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Multilinear strongly singular integral operators on non-homogeneous metric measure spaces

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Abstract

Let (X,d,μ) be a non-homogeneous metric measure space satisfying the geometrically and upper doubling measure conditions. In this paper, the boundedness in Lebesgue spaces for multilinear strongly singular integral operators on non-homogeneous metric measure spaces is proved. As an application, the boundedness in Morrey spaces for multilinear strongly singular integral operators is also obtained.

Keywords: Multilinear strongly singular integral; Non-homogeneous metric measure spaces; Lebesgue spaces; Morrey spaces

1 Introduction and main results

It is well known that a metric measure space (X,d,μ) equipped with a non-negative doubling measure μ is called a space of homogeneous type. μ is said to satisfy the doubling condition if there exists a constant $C>0$ such that $\mu(B(x,2r))\le C\mu(B(x,r))$ for all $x\in \operatorname{supp}\mu$ and $r>0$. In the case of non-doubling measures, a non-negative measure μ only should satisfy the polynomial growth condition, i.e., for all $x\in R^n$ and $r>0$, there exists a constant $C_0>0$ and $k\in(0,n]$ such that

$$\mu\bigl(B(x,r)\bigr)\le C_0r^k, \tag{1.1}$$

where $B(x,r)=\{y\in R^n:|y-x|<r\}$. This breakthrough brings rapid development in harmonic analysis (see [14, 15, 31, 34, 35, 37, 38] and their therein). And the analysis of non-doubling measures has important applications in solving the long-standing open Painlevé problem (see [35]).

Hytönen [17] stated that the measure satisfying (1.1) does not include the doubling measure as a special case. He introduced non-homogeneous metric measure spaces (X,d,μ), satisfying the geometrically and upper doubling measure conditions (see Definition 1.1 and 1.2). The highlight of this kind of spaces is that it includes both the homogeneous and metric spaces with polynomial growth measures as special cases. From then on, some results on non-homogeneous metric measure spaces were obtained. Hytönen et al. [20] and

Bui and Duong [3] independently introduced the atomic Hardy space $H^1(\mu)$ and proved that the dual space of $H^1(\mu)$ is RBMO(μ). In [3], the authors also proved that the Calderón-Zygmund operator and commutators of the Calderón-Zygmund operators and RBMO functions are bounded in $L^p(\mu)$ for $1 < p < +\infty$. Recently, some equivalent characterizations have been established by Liu et al. [29] for the boundedness of Carderón-Zygmund operators on $L^p(\mu)$ for $1 < p < +\infty$. In [9], Fu et al. established boundedness of multilinear commutators of the Calderón-Zygmund operators on the Orlicz spaces on non-homogeneous spaces. More results on non-homogeneous metric measure spaces have also been obtained in [4, 5, 10, 11, 18–24] and the references therein.

Some researchers considered the theory of multilinear singular integral operators; for example, in [7], Coifman and Meyers firstly established the theory of bilinear Calderón-Zygmund operators. Later, Grafakos and Torres [12, 13] demonstrated the boundedness of multilinear singular integral on the product Lebesgue spaces and Hardy spaces. The boundedness of multilinear singular integrals and commutators on non-doubling measures spaces (R^n, μ) was established by Xu [38, 39]. Weighted norm inequalities for multilinear Calderón-Zygmund operators on non-homogeneous metric measure spaces were also constructed in [16]. The boundedness for commutators of multilinear Calderón-Zygmund operators and multilinear fractional integral operators on non-homogeneous metric measure spaces was also established in [11, 36].

The introduction of the strongly singular integral operator is motivated by a class of multiplier operators whose symbol is given by $e^{i|\xi|^\alpha}/|\xi|\beta$ away from the origin, where $0 < \alpha < 1$ and $\beta > 0$. Fefferman and Stein [8] enlarged the multiplier operators onto a class of convolution operators. Coifman [6] also considered a related class of operators for $n = 1$. The strongly singular non-convolution operators were introduced and researched by Alvarez and Milman [1, 2], whose properties are similar to those of Calderón-Zygmund operators, but the kernel is more singular near the diagonal than those of the standard case. Furthermore, Lin and Lu [25–28] obtained the boundedness for the strongly singular integral and its commutators on Lebesgue spaces, Morrey spaces, and Hardy spaces.

