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Lyapunov-type inequalities for fractional Langevin-type equations involving Caputo-Hadamard fractional derivative

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Abstract

In this study, some new Lyapunov-type inequalities are presented for Caputo-Hadamard fractional Langevin-type equations of the forms

$$ {}_{H}^{C}D_{a+}^{\beta}({}_{H}^{C}D_{a+}^{\alpha}+p(t))x(t)+q(t)x(t)=0,\quad 0<a<t<b, $$

and

$$ {}_{H}^{C}D_{a+}^{\eta}\phi_p\big[({}_{H}^{C}D_{a+}^{\gamma}+u(t))x(t)\big]+v(t)\phi_p(x(t))=0,\quad 0<a<t<b, $$

subject to mixed boundary conditions, respectively, where $p(t)$, $q(t)$, $u(t)$, $v(t)$ are real-valued functions and $0<\beta<1<\alpha<2$, $1<\gamma,\eta<2$, $\phi_p(s)=|s|^{p-2}s$, $p>1$. The boundary value problems of fractional Langevin-type equations were firstly converted into the equivalent integral equations with corresponding kernel functions, and then the Lyapunov-type inequalities were derived by the analytical method. Noteworthy, the Langevin-type equations are multi-term differential equations, creating significant challenges and difficulties in investigating the problems. Consequently, this study provides new results that can enrich the existing literature on the topic.

MSC: 34B05; 34A08; 26A33

Keywords: Lyapunov-type inequality; Langevin-type equation; Caputo-Hadamard fractional derivative; Mixed boundary condition

1 Introduction

The study of the Lyapunov inequality can be traced back to 1892 when Lyapunov proved the following result:

Theorem 1.1 *Let $q(t)\in C([a,b],\mathbb{R})$. If the Hill differential equation*

$$ x''(t)+q(t)x(t)=0,\quad t\in(a,b), $$

subject to the Dirichlet boundary conditions

$$x(a) = x(b) = 0,$$

has a nontrivial solution, then $q(t)$ *satisfies the following inequality*

$$\int_a^b \left|q(s)\right| ds > \frac{4}{b-a}. \tag{1.1}$$

This striking inequality is known as a Lyapunov inequality [1]. The inequality (1.1) and its generalizations have been applied in various mathematical problems, involving stability problems, oscillation theory, and eigenvalue bounds for ordinary differential equations [2–4]. For some improved and generalized forms, such as Lyapunov-type inequalities for higher-order differential equations, p-Laplacian differential equations, partial differential equations, difference equations, impulsive differential equations, dynamic equations on time scales, fractional differential equations, some literature studies [5–10] and the monographs [11, 12] should be referred to for better comprehensive understanding. Noteworthy, a result of fractional Lyapunov-type inequality was first presented by Ferreira. In 2013, Ferreira [9] extended inequality (1.1) to the fractional case in the sense of the Riemann-Liouville fractional derivative and obtained the following classical result:

Theorem 1.2 *Let* $q(t) \in C([a,b],\mathbb{R})$. *If the fractional boundary value problem (BVP)*

$$\begin{cases} ({}_aD^{\alpha}x)(t) + q(t)x(t) = 0, & t \in (a,b), \\ x(a) = x(b) = 0, \end{cases}$$

has a nontrivial solution, where ${}_aD^{\alpha}$ *is the Riemann-Liouville fractional derivative of order* α, $1 < \alpha \le 2$, *then* $q(t)$ *satisfies the following inequality*

$$\int_a^b \left|q(s)\right| ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}. \tag{1.2}$$

One year later, the same author obtained the analogous Lyapunov-type inequality for the fractional BVP, involving Caputo fractional derivative (see [10]).

Theorem 1.3 *Let* $q(t) \in C([a,b],\mathbb{R})$. *If the fractional BVP*

$$\begin{cases} ({}_a^CD^{\alpha}x)(t) + q(t)x(t) = 0, & t \in (a,b), \\ x(a) = x(b) = 0, \end{cases}$$

has a nontrivial solution, where ${}_a^CD^{\alpha}$ *is the Caputo fractional derivative of order* α, $1 < \alpha \le 2$, *then* $q(t)$ *satisfies the following inequality*

$$\int_a^b \left|q(s)\right| ds > \frac{\alpha^{\alpha}\Gamma(\alpha)}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{1.3}$$

Inequalities (1.2) and (1.3) are the generalizations of inequality (1.1) in the sense of fractional derivative.

Based on the above-mentioned two studies, the subject of fractional Lyapunov-type inequalities has received significant research attention, and a variety of interesting results have been established. For some recent works on the topic, we refer the reader to the works [13–26], the survey paper [27] and the references cited therein. For example, according to the literature report [13], the authors generalized Lyapunov-type inequality (1.2) to the p-Laplacian problem:

$$\begin{cases} D^{\beta}_{a+}(\phi_p(D^{\alpha}_{a+}x(t))) + q(t)\phi_p(x(t)) = 0, & t \in (a,b), \\ x(a) = x'(a) = x'(b) = 0, & D^{\alpha}_{a+}x(a) = D^{\alpha}_{a+}x(b) = 0, \end{cases} \tag{1.4}$$

where D^{k}_{a+} is the Riemann-Liouville fractional derivative of order k ($k = \alpha, \beta$), $2 < \alpha \leq 3$, $1 < \beta \leq 2$; $\phi_p(s) = |s|^{p-2}s$, $p > 1$ is the p-Laplacian operator. The Lyapunov-type inequality for the BVP (1.4) is stated in the following result.

