(x)u\phi\right)\mathrm{d}x - \int_{\mathbb{R}^N} f(x,u)\phi\,\mathrm{d}x, \quad \forall u,\phi \in E.$$

Moreover, the critical points of J are weak solutions of (1.1).

To complete the proofs of our theorems, we need the following critical point theorems in [16–19].

Definition 2.7 Let $(X,\|\cdot\|)$ be a real Banach space, $J \in C^1(X,\mathbb{R})$. We say that J satisfies the $(\mathrm{C_c})$ condition if any sequence $\{u_n\} \subset X$ such that $J(u_n) \to c$ and $\|J'(u_n)\|(1+\|u_n\|) \to 0$ as $n \to \infty$ has a convergent subsequence.

Let X be a Banach space equipped with the norm $\|\cdot\|$ and $X = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$, where $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \bigoplus_{j=1}^{k} X_j$ and $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$.

Lemma 2.8 *Let $(X, \|\cdot\|)$ be a real reflexive Banach space, $J \in C^1(X, \mathbb{R})$ satisfies the* $(\mathrm{C_c})$ *condition for any $c > 0$ and J is even. If for each sufficiently large $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that the following conditions hold*:

(i) $a_k := \inf_{\{u\in Z_k, \|u\|=r_k\}} J(u) \to +\infty$ *as* $k \to \infty$,
(ii) $b_k := \max_{\{u\in Y_k, \|u\|=\rho_k\}} J(u) \le 0$,

then the functional J has an unbounded sequence of critical values, i.e., there exists a sequence $\{u_k\} \subset X$ such that $J'(u_k) = 0$ and $J(u_k) \to +\infty$ as $k \to \infty$.

In the following, we will introduce a variant fountain theorem by Zou [16]. Let X and the subspace Y_k and Z_k be defined above. Consider the following C^1-functional $J_\lambda : X \to \mathbb{R}$ defined by

$$J_\lambda(u) := A(u) - \lambda B(u), \quad \lambda \in [1,2]. \tag{2.8}$$

Lemma 2.9 *If the functional J_λ satisfies*

(T1) *J_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$, and, moreover, $J_\lambda(-u) = J_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times X$,*
(T2) *$B(u) \ge 0$ for all $u \in X$; moreover, $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$,*
(T3) *there exist $r_k > \rho_k > 0$ such that*

$$a_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} J_\lambda(u) > b_k(\lambda) := \max_{u\in Y_k, \|u\|=r_k} J_\lambda(u), \quad \forall \lambda \in [1,2],$$

then

$$a_k(\lambda) \le \zeta_k(\lambda) = \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} J_\lambda\bigl(\gamma(u)\bigr), \quad \forall \lambda \in [1,2],$$

where $B_k = \{u \in Y_k : \|u\| \le r_k\}$ and $\Gamma_k = \{\gamma \in C(B_k, X) : \gamma$ is odd, $\gamma|_{\partial B_k} = id\}$. Moreover, for a.e. $\lambda \in [1,2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^{\infty}$ such that

$$\sup_n \bigl\|u_n^k(\lambda)\bigr\| < \infty, \qquad J'_\lambda\bigl(u_n^k(\lambda)\bigr) \to 0 \quad \text{and} \quad J_\lambda\bigl(u_n^k(\lambda)\bigr) \to \zeta_k(\lambda) \quad \text{as } n \to \infty.$$

Remark 2.10 As mentioned in [6], E is a Hilbert space. Let $\{e_j\}$ be an orthonormal basis of E and define $X_j := \operatorname{span}\{e_j\}$, $Y_k := \bigoplus_{j=1}^{k} X_j$, and $Z_k := \overline{\bigoplus_{j=k+1}^{\infty} X_j}$, $k \in \mathbb{N}$. Clearly, $E = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for all $j \in \mathbb{N}$.