In this paper, we first introduce the multilinear strongly singular integral operators on non-homogeneous metric spaces. Then we will also prove that it is bounded in m-multiple Lebesgue spaces, provided that multilinear strongly singular integrals are bounded from m-multiple $L^1(\mu) \times \cdots \times L^1(\mu)$ to $L^{1/m,\infty}(\mu)$, where $L^p(\mu)$ and $L^{p,\infty}(\mu)$ denote the Lebesgue space and weak Lebesgue space, respectively. As an application, the boundedness in Morrey spaces for multilinear strongly singular integral on non-homogeneous metric spaces is obtained. A variant of sharp maximal operator $M^\sharp$, Kolmogorov's theorem and some good properties of the dominating function λ (see Definition 1.2) are the main tools for proving the results in this paper.

Before stating the main results of this paper, we first recall some notations and definitions.

Definition 1.1 ([17]) A metric space (X, d) is called geometrically doubling if there exists some $N_0 \in \mathbf{N}$ such that, for any ball $B(x, r) \subset X$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

Definition 1.2 ([17]) A metric measure space (X, d, μ) is said to be upper doubling if μ is a Borel measure on X and there exists a dominating function $\lambda : X \times (0, +\infty) \to (0, +\infty)$, and a constant $C_\lambda > 0$ such that for each $x \in X, r \longmapsto (x,r)$ is non-decreasing, and for all $x \in X$, $r > 0$,

$$\mu\bigl(B(x,r)\bigr) \le \lambda(x,r) \le C_\lambda \lambda(x,r/2). \tag{1.2}$$

Remark 1.3 (i) A space of homogeneous type is a special case of upper doubling spaces, where one can take the dominating function $\lambda(x,r) = \mu(B(x,r))$. On the other hand, a metric space (X,d,μ) satisfying the polynomial growth condition (1.1) (in particular, $(X,d,\mu) = (R^n, |\cdot|, \mu)$ with μ satisfying (1.1) for some $k \in (0,n]$)) is also an upper doubling measure space if we take $\lambda(x,r) = Cr^k$.

(ii) Let (X,d,μ) be an upper doubling space and λ be a dominating function on $X \times (0,+\infty)$ as in Definition 1.2. In [20], it was showed that there exists another dominating function $\tilde{\lambda}$ such that for all $x,y \in X$ with $d(x,y) \le r$,

$$\tilde{\lambda}(x,r) \le \tilde{C}\tilde{\lambda}(y,r). \tag{1.3}$$

Thus, in this paper, we suppose that λ always satisfies (1.3).

Definition 1.4 ([3]) Let $1 < \alpha, \beta < +\infty$. A ball $B \subset X$ is called (α,β)-doubling if $\mu(\alpha B) \le \beta\mu(B)$.

Remark 1.5 As pointed out in Lemma 2.3 in [3], there exist plenty of doubling balls with small radii and with large radii. For the rest of this paper, unless α and β are specified otherwise, by an (α,β) doubling ball, we mean a $(6,\beta_0)$-doubling with a fixed number $\beta_0 > \max\{C_\lambda^{3\log_2 6}, 6^n\}$, where $n = \log_2 N_0$ is viewed as a geometric dimension of the space.

Definition 1.6 ([3]) Let $N_{B,Q}$ be the smallest integer satisfying $6^{N_{B,Q}} r_B \ge r_Q$ denote

$$K_{B,Q} = 1 + \sum_{k=1}^{N_{B,Q}} \frac{\mu(6^k B)}{\lambda(x_B, 6^k r_B)}, \tag{1.4}$$

where x_B and r_B are center and radius of B, respectively.

Let us first introduce m-linear strongly singular integral kernel.