Theorem 1.4 *Let $q(t) \in C([a,b],\mathbb{R})$. If there exists a nontrivial continuous solution of the fractional BVP* (1.4), *then*

$$\begin{aligned} &\int_a^b (b-s)^{\beta-1}(s-a)^{\beta-1}\big|q(s)\big|\,ds \\ &\quad \geq \big[\Gamma(\alpha)\big]^{p-1}\Gamma(\beta)(b-a)^{\beta-1}\left(\int_a^b (b-s)^{\alpha-2}(s-a)\,ds\right)^{1-p}. \end{aligned}$$

Recently, Laadjal et al. [15] established Lyapunov-type inequalities for the following Hadamard fractional differential equation with Dirichlet boundary conditions:

$$\begin{cases} {}^{H}D^{\alpha}_{a+}x(t) - q(t)x(t) = 0, & 1 \leq a < t < b, 1 < \alpha \leq 2, \\ x(a) = x(b) = 0, \end{cases} \tag{1.5}$$

where ${}^{H}D^{\alpha}_{a+}$ is the Hadamard fractional derivative of order α, and $q : [a,b] \to \mathbb{R}$ is a continuous function. The Lyapunov-type inequality for the BVP (1.5) is described in the following theorem.

Theorem 1.5 *If a nontrivial continuous solution to the Hadamard fractional BVP* (1.5) *exists, then*

$$\int_a^b \big|q(s)\big|\,ds \geq 4^{\alpha-1}\Gamma(\alpha+1)\left(\ln\frac{b}{a}\right)^{1-\alpha}.$$

Recently, Wang et al. [16] have derived new Lyapunov-type inequality for the fractional BVP involving Caputo-Hadamard fractional derivative subject to m-point boundary conditions:

$$\begin{cases} {}^{C}_{H}D^{\alpha}_{a+}x(t) + q(t)x(t) = 0, & 0 < a < t < b, 1 < \alpha < 2, \\ x(a) = 0, & x(b) = \sum_{i=1}^{m-2}\beta_i x(\xi_i), \end{cases} \tag{1.6}$$

where ${}^{C}_{H}D^{\alpha}_{a+}$ denotes the Caputo-Hadamard fractional derivative of order α; $\beta_i \geq 0$, $a < \xi_i < b$, $(i = 1,2,\ldots,m-2)$, with $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, $0 \leq \sum_{i=1}^{m-2}\beta_i < 1$. The Lyapunov-type inequality for the BVP (1.6) is expressed in the following theorem.

Theorem 1.6 *Let $q(t) \in C([a,b],\mathbb{R})$. If there exists a nontrivial continuous solution of the Caputo-Hadamard fractional BVP* (1.6), *then*

$$\int_a^b |q(s)|\,ds \geq \frac{a\alpha^{\alpha}\Gamma(\alpha)}{[(\alpha-1)(\ln b-\ln a)]^{\alpha-1}}\cdot\frac{\ln\frac{b}{a}-\sum_{i=1}^{m-2}\beta_i\ln\frac{\xi_i}{a}}{\ln\frac{b}{a}+\sum_{i=1}^{m-2}\beta_i\ln\frac{b}{\xi_i}}.$$

Although the fractional Lyapunov-type inequalities have been studied by many authors, the fractional multi-term differential equations have rarely been studied to date [17, 18]. Pourhadi and Mursaleen [17] analyzed a Lyapunov-type inequality for a multi-term differential equation involving Caputo fractional derivative subject to mixed boundary conditions:

$$\begin{cases} ({}_a^C D^{\alpha} y)(t) + p(t)y'(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = y'(a) = y(b) = 0, \end{cases} \tag{1.7}$$

where ${}_a^C D^{\alpha}$ denotes the Caputo fractional derivative of order α, $2 < \alpha \leq 3$. The Lyapunov-type inequality for the BVP (1.7) is given as follows.

Theorem 1.7 *Let $p(t) \in C^1([a,b])$ and $q(t) \in C([a,b])$. If there exists a nontrivial continuous solution of the fractional BVP* (1.7), *then*

$$\int_a^b \left(|p(s)| + |q(s)| + |p'(s)|\right) ds \geq \frac{\Gamma(\alpha)(b-a)^{1-\alpha}}{\max\{g(\alpha), h(\alpha), A(\alpha+1)\}}$$

if $\alpha \leq b-a+1$ and

$$\int_a^b \left(|p(s)| + |q(s)| + |p'(s)|\right) ds \geq \frac{\Gamma(\alpha)(b-a)^{2-\alpha}}{(\alpha-1)\max\{g(\alpha), h(\alpha), A(\alpha+1)\}}$$

if $\alpha \geq b-a+1$, where

$$g(\alpha) = \frac{1}{4}(4-\alpha)^2, \qquad A(\alpha) = 4\alpha^{-\alpha}(\alpha-2)^{\alpha-2},$$

$$h(\alpha) = \left(\frac{\alpha-2}{2}\right)^{\frac{(\alpha-2)(3-\alpha)}{(4-\alpha)}} - \left(\frac{\alpha-2}{2}\right)^{\frac{2-(\alpha-2)^2}{4-\alpha}}.$$