3 Existence of weak solutions for (1.1)

Theorem 3.1 *Assume that* (V), (H1)-(H4), *and* (H6) *hold. Then* (1.1) *has infinitely many weak solutions $\{u_k\}$ satisfying*

$$\frac{1}{2}\int_{\mathbb{R}^N} \bigl(\bigl|(-\Delta)^{\frac{\alpha}{2}} u_k\bigr|^2 + V(x)u_k^2\bigr)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(x, u_k)\,\mathrm{d}x \to +\infty \quad \text{as } k \to \infty.$$

Proof We first prove that J satisfies the $(\mathrm{C_c})$ condition for any $c > 0$. Let $\{u_n\} \subset E$ be a $(\mathrm{C_c})$ sequence, *i.e.*,

$$J(u_n) \to c > 0, \qquad \|J'(u_n)\|\big(1 + \|u_n\|\big) \to 0 \quad \text{when } n \to \infty, \tag{3.1}$$

which implies that

$$c = J(u_n) + o(1), \qquad J'(u_n)u_n = o(1) \quad \text{as } n \to \infty. \tag{3.2}$$

In what follows, we shall show that $\{u_n\}$ is bounded. Otherwise, up to a subsequence, $\{u_n\}$ is unbounded in E, and we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. We define the sequence $\{w_n\}$ by $w_n = \frac{u_n}{\|u_n\|}$, $n = 1, 2, \ldots$. Clearly, $\{w_n\} \subset E$ and $\|w_n\| = 1$ for any n. Going over, if necessary, to a subsequence, we may assume that

$$\begin{aligned} &w_n \rightharpoonup w \text{ weakly in } E, \\ &w_n \to w \text{ strongly in } L^p\big(\mathbb{R}^N\big) \text{ for } p \in [2, 2_s^*), \\ &w_n(x) \to w(x) \text{ a.e. } x \in \mathbb{R}^N. \end{aligned} \tag{3.3}$$

Suppose that $w \neq 0$ in E. Dividing by $\|u_n\|^2$ in both sides of (2.7), noting that $J(u_n) \to c$, we obtain

$$\int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2} \,\mathrm{d}x = \frac{1}{2} + o\big(\|u_n\|^{-2}\big) < +\infty. \tag{3.4}$$

On the other hand, denote $\Omega_{\neq} := \{x \in \mathbb{R}^N : w(x) \neq 0\}$, by (H2), for all $x \in \Omega_{\neq}$, and we find

$$\frac{F(x, u_n)}{\|u_n\|^2} = \frac{F(x, u_n)}{|u_n|^2} \frac{|u_n|^2}{\|u_n\|^2} = \frac{F(x, u_n)}{|u_n|^2} |w_n|^2 \to +\infty \quad \text{when } n \to \infty.$$

If $|\Omega_{\neq}| > 0$, using Fatou's lemma, we obtain

$$\int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2} \,\mathrm{d}x \to +\infty \quad \text{as } n \to \infty.$$

This contradicts (3.4). Hence, $\Omega_{\neq}$ has zero measure, *i.e.*, $w = 0$ a.e. in $\mathbb{R}^N$. Let $t_n \in [0, 1]$ such that

$$J(t_n u_n) = \max_{t \in [0,1]} J(t u_n).$$

Then we claim $J(t_n u_n)$ is bounded. If $t_n = 0$, $J(0) = 0$; if $t_n = 1$, $J(t_n u_n) = J(u_n) \to c$. Therefore, $J(t_n u_n)$ is bounded when $t_n = 0, 1$. If $0 < t_n < 1$ for n large enough

$$\begin{aligned} &\int_{\mathbb{R}^N} \big((-\Delta)^{\frac{\alpha}{2}} t_n u_n (-\Delta)^{\frac{\alpha}{2}} t_n u_n + V(x) t_n u_n \cdot t_n u_n\big) \,\mathrm{d}x - \int_{\mathbb{R}^N} f(x, t_n u_n) t_n u_n \,\mathrm{d}x \\ &\quad = \big(J'(t_n u_n), t_n u_n\big) = t_n \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t = t_n} J(t u_n) = 0. \end{aligned}$$