Definition 1.7 A kernel $K(\cdot,\dots,\cdot) \in L^1_{\mathrm{loc}}((X)^{m+1} \backslash \{(x,y_1 \cdots, y_j, \dots, y_m) : x = y_1 = \cdots = y_j = \cdots = y_m\})$ is called an m-linear strongly singular integral kernel if it satisfies:

(i)

$$\bigl|K(x,y_1,\dots,y_j,\dots,y_m)\bigr| \le C\left[\sum_{j=1}^{m} \lambda\bigl(x, d(x,y_j)\bigr)\right]^{-m} \tag{1.5}$$

for all $(x, y_1 \cdots, y_j, \dots, y_m) \in (X)^{m+1}$ with $x \ne y_j$ for some j.

(ii) There exist $0 < \alpha < 1$ and $0 < \delta \le 1$ such that

$$\begin{aligned} &\bigl|K(x,y_1,\dots,y_j,\dots,y_m) - K\bigl(x',y_1,\dots,y_j,\dots,y_m\bigr)\bigr| \\ &\quad \le \frac{\mathrm{C}d(x,x')^\delta}{[\sum_{j=1}^m d(x,y_j)]^{\delta/\alpha}[\sum_{j=1}^m \lambda(x,d(x,y_j))]^m} \end{aligned} \tag{1.6}$$

provided that $Cd(x,x')^\alpha \leq \max_{1\leq j\leq m} d(x,y_j)$ and for each j,

$$\begin{aligned}&\left|K(x,y_1,\dots,y_j,\dots,y_m)-K\left(x,y_1,\dots,y_j',\dots,y_m\right)\right|\\&\quad\leq \frac{Cd(y_j,y_j')^\delta}{[\sum_{j=1}^m d(x,y_j)]^{\delta/\alpha}[\sum_{j=1}^m \lambda(x,d(x,y_j))]^m}\end{aligned} \tag{1.7}$$

provided that $Cd(y_j,y_j')^\alpha \leq \max_{1\leq j\leq m} d(x,y_j)$.

A multilinear operator T is called a multilinear strongly singular integral operator with the above kernel K, satisfying (1.5), (1.6), and (1.7), if for $f_1,\dots f_m$ are L^∞ functions with compact support and $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$,

$$T(f_1,\dots f_m)(x)=\int_{X^m} K(x,y_1,\dots y_m)f_1(y_1)\cdots f_m(y_m)\,d\mu(y_1)\cdots d\mu(y_m). \tag{1.8}$$

Definition 1.8 ([4]) Let $k>1$ and $1\leq q\leq p<+\infty$. The Morrey spaces are defined by

$$M_q^p(k,\mu):=\left\{f\in L_{\mathrm{loc}}^q(\mu):\|f\|_{M_q^p(k,\mu)}<+\infty\right\},$$

where

$$\|f\|_{M_q^p(k,\mu)}=\sup_{B\in X}\mu(kB)^{\frac{1}{p}-\frac{1}{q}}\left(\int_B |f|^q\,d\mu\right)^{\frac{1}{q}}.$$

Remark 1.9 The definition of the Morrey spaces are independent of the constant $k>1$, and the norms are equivalent for different $k>1$, see [4, 30, 32, 33, 40].

Definition 1.10 ([10]) Let $\epsilon\in(0,\infty)$. A dominating function λ is said to satisfy the ϵ-weak reverse doubling condition if, for all $r\in(0,2\operatorname{diam}(X))$ and $a\in(1,2\operatorname{diam}(X)/r)$, there exists a number $C(a)\in[1,\infty)$, depending only on a and X, such that for all $x\in X$,

$$\lambda(x,ar)\geq C(a)\lambda(x,r) \tag{1.9}$$

and

$$\sum_{k=1}^\infty \frac{1}{[C(a)]^k}<\infty. \tag{1.10}$$

For the sake of simplicity and without loss of generality, we only consider the case of $m=2$ in this paper. Let us state the main result as follows.

Theorem 1.11 *Let T be defined by (1.8). Assume $1<p_1,p_2,q<+\infty$ and $f_1\in L^{p_1}(\mu)$, $f_2\in L^{p_2}(\mu)$ with $\int_X T(f_1,f_2)(x)\,d\mu(x)=0$ if $\|\mu\|:=\mu(X)<\infty$. If T is bounded from $L^1(\mu)\times L^1(\mu)$ to $L^{1/2,\infty}(\mu)$, then there exists a constant $C>0$ such that*

$$\left\|T(f_1,f_2)\right\|_{L^q(\mu)}\leq C\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}, \tag{1.11}$$

where $\frac{1}{q}=\frac{1}{p_1}+\frac{1}{p_2}$.