On the other hand, in 1908, Langevin proposed the following differential equation in the study of particle Brownian motion:

$$m\frac{d^2x(t)}{dt^2} = -\zeta\frac{dx(t)}{dt} + F(t), \tag{1.8}$$

where $-\zeta\dot{x}(t)$ represents dynamical friction experienced by the particle, x is the displacement and ζ denotes the coefficient of friction, m is the mass of particle, and $F(t)$ is the fluctuating force. The Eq. (1.8) is called the Langevin equation, which is found to be an essential tool to describe the evolution of physical phenomena in fluctuating environments [28]. However, for systems with complex phenomena, it has been realized that the conventional integer Langevin equation does not provide an accurate description of the dynamical systems. Therefore, one way to overcome this disadvantage is to replace the integer derivative by the fractional derivative [29]. This gives rise to fractional Langevin-type equations. In recent years, fractional Langevin-type equations have been studied extensively, and further systematic explorations are still carried out [30–32]. For example, Ahmad et al. [31] proposed the investigation of Langevin-type equation involving two fractional orders:

$$ {}^{C}D_{0+}^{\beta}\big({}^{C}D_{0+}^{\alpha}+\lambda\big)x(t)=f\big(t,x(t)\big),\quad t\in(0,1), $$

where ${}^{C}D_{0+}^{\rho}$ is the Caputo fractional derivative of order ρ ($\rho=\alpha,\beta$), $0<\alpha\le 1$, $1<\beta\le 2$, $\lambda\in\mathbb{R}$.

In the past decades, in order to meet the research needs, the p-Laplacian equation was introduced into some BVPs [32, 33]. In particular, Zhou et al. [32] discussed the following fractional Langevin-type equation with the p-Laplacian operator of the form:

$$ {}^{C}D_{0+}^{\beta}\phi_p\big[\big({}^{C}D_{0+}^{\alpha}+\lambda\big)x(t)\big]=f\big(t,x(t),{}^{C}D_{0+}^{\alpha}x(t)\big),\quad t\in(0,1), $$

where ${}^{C}D_{0+}^{\varrho}$ is the Caputo fractional derivative of order ϱ ($\varrho=\alpha,\beta$), $0<\alpha,\beta\le 1$, $\lambda\ge 0$.

The above-mentioned studies indicate that the Langevin-type equations are multi-term differential equations. Since there is no result available in the literature that is concerned with the Lyapunov-type inequalities for fractional Langevin-type equations, the main objective of this study is to bridge the gap and establish Lyapunov-type inequalities for the fractional Langevin-type equations involving Caputo-Hadamard fractional derivative subject to mixed boundary conditions. Precisely, the Lyapunov-type inequalities for the following problems are investigated herein:

$$ \begin{cases} {}_{H}^{C}D_{a+}^{\beta}({}_{H}^{C}D_{a+}^{\alpha}+p(t))x(t)+q(t)x(t)=0, & 0<a<t<b,\\ x(a)={}_{H}^{C}D_{a+}^{\alpha}x(a)=0, & x(b)=0, \end{cases} \tag{1.9} $$

and

$$ \begin{cases} {}_{H}^{C}D_{a+}^{\eta}\phi_p[({}_{H}^{C}D_{a+}^{\gamma}+u(t))x(t)]+v(t)\phi_p(x(t))=0, & 0<a<t<b,\\ x(a)={}_{H}^{C}D_{a+}^{\gamma}x(a)=0, & x(b)={}_{H}^{C}D_{a+}^{\gamma}x(b)=0, \end{cases} \tag{1.10} $$

where ${}_{H}^{C}D_{a+}^{\kappa}$ denotes the Caputo-Hadamard fractional derivative of order κ ($\kappa=\alpha,\beta,\gamma,\eta$), $0<\beta<1<\alpha<2$, $1<\gamma,\eta<2$, $p(t),q(t),u(t),v(t)\in C([a,b],\mathbb{R})$. Clearly, there are two special cases of Eqs. (1.9) and (1.10), respectively; one is the $p(t)\equiv 0$ in Eq. (1.9) and $p=2$, $u(t)\equiv 0$ in Eq. (1.10), and then Eqs. (1.9) and (1.10) degenerate to the sequential fractional BVPs [24–26]; the other is the $p(t)=u(t)=\lambda\in\mathbb{R}$, and then Eqs. (1.9) and (1.10) degenerate to the classical fractional Langevin-type equations (see [30–32]).

The remaining part of the paper is organized as follows: In Sect. 2, we recall some definitions on the fractional integral and derivative, and related basic properties which are needed later. In Sect. 3, we transform the problems (1.9) and (1.10) into equivalent integral equations with kernel functions, respectively, and give the properties of kernel functions. In Sect. 4, we present the Lyapunov-type inequalities for problem (1.9) and (1.10), respectively. Finally, we summarize our results and specify new directions for the future works in Sect. 5.

2 Preliminaries

In this section, we recall some definitions and lemmas about fractional integral and fractional derivative that will be used in the rest of this paper. Let $x(t)$ be a function defined on (a,b), where $0 < a < b < \infty$. Define the space $AC_\delta^n[a,b]$ as follows

$$AC_\delta^n[a,b] = \left\{x : [a,b] \mapsto \mathbb{R} | \delta^{n-1}x(t) \in AC[a,b], \delta = t\,\frac{d}{dt}\right\},$$

and $AC[a,b]$ denote the space of all absolutely continuous real valued function on $[a,b]$.

Definition 2.1 ([34]) The left-sided Hadamard fractional integral of order $\alpha > 0$ for a function $x : [a,b] \to \mathbb{R}$, $(0 < a < b < \infty)$ is defined by

$${}^{H}I_{a+}^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} x(s)\,\frac{ds}{s},$$

provided that the integral exists.