Consequently, by (H3), noting that (3.1) and (3.2) hold, we have

$$\begin{aligned} J(tu_n) &\le J(t_n u_n) - \frac{1}{2}\big(J'(t_n u_n), t_n u_n\big) \\ &= \frac{1}{2}\int_{\mathbb{R}^N} f(x, t_n u_n) t_n u_n \,\mathrm{d}x - \int_{\mathbb{R}^N} F(x, t_n u_n)\,\mathrm{d}x \\ &\le \int_{\mathbb{R}^N} \left[\left(\frac{1}{2} u_n f(x, u_n) - F(x, u_n)\right) + \frac{d_3}{2}\varphi_1(x)\right]\mathrm{d}x \\ &= J(u_n) - \frac{1}{2}\big(J'(u_n), u_n\big) + \int_{\mathbb{R}^N} \frac{d_3}{2}\varphi_1(x)\,\mathrm{d}x \\ &\le d_4, \quad \forall t \in [0,1], \end{aligned} \tag{3.5}$$

where d_4 is a positive constant. But fixing any $m > d_4$, we let $\overline{w}_n = \sqrt{2m}\frac{u_n}{\|u_n\|} = \sqrt{2m} w_n$. Note that from (H1) we see that there exist $d_5 > 0$, $d_6 > 0$ such that

$$F(x,u) \le d_5 |u|^2 + d_6 |u|^p, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.6}$$

Then by (3.3) we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(x, \overline{w}_n)\,\mathrm{d}x \le \lim_{n\to\infty} \int_{\mathbb{R}^N} \big(d_5 |\overline{w}_n|^2 + d_6 |\overline{w}_n|^p\big)\,\mathrm{d}x = 0.$$

Then for n large enough,

$$J(t_n u_n) \ge J\left(\frac{\sqrt{2m}}{\|u_n\|} u_n\right) = J(\overline{w}_n) = m - \int_{\mathbb{R}^N} F(x, \overline{w}_n)\,\mathrm{d}x \ge m.$$

This also contradicts (3.5).

Now the sequence $\{u_n\}$ is bounded, as required. Next, we verify that $\{u_n\}$ has a convergent subsequence. Without loss of generality, we assume that

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly in } E, \\ &u_n \to u \text{ strongly in } L^p\big(\mathbb{R}^N\big) \text{ for } p \in [2, 2_s^*). \end{aligned} \tag{3.7}$$

Combining this with (H1) and the Hölder inequality, we see

$$\begin{aligned} &\left|\int_{\mathbb{R}^N} \big[f(x,u_n) - f(x,u)\big](u_n - u)\,\mathrm{d}x\right| \\ &\quad \le \int_{\mathbb{R}^N} \big[d_1\big(|u_n| + |u|\big) + d_2\big(|u_n|^{p-1} + |u|^{p-1}\big)\big]|u_n - u|\,\mathrm{d}x \\ &\quad \le d_1\big(\|u_n\|_2 + \|u\|_2\big)\|u_n - u\|_2 + d_2\big(\|u_n\|_p^{p-1} + \|u\|_p^{p-1}\big)\|u_n - u\|_p \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Consequently,

$$\|u_n - u\|^2 = \big(J'(u_n) - J'(u), u_n - u\big) + \int_{\mathbb{R}^N} \big[f(x,u_n) - f(x,u)\big](u_n - u)\,\mathrm{d}x \to 0$$

with the fact that $(J'(u_n) - J'(u), u_n - u) \to 0$ when $n \to \infty$. Therefore, we prove that J satisfies the $(\mathrm{C_c})$ condition for any $c > 0$.

Clearly, $J(u) = J(-u)$ by (H6). It remains to prove that the conditions (i) and (ii) of Lemma 2.8 hold. Let $\beta_r(k) := \sup_{\{u\in Z_k,\|u\|=1\}} \|u\|_r$ with $r \in [2, 2^*_\alpha)$, where Z_k is defined in Remark 2.10. Then by Lemma 3.8 of [19], $\beta_r(k) \to 0$ as $k \to \infty$ for the fact that $E \hookrightarrow\hookrightarrow L^r(\mathbb{R}^N)$.