Definition 2.2 ([34]) Let $\alpha > 0$, $n = [\alpha] + 1$. The left-side Hadamard fractional derivative of order α for a function $x : [a,b] \to \mathbb{R}$, $(0 < a < b < \infty)$ is defined by

$${}^{H}D_{a+}^{\alpha}x(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\,\frac{d}{dt}\right)^n \int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1} x(s)\,\frac{ds}{s}\,ds.$$

Definition 2.3 ([35]) Let $\alpha > 0$, $n = [\alpha] + 1$. The left-side Caputo-Hadamard fractional derivative of order α for a function $x(t) \in AC_\delta^n[a,b]$ is defined by

$${}_{H}^{C}D_{a+}^{\alpha}x(t) = \left({}^{H}I_{a+}^{n-\alpha}\delta^n x\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln\frac{t}{s}\right)^{n-\alpha-1} \delta^n x(s)\,\frac{ds}{s}\,ds.$$

Lemma 2.1 ([34]) *Let $\alpha, \beta > 0$, for Hadamard fractional integrals, the semigroup property holds*:

$$\left({}^{H}I_{a+}^{\alpha}\,{}^{H}I_{a+}^{\beta}x\right)(t) = \left({}^{H}I_{a+}^{\alpha+\beta}x\right)(t).$$

Lemma 2.2 ([35]) *Let $\alpha > 0$, $n = [\alpha] + 1$, $x(t) \in AC_\delta^n[a,b]$. Then*

$$\left({}^{H}I_{a+}^{\alpha}\,{}_{H}^{C}D_{a+}^{\alpha}x\right)(t) = x(t) - \sum_{k=0}^{n-1} \frac{\delta^k x(a)}{k!} \left(\ln\frac{t}{a}\right)^k.$$

3 Green's functions of BVPs (1.9) and (1.10)

In this subsection, we discuss Green's functions of problems (1.9) and (1.10) and present some of their properties.

Lemma 3.1 *$x(t) \in C[a,b]$ is a solution of the BVP (1.9) if and only if $x(t)$ satisfies the integral equation*

$$x(t) = \int_a^b G_1(t,s)q(s)x(s)\,ds + \int_a^b G_2(t,s)p(s)x(s)\,ds, \tag{3.1}$$

where kernel functions $G_1(t,s)$ and $G_2(t,s)$ are given by

$$G_1(t,s) = \frac{1}{s\Gamma(\alpha+\beta)} \begin{cases} \frac{\ln(t/a)}{\ln(b/a)}(\ln\frac{b}{s})^{\alpha+\beta-1} - (\ln\frac{t}{s})^{\alpha+\beta-1}, & 0 < a \le s \le t \le b, \\ \frac{\ln(t/a)}{\ln(b/a)}(\ln\frac{b}{s})^{\alpha+\beta-1}, & 0 < a \le t \le s \le b, \end{cases}$$

and

$$G_2(t,s) = \frac{1}{s\Gamma(\alpha)} \begin{cases} \frac{\ln(t/a)}{\ln(b/a)}(\ln\frac{b}{s})^{\alpha-1} - (\ln\frac{t}{s})^{\alpha-1}, & 0 < a \le s \le t \le b, \\ \frac{\ln(t/a)}{\ln(b/a)}(\ln\frac{b}{s})^{\alpha-1}, & 0 < a \le t \le s \le b. \end{cases}$$

Proof Applying the operator ${}^{H}I_{a+}^{\beta}$ to both sides of Eq. (1.9) and using Lemma 2.2, we get

$${}_{H}^{C}D_{a+}^{\alpha}x(t) + p(t)x(t) = -{}^{H}I_{a+}^{\beta}q(t)x(t) + c_0,$$

for some $c_0 \in \mathbb{R}$. From the boundary conditions $x(a) = {}_{H}^{C}D_{a+}^{\alpha}x(a) = 0$, we obtain $c_0 = 0$, then

$${}_{H}^{C}D_{a+}^{\alpha}x(t) + p(t)x(t) = -{}^{H}I_{a+}^{\beta}q(t)x(t). \tag{3.2}$$

In view of Lemma 2.1 and Lemma 2.2, a general solution of the fractional Eq. (3.2) is given by

$$x(t) = -{}^{H}I_{a+}^{\alpha}p(t)x(t) - {}^{H}I_{a+}^{\alpha+\beta}q(t)x(t) + c_1 + c_2\ln(t/a), \tag{3.3}$$

for some $c_1, c_2 \in \mathbb{R}$. Now using the conditions $x(a) = 0$ and $x(b) = 0$, we obtain

$$c_1 = 0, \qquad c_2 = \frac{1}{\ln(b/a)}\left[{}^{H}I_{a+}^{\alpha}p(t)x(t)|_{t=b} + {}^{H}I_{a+}^{\alpha+\beta}y(t)|_{t=b}\right].$$

Substituting the values c_1 and c_2 in (3.3), we have

$$\begin{aligned} x(t) &= -{}^{H}I_{a+}^{\alpha}p(t)x(t) - {}^{H}I_{a+}^{\alpha+\beta}q(t)x(t) + \frac{\ln(t/a)}{\ln(b/a)}\left[{}^{H}I_{a+}^{\alpha}p(t)x(t)|_{t=b} + {}^{H}I_{a+}^{\alpha+\beta}q(t)x(t)|_{t=b}\right] \\ &= \frac{\ln(t/a)}{\ln(b/a)}{}^{H}I_{a+}^{\alpha+\beta}q(t)x(t)|_{t=b} - {}^{H}I_{a+}^{\alpha+\beta}q(t)x(t) \\ &\quad + \frac{\ln(t/a)}{\ln(b/a)}{}^{H}I_{a+}^{\alpha}p(t)x(t)|_{t=b} - {}^{H}I_{a+}^{\alpha}p(t)x(t) \\ &= \frac{\ln(t/a)}{\ln(b/a)}\frac{1}{\Gamma(\alpha+\beta)}\int_a^b \left(\ln\frac{b}{s}\right)^{\alpha+\beta-1} q(s)x(s)\frac{ds}{s} \\ &\quad - \frac{1}{\Gamma(\alpha+\beta)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha+\beta-1} q(s)x(s)\frac{ds}{s} \\ &\quad + \frac{\ln(t/a)}{\ln(b/a)}\frac{1}{\Gamma(\alpha)}\int_a^b \left(\ln\frac{b}{s}\right)^{\alpha-1} p(s)x(s)\frac{ds}{s} - \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} p(s)x(s)\frac{ds}{s} \\ &= \int_a^b G_1(t,s)q(s)x(s)\,ds + \int_a^b G_2(t,s)p(s)x(s)\,ds. \end{aligned}$$

By direct computation, one can obtain the converse of the lemma. The proof is completed. □

Lemma 3.2 *Let* $\frac{1}{p} + \frac{1}{q} = 1$, *then* $x(t) \in C[a,b]$ *is a solution of the BVP* (1.10) *if and only if* $x(t)$ *satisfies the integral equation*

$$x(t) = \int_a^b G(t,s)u(s)x(s)\,ds - \int_a^b G(t,s)\phi_q\left(\int_a^b H(s,\tau)v(\tau)\phi_p\big(x(\tau)\big)\,d\tau\right)ds, \tag{3.4}$$

where kernel function $G(t,s)$ *and* $H(s,\tau)$ *are defined by*

$$G(t,s) = \frac{1}{s\Gamma(\gamma)}\begin{cases} \frac{\ln(t/a)}{\ln(b/a)}(\ln\frac{b}{s})^{\gamma-1} - (\ln\frac{t}{s})^{\gamma-1}, & 0 < a \le s \le t \le b, \\ \frac{\ln(t/a)}{\ln(b/a)}(\ln\frac{b}{s})^{\gamma-1}, & 0 < a \le t \le s \le b, \end{cases}$$

and

$$H(s,\tau) = \frac{1}{\tau\Gamma(\eta)}\begin{cases} \frac{\ln(s/a)}{\ln(b/a)}(\ln\frac{b}{\tau})^{\eta-1} - (\ln\frac{s}{\tau})^{\eta-1}, & 0 < a \le \tau \le s \le b, \\ \frac{\ln(s/a)}{\ln(b/a)}(\ln\frac{b}{\tau})^{\eta-1}, & 0 < a \le s \le \tau \le b. \end{cases}$$

Proof Let $y(t) = \phi_p[({}^C_H D^{\gamma}_{a+} + u(t))x(t)]$. Then BVP (1.10) can be turned into the following coupled BVPs:

$$\begin{cases} {}^C_H D^{\eta}_{a+} y(t) = -v(t)\phi_p(x(t)), & 0 < a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{3.5}$$

and

$$\begin{cases} {}^C_H D^{\gamma}_{a+} x(t) + u(t)x(t) = \phi_q(y(t)), & 0 < a < t < b. \\ x(a) = x(b) = 0. \end{cases} \tag{3.6}$$

As in the proof of Lemma 3.1, we see that BVP (3.5) has a unique solution, which is given by

$$y(t) = \int_a^b H(t,s)v(s)\phi_p\big(x(s)\big)\,ds, \tag{3.7}$$

and BVP (3.6) has a unique solution, which is given by

$$x(t) = \int_a^b G(t,s)u(s)x(s)\,ds - \int_a^b G(t,s)\phi_q\big(y(s)\big)\,ds. \tag{3.8}$$

Substitute (3.7) into (3.8), we see that BVP (1.10) has a unique solution that is given by (3.4). Conversely, by direct computation, it can be established that (3.4) satisfies the problem (1.10). This completes the proof. □

Lemma 3.3 ([16]) *Let* $1 < \rho < 2$, $t,s \in [a,b]$ *then the function*

$$K(t,s) = \frac{1}{s\Gamma(\rho)}\begin{cases} \frac{\ln(t/a)}{\ln(b/a)}(\ln\frac{b}{s})^{\rho-1} - (\ln\frac{t}{s})^{\rho-1}, & 0 < a \le s \le t \le b, \\ \frac{\ln(t/a)}{\ln(b/a)}(\ln\frac{b}{s})^{\rho-1}, & 0 < a \le t \le s \le b, \end{cases}$$

satisfies the following property:

$$\left|K(t,s)\right| \le \frac{1}{a}\frac{(\rho-1)^{\rho-1}}{\rho^{\rho}\Gamma(\rho)}\left(\ln\frac{b}{a}\right)^{\rho-1}.$$

Lemma 3.4 *The function $G_1(t,s)$ given by Lemma 3.1 satisfies the following properties*:
(i) *$G_1(t,s)$ is a nonnegative continuous function in $[a,b]\times[a,b]$*;
(ii) *$G_1(t,s) \le \frac{(\ln(b/a))^{\alpha+\beta-1}}{a\Gamma(\alpha+\beta)}$ for any $(t,s)\in[a,b]\times[a,b]$.*

Proof (i) Continuity is obvious. We now prove nonnegativity. To this end, we define

$$g_{11}(t,s) = \frac{\ln(t/a)}{\ln(b/a)}\left(\ln\frac{b}{s}\right)^{\alpha+\beta-1} - \left(\ln\frac{t}{s}\right)^{\alpha+\beta-1}, \quad 0 < a \le s \le t \le b,$$
$$g_{12}(t,s) = \frac{\ln(t/a)}{\ln(b/a)}\left(\ln\frac{b}{s}\right)^{\alpha+\beta-1}, \quad 0 < a \le t \le s \le b.$$