Now for $u \in Z_k$ with $\|u\| = r_k = (\beta_2(k) + \beta_p(k))^{-1}$, we obtain

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} \left(d_5|u|^2 + d_6|u|^p\right) \mathrm{d}x = \frac{1}{2}\|u\|^2 - d_5\|u\|_2^2 - d_6\|u\|_p^p \\ &\geq \frac{1}{2}\|u\|^2 - d_5\beta_2^2(k)\|u\|^2 - d_6\beta_p^p(k)\|u\|^p \geq \frac{1}{2}r_k^2 - d_5 - d_6 \to \infty \quad \text{as } k \to \infty. \end{aligned}$$

Hence,

$$a_k := \inf_{u\in Z_k,\|u\|=r_k} J(u) \to +\infty \quad \text{as } k \to \infty. \tag{3.8}$$

Next we shall prove that, for any finite dimensional subspace $\mathscr{X} \subset E$, we have

$$J(u) \to -\infty \quad \text{as } \|u\| \to \infty, u \in \mathscr{X}. \tag{3.9}$$

Suppose the contrary. For some sequence $\{u_n\} \subset \mathscr{X}$ with $\|u_n\| \to \infty$, there is a $M > 0$ such that $J(u_n) \geq -M$ for all $n \in \mathbb{N}$. Put $v_n = \frac{u_n}{\|u_n\|}$ and then $\|v_n\| = 1$. Up to a subsequence, assume that $v_n \rightharpoonup v$ weakly in E. Since $\dim \mathscr{X} < \infty$, $v_n \to v \in \mathscr{X}$ in E, $v_n \to v$ a.e. on $\mathbb{R}^N$, and $\|v\| = 1$. Denote $\Omega := \{x \in \mathbb{R}^N : v(x) \neq 0\}$, then $\mathrm{meas}(\Omega) > 0$ and for a.e. $x \in \Omega$, $\lim_{n\to\infty} |u_n(x)| \to \infty$. It follows from (2.7) that

$$\lim_{n\to\infty} \frac{\int_{\mathbb{R}^N} F(x,u_n)\,\mathrm{d}x}{\|u_n\|^2} = \lim_{n\to\infty} \frac{\frac{1}{2}\|u_n\|^2 - J(u_n)}{\|u_n\|^2} \leq d_7 \quad \text{with a constant } d_7 > 0. \tag{3.10}$$

But, for large n, on account of F being nonnegative, (H2) and Fatou's Lemma enable us to obtain

$$\begin{aligned} \lim_{n\to\infty} \frac{\int_{\mathbb{R}^N} F(x,u_n)\,\mathrm{d}x}{\|u_n\|^2} &\geq \lim_{n\to\infty} \int_\Omega \frac{F(x,u_n)v_n^2}{u_n^2}\,\mathrm{d}x \geq \liminf_{n\to\infty} \int_\Omega \frac{F(x,u_n)v_n^2}{u_n^2}\,\mathrm{d}x \\ &\geq \int_\Omega \liminf_{n\to\infty} \frac{F(x,u_n)v_n^2}{u_n^2}\,\mathrm{d}x \\ &= \int_\Omega \liminf_{n\to\infty} \frac{F(x,u_n)}{u_n^2}\left[\chi_\Omega(x)\right]v_n^2\,\mathrm{d}x \to \infty \end{aligned}$$

as $n \to \infty$. This contradicts (3.10). Consequently, (3.9) holds, as required. Note that $\dim Y_k < \infty$ in Remark 2.10, and there exist positive constants d_k such that

$$J(u) \leq 0, \quad \text{for each } u \in Y_k \text{ and } \|u\| \geq d_k. \tag{3.11}$$

Combining this and (3.8), we can take $\rho_k := \max\{d_k, r_k + 1\}$, and thus $b_k := \max_{\{u\in Y_k,\|u\|=\rho_k\}} J(u) \leq 0$. Until now, we have proved the functional J satisfies all the conditions of Lemma 2.8. Hence, J has an unbounded sequence of critical values, *i.e.*, there exists a sequence $\{u_k\} \subset E$ such that $J'(u_k) = 0$ and $J(u_k) \to +\infty$ as $k \to \infty$. This completes the proof. $\square$

We prove that there exists $a_0 = a_0(r_0) > 0$ (r_0 is determined in (H5)) such that

$$\bigl|-\beta F(x,s) + sf(x,s)\bigr| \leq a_0|s|^2, \quad |s| \leq r_0, \text{ for all } x \in \mathbb{R}^N. \tag{3.12}$$