Clearly, we have

$$0 \le g_{12}(t,s) \le g_{12}(s,s) = \frac{\ln(s/a)}{\ln(b/a)}\left(\ln\frac{b}{s}\right)^{\alpha+\beta-1},$$

for any $t,s \in [a,b]$. On the other hand, for $a \le s \le t \le b$, it is easy to see that $\ln(t/a)\cdot\ln(b/s) \ge \ln(b/a)\cdot\ln(t/s)$. Hence,

$$\begin{aligned} g_{11}(t,s) &= \frac{\ln(t/a)}{\ln(b/a)}\left(\ln\frac{b}{s}\right)^{\alpha+\beta-1} - \left(\ln\frac{t}{s}\right)^{\alpha+\beta-1} \\ &\ge \left[\frac{\ln(t/a)}{\ln(b/a)}\left(\ln\frac{b}{s}\right)\right]^{\alpha+\beta-1} - \left(\ln\frac{t}{s}\right)^{\alpha+\beta-1} \ge 0. \end{aligned}$$

As a consequence, we get $G_1(t,s) \ge 0$. Now we show that property (ii) holds. Let $\nu = \alpha + \beta$, then $2 < \nu < 3$. In this way, the function $g_{11}(t,s)$ can be rewritten as follows:

$$g_{11}(t,s) = \frac{\ln(t/a)}{\ln(b/a)}\left(\ln\frac{b}{s}\right)^{\nu-1} - \left(\ln\frac{t}{s}\right)^{\nu-1}, \quad a \le s \le t \le b.$$

Differentiating $g_{11}(t,s)$ with respect to t for every fixed $s \in [a,b]$, we obtain

$$\partial_t g_{11}(t,s) = \frac{1}{t}\left[\frac{(\ln(b/s))^{\nu-1}}{\ln(b/a)} - (\nu-1)\left(\ln\frac{t}{s}\right)^{\nu-2}\right].$$

It follows that

$$\partial_t g_{11}(t,s) = 0 \quad \Leftrightarrow \quad t_s^* = s\exp\left[\frac{(\ln(b/s))^{\nu-1}}{(\nu-1)\ln(b/a)}\right]^{1/(\nu-2)} \in [s,b].$$

Easily, we can check that

$$\partial_t g_{11}(t,s) \begin{cases} \ge 0, & \text{if } t \le t_s^*, \\ \le 0, & \text{if } t \ge t_s^*. \end{cases}$$

This implies

$$\begin{aligned}
\max_{t\in[s,b]} g_{11}(t,s) &= g_{11}\left(t_s^*,s\right) \\
&= \frac{\ln(t_s^*/a)}{\ln(b/a)}\left(\ln\frac{b}{s}\right)^{\nu-1} - \left(\ln\frac{t_s^*}{s}\right)^{\nu-1} \\
&= \frac{(\ln(s/a))(\ln(b/s))^{\nu-1}}{\ln(b/a)} + \frac{(\ln(b/s))^{\nu-1}}{\ln(b/a)}\left[\frac{(\ln(b/s))^{\nu-1}}{(\nu-1)\ln(b/a)}\right]^{1/(\nu-2)} \\
&\quad - \left[\frac{(\ln(b/s))^{\nu-1}}{(\nu-1)\ln(b/a)}\right]^{(\nu-1)/(\nu-2)} \\
&= \frac{(\ln(s/a))(\ln(b/s))^{\nu-1}}{\ln(b/a)} + (\nu-2)\left[\frac{(\ln(b/s))^{\nu-1}}{(\nu-1)\ln(b/a)}\right]^{(\nu-1)/(\nu-2)} \\
&= \left(\ln\frac{b}{s}\right)^{\nu-1}\left[\frac{\ln(s/a)}{\ln(b/a)} + \frac{(\nu-2)(\ln(b/s))^{(\nu-1)/(\nu-2)}}{[(\nu-1)\ln(b/a)]^{(\nu-1)/(\nu-2)}}\right] \\
&\geq g_{12}(s,s) = \frac{\ln(s/a)}{\ln(b/a)}\left(\ln\frac{b}{s}\right)^{\nu-1}.
\end{aligned} \tag{3.9}$$

Denote

$$\begin{aligned}
&m_1 = \frac{1}{\ln(b/a)}, \qquad m_2 = \frac{\nu-2}{[(\nu-1)\ln(b/a)]^{(\nu-1)/(\nu-2)}}, \\
&g(s) = m_1\ln\frac{s}{a} + m_2\left(\ln\frac{b}{s}\right)^{(\nu-1)/(\nu-2)}, \quad s\in[a,b].
\end{aligned}$$

Differentiating $g(s)$ on (a,b), we get

$$g'(s) = \frac{1}{s}\left[m_1 - m_2\frac{\nu-1}{\nu-2}\left(\ln\frac{b}{s}\right)^{1/(\nu-2)}\right].$$

From which we can derive that

$$g'(s) = 0 \quad\Leftrightarrow\quad s^* = \frac{a^{\nu-1}}{b^{\nu-2}} < a \quad\text{and}\quad g'(s)\geq 0, \quad \text{on } \left[s^*,+\infty\right).$$

This means, $g(s)$ is a monotone increasing function on $[a,b]$, that is,

$$\max_{s\in[a,b]} g(s) \leq g(b) = 1. \tag{3.10}$$

From Eqs. (3.9) and (3.10), we get that

$$\max_{s\in[a,b]} g_{11}\left(t_s^*,s\right) \leq \max_{s\in[a,b]}\left(\ln\frac{b}{s}\right)^{\nu-1} = \left(\ln\frac{b}{a}\right)^{\nu-1}.$$

Hence, for any $t,s\in[a,b]$,

$$0 \leq G_1(t,s) \leq \frac{(\ln(b/a))^{\alpha+\beta-1}}{s\Gamma(\alpha+\beta)} \leq \frac{(\ln(b/a))^{\alpha+\beta-1}}{a\Gamma(\alpha+\beta)}.$$

The lemma is proved. □

4 Lyapunov-type inequalities for BVP (1.9) and (1.10)

In this section, we present the Lyapunov-type inequalities for problems (1.9) and (1.10), respectively. To show this, we define $X = C[a,b]$ as the Banach space endowed with norm $\|x\|_\infty = \max_{t\in[a,b]}|x(t)|$.

Theorem 4.1 *If the BVP* (1.9) *has a nontrivial continuous solution* $x(t)\in X$, *where* $q(t)$ *is a real and continuous function in* $[a,b]$, *then*

$$\int_a^b \left(\left|q(s)\right| + \left|p(s)\right|\right) ds \geq \frac{a(\ln(b/a))^{1-\alpha}}{\max\{\frac{(\ln(b/a))^\beta}{\Gamma(\alpha+\beta)}, \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha\Gamma(\alpha)}\}}. \tag{4.1}$$

Proof According to Lemma 3.1 and Eq. (3.1), if $x(t)\in X$ is a nontrivial solution of the BVP (1.9), then

$$\left|x(t)\right| \leq \int_a^b G_1(t,s)\left|q(s)x(s)\right| ds + \int_a^b \left|G_2(t,s)\right|\left|p(s)x(s)\right| ds.$$

Hence, we derive immediately,

$$\left|x(t)\right| \leq \left[\int_a^b G_1(t,s)\left|q(s)\right| ds + \int_a^b \left|G_2(t,s)\right|\left|p(s)\right| ds\right]\|x\|_\infty.$$

This in combination with the Lemma 3.3 and Lemma 3.4 shows that

$$\begin{aligned}\|x\|_\infty &\leq \left[\frac{(\ln(b/a))^{\alpha+\beta-1}}{a\Gamma(\alpha+\beta)}\int_a^b \left|q(s)\right| ds + \frac{(\alpha-1)^{\alpha-1}}{a\alpha^\alpha\Gamma(\alpha)}\left(\ln\frac{b}{a}\right)^{\alpha-1}\int_a^b \left|p(s)\right| ds\right]\|x\|_\infty \\ &\leq \frac{(\ln(b/a))^{\alpha-1}}{a}\max\left\{\frac{(\ln(b/a))^\beta}{\Gamma(\alpha+\beta)}, \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha\Gamma(\alpha)}\right\}\int_a^b \left(\left|q(s)\right| + \left|p(s)\right|\right) ds\|x\|_\infty,\end{aligned}$$

from which the inequality (4.1) follows. Thus, Theorem 4.1 is proved. □

Theorem 4.2 *If the BVP* (1.10) *has a nontrivial continuous solution* $x(t)\in X$, *where* $q(t)$ *is a real and continuous function in* $[a,b]$, *then either*

(I) $\int_a^b |u(s)|\,ds \geq \frac{a\gamma^\gamma\Gamma(\gamma)}{[(\gamma-1)\ln(b/a)]^{\gamma-1}}$, *or*

(II) $\int_a^b |v(s)|\,ds \geq \phi_p\{\frac{a\gamma^\gamma\Gamma(\gamma)-[(\gamma-1)\ln(b/a)]^{\gamma-1}\int_a^b|u(s)|\,ds}{(b-a)[(\alpha-1)\ln(b/a)]^{\gamma-1}}\}\frac{a\eta^\eta\Gamma(\eta)}{[(\eta-1)\ln(b/a)]^{\eta-1}}$.

Proof From Lemma 3.2 and Eq. (3.4), we know that if $x(t)\in X$ is a nontrivial solution of the BVP (1.10), then

$$\left|x(t)\right| \leq \int_a^b \left|G(t,s)\right|\left|u(s)x(s)\right| ds + \int_a^b \left|G(t,s)\right|\phi_q\left(\int_a^b \left|H(s,\tau)\right|\left|v(\tau)\phi_p\left(x(\tau)\right)\right| d\tau\right) ds,$$

from which we deduce that

$$\left|x(t)\right| \leq \left[\int_a^b \left|G(t,s)\right|\left|u(s)\right| ds + \int_a^b \left|G(t,s)\right|\phi_q\left(\int_a^b \left|H(s,\tau)\right|\left|v(\tau)\right| d\tau\right) ds\right]\|x\|_\infty.$$