Indeed, by (3.6) we see

$$\bigl|F(x,s)\bigr| \leq d_5|s|^2 + d_6|s|^p \leq \bigl(d_5 + d_6 r_0^{p-2}\bigr)|s|^2, \quad |s| \leq r_0, \forall x \in \mathbb{R}^N.$$

This, together with (H1), implies that

$$\begin{aligned}\bigl|-\beta F(x,s) + sf(x,s)\bigr| &\leq \beta\bigl|F(x,s)\bigr| + \bigl|sf(x,s)\bigr| \leq \beta\bigl(d_5 + d_6 r_0^{p-2}\bigr)|s|^2 + \bigl|s\bigl(d_1|s| + d_2|s|^{p-1}\bigr)\bigr| \\ &\leq \bigl(d_1 + \beta d_5 + (d_2 + \beta d_6) r_0^{p-2}\bigr)|s|^2, \quad \text{if } |s| \leq r_0.\end{aligned}$$

Clearly, (3.12) holds true with $a_0 = d_1 + \beta d_5 + (d_2 + \beta d_6) r_0^{p-2}$. In the following theorem, we make the following assumption instead of (V):

(V′) $V \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{x\in\mathbb{R}^N} V(x) = V_0 \geq [\frac{1}{2a_0+1}(\frac{\beta}{2} - 1)]^{-1} > 0$, where a_0 in (3.12), β in (H5).

Especially, by (V′), we obtain

$$\|u\|_2^2 \leq \frac{1}{2a_0 + 1}\left(\frac{\beta}{2} - 1\right)\|u\|^2, \quad u \in E. \tag{3.13}$$

Now, we define a class of functionals on E by

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x = A(u) - \lambda B(u), \quad \lambda \in [1,2].$$

It is easy to know that $J_\lambda \in C^1(E,\mathbb{R})$ for all $\lambda \in [1,2]$ and the critical points of J_1 correspond to the weak solutions of problem (1.1). Note that $J_1 = J$, where J is the functional defined in (2.7).

Theorem 3.2 *Assume that* (V′), (H1), *and* (H4)-(H6) *hold. Then* (1.1) *possesses infinitely many weak solutions.*

Proof We first prove that there exist a positive integer k_1 and two sequences $r_k > \rho_k \to \infty$ as $k \to \infty$ such that

$$a_k(\lambda) = \inf_{u\in Z_k, \|u\|=\rho_k} J_\lambda(u) > 0, \quad \forall k \geq k_1, \tag{3.14}$$

$$b_k(\lambda) = \max_{u\in Y_k, \|u\|=r_k} J_\lambda(u) < 0, \quad \forall k \in \mathbb{N}, \tag{3.15}$$

where Y_k and Z_k are defined in Remark 2.10.

Step 1. We claim that (3.14) is true.

By (3.6) and (H4) we have

$$\begin{aligned} J_\lambda(u) &= \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \geq \frac{1}{2}\|u\|^2 - 2\int_{\mathbb{R}^N} F(x,u)\,\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|^2 - 2\int_{\mathbb{R}^N} \bigl(d_5|u|^2 + d_6|u|^p\bigr)\,\mathrm{d}x \geq \frac{1}{2}\|u\|^2 - 2d_5\|u\|_2^2 - 2d_6\|u\|_p^p. \end{aligned} \tag{3.16}$$

Since $E \hookrightarrow\hookrightarrow L^r(\mathbb{R}^N)$ with $r \in [2, 2_\alpha^*)$, and from Theorem 3.1 we have

$$J_\lambda(u) \geq \frac{1}{2}\|u\|^2 - 2d_5\beta_2^2(k)\|u\|^2 - 2d_6\beta_p^p(k)\|u\|^p.$$

Let $\rho_k = \frac{1}{\beta_2(k)+\beta_p(k)} \to \infty$ as $k \to \infty$. Then there exists k_1 such that $\frac{1}{2}\rho_k^2 - 2d_5 - 2d_6 > 0$, $\forall k \geq k_1$. Therefore,

$$a_k(\lambda) = \inf_{u\in Z_k, \|u\|=\rho_k} J_\lambda(u) \geq \frac{1}{2}\rho_k^2 - 2d_5 - 2d_6 > 0, \quad \forall k \geq k_1.$$

Step 2. We show that (3.15) is true.