Combining this with Lemma 3.3 gives

$$\begin{aligned}
\|x\|_\infty \leq{} & \left\{ \frac{(\gamma-1)^{\gamma-1}}{a\gamma^\gamma\Gamma(\gamma)}\left(\ln\frac{b}{a}\right)^{\gamma-1}\int_a^b |u(s)|\,ds \right. \\
& + \frac{b-a}{a}\frac{(\gamma-1)^{\gamma-1}}{\gamma^\gamma\Gamma(\gamma)}\left(\ln\frac{b}{a}\right)^{\gamma-1} \\
& \left. \times \phi_q\left[\frac{(\eta-1)^{\eta-1}}{a\eta^\eta\Gamma(\eta)}\left(\ln\frac{b}{a}\right)^{\eta-1}\right]\phi_q\left(\int_a^b |v(\tau)|\,d\tau\right)\right\}\|x\|_\infty \\
\leq{} & \left\{ \frac{(\gamma-1)^{\gamma-1}}{a\gamma^\gamma\Gamma(\gamma)}\left(\ln\frac{b}{a}\right)^{\gamma-1}\int_a^b |u(s)|\,ds \right. \\
& + \frac{b-a}{a}\frac{(\gamma-1)^{\gamma-1}}{\gamma^\gamma\Gamma(\gamma)}\left(\ln\frac{b}{a}\right)^{\gamma-1} \\
& \left. \times \phi_q\left[\frac{(\eta-1)^{\eta-1}}{a\eta^\eta\Gamma(\eta)}\left(\ln\frac{b}{a}\right)^{\eta-1}\right]\phi_q\left(\int_a^b |v(\tau)|\,d\tau\right)\right\}\|x\|_\infty. \qquad (4.2)
\end{aligned}$$

In order to prove the inequality (4.2), now we divide the proof into two cases.

Case 1. If the following inequality holds

$$\frac{(\gamma-1)^{\gamma-1}}{a\gamma^\gamma\Gamma(\gamma)}\left(\ln\frac{b}{a}\right)^{\gamma-1}\int_a^b |u(s)|\,ds \geq 1, \qquad (4.3)$$

then inequality (4.2) holds for any $v(t)\in C[a,b]$, which implies (I).

Case 2. If the inequality (4.3) is untenable, that is,

$$\frac{(\gamma-1)^{\gamma-1}}{a\gamma^\gamma\Gamma(\gamma)}\left(\ln\frac{b}{a}\right)^{\gamma-1}\int_a^b |u(s)|\,ds < 1,$$

then from Eq. (4.2), we get (II) immediately. Therefore, we finish the proof of Theorem 4.2. □

As special cases of Theorem 4.1 and Theorem 4.2, we have the following corollaries:

Corollary 4.1 *Consider the following fractional Langevin equation*:

$$\begin{cases} {}_H^CD_{a+}^\beta({}_H^CD_{a+}^\alpha + \lambda)x(t) + \mu x(t) = 0, & 0 < a < t < b, 0 < \beta < 1 < \alpha < 2, \\ x(a) = {}_H^CD_{a+}^\alpha x(a) = 0, & x(b) = 0, \end{cases} \qquad (4.4)$$

where ${}_H^CD_{a+}^\kappa$ *denotes the Caputo-Hadamard fractional derivative of order* κ ($\kappa = \alpha,\beta$), $\lambda,\mu \in \mathbb{R}$. *If* (4.4) *has a nontrivial continuous solution, then*

$$|\lambda| + |\mu| \geq \frac{a(\ln(b/a))^{1-\alpha}}{(b-a)\max\{\frac{(\ln(b/a))^\beta}{\Gamma(\alpha+\beta)}, \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha\Gamma(\alpha)}\}}.$$

Corollary 4.2 *Consider the following p-Laplacian fractional Langevin equation*:

$$\begin{cases} {}_H^CD_{a+}^\eta\phi_p[({}_H^CD_{a+}^\gamma + \lambda)x(t)] + \mu\phi_p(x(t)) = 0, & 0 < a < t < b, 1 < \gamma,\eta < 2, \\ x(a) = {}_H^CD_{a+}^\gamma x(a) = 0, & x(b) = {}_H^CD_{a+}^\gamma x(b) = 0, \end{cases} \qquad (4.5)$$

where ${}^{C}_{H}D^{\kappa}_{a+}$ *denotes the Caputo-Hadamard fractional derivative of order* κ $(\kappa = \gamma, \eta)$, $\lambda, \mu \in \mathbb{R}$. *If* (4.5) *has a nontrivial continuous solution, then either*

(III) $|\lambda| \geq \frac{a\gamma^{\gamma}\Gamma(\gamma)}{(b-a)[(\gamma-1)\ln(b/a)]^{\gamma-1}}$, *or*

(IV) $|\mu| \geq \phi_p\{\frac{a\gamma^{\gamma}\Gamma(\gamma)-(b-a)[(\gamma-1)\ln(b/a)]^{\gamma-1}|\lambda|}{(b-a)[(\gamma-1)\ln(b/a)]^{\gamma-1}}\}\frac{a\eta^{\eta}\Gamma(\eta)}{(b-a)[(\eta-1)\ln(b/a)]^{\eta-1}}$.

Especially for $\lambda = 0$, *if* (4.5) *has a nontrivial continuous solution, then*

$$|\mu| \geq \phi_p\left\{\frac{a\gamma^{\gamma}\Gamma(\gamma)}{(b-a)[(\gamma-1)\ln(b/a)]^{\gamma-1}}\right\}\frac{a\eta^{\eta}\Gamma(\eta)}{(b-a)[(\eta-1)\ln(b/a)]^{\eta-1}}.$$

5 Conclusion

In this study, Lyapunov-type inequalities were obtained for the two types of fractional Langevin-type equations in the frame of Caputo-Hadamard fractional derivative. In recent years, the fractional Langevin-type equations and Lyapunov-type inequalities are one of the research hot spots on fractional calculus theory. Therefore, this research is valuable and meaningful. Noteworthy, this is the first article to consider Lyapunov-type inequalities for fractional Langevin-type equations. However, a lot more explorations are still required in the future, such as discussing the Lyapunov-type inequalities for nonlinear fractional Langevin-type equations associated with the anti-periodic boundary conditions or other general boundary conditions.

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WZ was a major contributor to writing the manuscript and funding acquisition. JZ and JN made the formal analysis, writing–review, and editing. All authors read and approved the final manuscript.

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