We apply the method in Lemma 2.6 of [20] to verify the claim. First, we prove that there exists $\varepsilon > 0$ such that

$$\operatorname{meas}\bigl(x \in \mathbb{R}^N : |u(x)| \geq \varepsilon\|u\|\bigr) \geq \varepsilon, \quad \forall u \in \mathscr{X}\setminus\{0\}, \forall \mathscr{X} \subset E \text{ and } \dim \mathscr{X} < \infty. \tag{3.17}$$

There would otherwise exist a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{X}\setminus\{0\}$ such that

$$\operatorname{meas}\left(x \in \mathbb{R}^N : |u_n(x)| \geq \frac{\|u_n\|}{n}\right) < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{3.18}$$

For each $n \in \mathbb{N}$, let $v_n := \frac{u_n}{\|u_n\|} \in \mathscr{X}$. Then $\|v_n\| = 1$, $\forall n \in \mathbb{N}$ and

$$\operatorname{meas}\left(x \in \mathbb{R}^N : |v_n(x)| \geq \frac{1}{n}\right) < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{3.19}$$

Passing to a subsequence if necessary, we may assume $v_n \to v_0$ in E for some $v_0 \in \mathscr{X}$ since $\mathscr{X}$ is of finite dimension. We easily find $\|v_0\| = 1$. Consequently, there exists a constant $\sigma_0 > 0$ such that

$$\operatorname{meas}\bigl(x \in \mathbb{R}^N : |v_0(x)| \geq \sigma_0\bigr) \geq \sigma_0. \tag{3.20}$$

Indeed, if not, then we have

$$\operatorname{meas}\left(x \in \mathbb{R}^N : |v_0(x)| \geq \frac{1}{n}\right) = 0, \quad \forall n \in \mathbb{N}, \tag{3.21}$$

which implies

$$0 \leq \int_{\mathbb{R}^N} |v_0(x)|^3\,\mathrm{d}x \leq \frac{\|v_0\|_2^2}{n} \to 0 \quad \text{as } n \to \infty.$$

This leads to $v_0 = 0$, contradicting $\|v_0\| = 1$. In view of $E \hookrightarrow\hookrightarrow L^p(\mathbb{R}^N)$ and the equivalence of any two norms on $\mathscr{X}$, we have

$$\int_{\mathbb{R}^N} |v_n - v_0|^2 \, \mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{3.22}$$

For every $n \in \mathbb{N}$, denote

$$\mathscr{N} := \left\{ x \in \mathbb{R}^N : \left|v_n(x)\right| < \frac{1}{n} \right\} \quad \text{and} \quad \mathscr{N}^c := \left\{ x \in \mathbb{R}^N : \left|v_n(x)\right| \geq \frac{1}{n} \right\},$$

and $\mathscr{N}_0 := \{x \in \mathbb{R}^N : |v_0(x)| \geq \sigma_0\}$, where σ_0 is defined by (3.20). Then for n large enough, by (3.20), we see

$$\operatorname{meas}(\mathscr{N} \cap \mathscr{N}_0) \geq \operatorname{meas}(\mathscr{N}_0) - \operatorname{meas}\left(\mathscr{N}^c\right) \geq \sigma_0 - \frac{1}{n} \geq \frac{3\sigma_0}{4}.$$

Consequently, for n large enough, we find

$$\begin{aligned} \int_{\mathbb{R}^N} |v_n - v_0|^2 \, \mathrm{d}x &\geq \int_{\mathscr{N} \cap \mathscr{N}_0} |v_n - v_0|^2 \, \mathrm{d}x \geq \frac{1}{4} \int_{\mathscr{N} \cap \mathscr{N}_0} |v_0|^2 \, \mathrm{d}x - \int_{\mathscr{N} \cap \mathscr{N}_0} |v_n|^2 \, \mathrm{d}x \\ &\geq \left(\frac{\sigma_0^2}{4} - \frac{1}{n^2} \right) \operatorname{meas}(\mathscr{N} \cap \mathscr{N}_0) \geq \frac{9\sigma_0^3}{64} > 0. \end{aligned}$$

This contradicts (3.22). Therefore, (3.17) holds. For the ε given in (3.17), we let

$$\mathscr{N}_u := \left\{ x \in \mathbb{R}^N : \left|u(x)\right| \geq \varepsilon \|u\| \right\}, \quad \forall u \in \mathscr{X} \setminus \{0\}.$$

Then by (3.17), we find

$$\operatorname{meas}(\mathscr{N}_u) \geq \varepsilon, \quad \forall u \in \mathscr{X} \setminus \{0\}. \tag{3.23}$$

As is well known, (H5) implies (H2), and hence for any $k \in \mathbb{N}$, there is a constant $S_k > 0$ such that

$$F(x, u) \geq \frac{|u|^2}{\varepsilon^3}, \quad \forall |u| \geq S_k,$$

where ε is determined in (3.17). Therefore,

$$J_\lambda(u) = \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}^N} F(x, u) \, \mathrm{d}x \leq \frac{1}{2}\|u\|^2 - \int_{\mathscr{N}_u} \frac{|u|^2}{\varepsilon^3} \, \mathrm{d}x \leq \left(\frac{1}{2} - 1 \right) \|u\|^2.$$

Now for any $k \in \mathbb{N}$, if we take $r_k > \max\{\rho_k, \frac{S_k}{\varepsilon}\}$, so $\|u\| = r_k$ is large enough, we have

$$b_k(\lambda) = \max_{u \in Y_k, \|u\| = r_k} J_\lambda(u) < 0, \quad \forall k \in \mathbb{N}.$$

Step 3. Clearly, $J_\lambda \in C^1(E, \mathbb{R})$ implies that J_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. In view of (H6), $J_\lambda(-u) = J_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$. Thus the condition (T1) of Lemma 2.9 holds. Besides, the condition (T2) of Lemma 2.9 holds for the fact that $A(u) = \frac{1}{2}\|u\|^2 \to \infty$ as $\|u\| \to \infty$ and $B(u) \geq 0$ since $F(x,u) \geq 0$. Evidently, Step 1 and Step 2 imply that the condition (T3) of Lemma 2.9 also holds for all $k \geq k_1$. Consequently, Lemma 2.9 implies that for any $k \geq k_1$ and a.e. $\lambda \in [1,2]$, there exists a sequence $\{u_n^k(\lambda)\}_{n=1}^{\infty}$ such that

$$\sup_n \|u_n^k(\lambda)\| < \infty, \qquad J_\lambda'\big(u_n^k(\lambda)\big) \to 0 \quad \text{and} \quad J_\lambda\big(u_n^k(\lambda)\big) \to \zeta_k(\lambda) \quad \text{as } n \to \infty,$$

where

$$\begin{aligned} &B_k = \{u \in Y_k : \|u\| \leq r_k\}, \qquad \Gamma_k = \{\gamma \in C(B_k, W) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = \mathrm{id}\}, \\ &\zeta_k(\lambda) = \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} J_\lambda\big(\gamma(u)\big), \quad \forall \lambda \in [1,2]. \end{aligned}$$

Furthermore, we easily have $\zeta_k(\lambda) \in [\overline{a}_k, \overline{\zeta}_k]$, $\forall k \geq k_1$, where $\overline{\zeta}_k := \max_{u\in B_k} J_\lambda(\gamma(u))$ and $\overline{a}_k := \frac{1}{2}\rho_k^2 - 2d_5 - 2d_6 \to \infty$ as $k \to \infty$.

Claim 1. $\{u_n^k(\lambda)\}_{n=1}^{\infty} \subset E$ possesses a strong convergent subsequence in E, a.e. $\lambda \in [1,2]$ and $k \geq k_1$. In fact, by the boundedness of $\{u_n^k(\lambda)\}_{n=1}^{\infty}$, passing to a subsequence, as $n \to \infty$, we may assume $u_n^k(\lambda) \rightharpoonup u^k(\lambda)$ in E. By the method of Theorem 3.1, we easily prove that $u_n^k(\lambda) \to u^k(\lambda)$ strongly in E.

Thus, for each $k \geq k_1$, we can choose $\lambda_l \to 1$ such that for the sequence $\{u_n^k(\lambda_l)\}_{n=1}^{\infty}$ we have obtained a convergent subsequence, and passing again to a subsequence, we may assume

$$\lim_{n\to\infty} u_n^k(\lambda_l) = u_l^k \text{ in } E, \quad \forall l \in \mathbb{N} \text{ and } k \geq k_1.$$

Thus we obtain

$$J_{\lambda_l}'\big(u_l^k\big) = 0 \quad \text{and} \quad J_{\lambda_l}\big(u_l^k\big) \in [\overline{a}_k, \overline{\zeta}_k], \quad \forall l \in \mathbb{N} \text{ and } k \geq k_1. \tag{3.24}$$

Claim 2. $\{u_l^k\}$ is bounded in E and has a convergent subsequence with the limit $u^k \in E$ for all $k \geq k_1$. For convenience, we set $u_l^k = u_l$ for all $l \in \mathbb{N}$. Consequently, (3.12) and (H5) imply that

$$\begin{aligned} &\beta J_{\lambda_l}(u_l) - \big(J_{\lambda_l}'(u_l), u_l\big) \\ &\quad = \left(\frac{\beta}{2} - 1\right)\|u_l\|^2 + \lambda_l \int_{\mathbb{R}^N} \big[-\beta F\big(x, u_l(x)\big) + f\big(x, u_l(x)\big)u_l(x)\big]\,\mathrm{d}x \\ &\quad = \left(\frac{\beta}{2} - 1\right)\|u_l\|^2 + \lambda_l \int_{|u_l|\leq r_0} \big[-\beta F\big(x, u_l(x)\big) + f\big(x, u_l(x)\big)u_l(x)\big]\,\mathrm{d}x \\ &\qquad + \lambda_l \int_{|u_l|>r_0} \big[-\beta F\big(x, u_l(x)\big) + f\big(x, u_l(x)\big)u_l(x)\big]\,\mathrm{d}x \\ &\quad \geq \left(\frac{\beta}{2} - 1\right)\|u_l\|^2 - \lambda_l \int_{|u_l|\leq r_0} \big|-\beta F\big(x, u_l(x)\big) + f\big(x, u_l(x)\big)u_l(x)\big|\,\mathrm{d}x \\ &\quad \geq \left(\frac{\beta}{2} - 1\right)\|u_l\|^2 - \lambda_l a_0 \|u_l\|_2^2 \geq \left(\frac{\beta}{2} - 1\right)\|u_l\|^2 - 2a_0\|u_l\|_2^2 \\ &\quad \geq \left(\frac{\beta}{2} - 1\right)\|u_l\|^2 - \frac{2a_0}{2a_0+1}\left(\frac{\beta}{2} - 1\right)\|u_l\|^2 = \frac{1}{2a_0+1}\left(\frac{\beta}{2} - 1\right)\|u_l\|^2. \end{aligned}$$

Therefore, $\{u_l\}_{l=1}^{\infty}$ is bounded in E. By Claim 1, we see that $\{u_l\}_{l=1}^{\infty}$ has a convergent subsequence, which converges to an element $u^k \in W$ for all $k \geq k_1$.

Hence, passing to the limit in (3.24), we see $J_1'(u^k) = 0$ and $J_1(u^k) \in [\overline{a}_k, \overline{\zeta}_k]$, $\forall l \in \mathbb{N}$ and $k \geq k_1$. Since $\overline{a}_k \to \infty$ as $k \to \infty$, we get infinitely many nontrivial critical points of $J_1 = J$. Therefore (1.1) possesses infinitely many nontrivial solutions by Lemma 2.9. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
WD and JX obtained the results in a joint research. All the authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Hebei University of Engineering, Handan, Hebei 056038, China. [2]School of Mathematics, Shandong University, Jinan, Shandong 250100, China. [3]Department of Mathematics, Shandong Jianzhu University, Jinan, Shandong 250101, China.

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