As an application of the main result in this paper, we obtain the following result.

Theorem 1.12 *Let T be defined by (1.8). Assume that $1 < q_1 \le p_1 < +\infty$, $1 < q_2 \le p_2 < +\infty$, and $f_1 \in L^{p_1}(\mu), f_2 \in L^{p_2}(\mu)$ with $\int_X T(f_1, f_2)(x)\,d\mu(x) = 0$ if $\|\mu\| < \infty$. Assume that λ satisfies the ϵ-weak reverse doubling condition. If T is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{1/2,\infty}(\mu)$, then there exists a constant $C > 0$ such that*

$$\|T(f_1, f_2)\|_{M^p_q(\mu)} \le C\|f_1\|_{M^{p_1}_{q_1}(\mu)}\|f_2\|_{M^{p_2}_{q_2}(\mu)}, \tag{1.12}$$

where $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

Throughout the paper, C denotes a positive constant independent of the main parameters involved, but it may be different in different places.

2 Proof of main results

To prove Theorem 1.11, we first give some notations and lemmas.

Let $f \in L^1_{\mathrm{loc}}(\mu)$, the sharp maximal operator is defined as follows

$$M^\sharp f(x) = \sup_{B\ni x} \frac{1}{\mu(6B)} \int_B \left|f(y) - m_{\widetilde{B}}(f)\right| d\mu(y) + \sup_{(B,Q)\in\Delta_x} \frac{|m_B(f) - m_Q(f)|}{K_{B,Q}}, \tag{2.1}$$

where $\Delta_x := \{(B, Q) : x \in B \subset Q$ and B, Q are doubling balls$\}$. And the non-centered doubling maximal operator is denoted as follows

$$Nf(x) = \sup_{\substack{B\ni x,\\ B \text{ doubling}}} \frac{1}{\mu(B)} \int_B \left|f(y)\right| d\mu(y).$$

For any $0 < \delta < 1$, we define that

$$M^\sharp_\delta f(x) = \left\{M^\sharp\left(|f|^\delta\right)(x)\right\}^{1/\delta}, \qquad N_\delta f(x) = \left\{N\left(|f|^\delta\right)(x)\right\}^{1/\delta}.$$

We can obtain that for any $f \in L^1_{\mathrm{loc}}(\mu)$,

$$\left|f(x)\right| \le N_\delta f(x) \tag{2.2}$$

for $\mu - a.e.\ x \in X$ (see [36]).

Let $\rho > 1$ and $1 < p < \infty$, the non-centered maximal operator $M_{(\rho)}f$ is defined as follows

$$M_{(\rho)}f(x) = \sup_{B\ni x}\left\{\frac{1}{\mu(\rho B)} \int_B \left|f(y)\right| d\mu(y)\right\}. \tag{2.3}$$

The operator $M_{(\rho)}f$ is bounded on $L^p(\mu)$ for $\rho \ge 5$ and $p > 1$ (see [3]).

Lemma 2.1 ([3, 36]) *Let $f \in L^1_{\mathrm{loc}}(\mu)$ with $\int_X f(x)\,d\mu(x) = 0$ if $\|\mu\| < \infty$. For $1 < p < \infty$ and $0 < \delta < 1$, if $\inf(1, N_\delta f) \in L^p(\mu)$, then there exists a constant $C > 0$ such that*

$$\left\|N_\delta(f)\right\|_{L^p(\mu)} \le C\left\|M^\sharp_\delta(f)\right\|_{L^p(\mu)}. \tag{2.4}$$

Lemma 2.2 ([38]) *Let (X, d, μ) be probability measure spaces and let $0 < s < t < +\infty$, then there exists a constant C such that for any measurable function f,*

$$\|f\|_{L^s(\mu)} \le C\|f\|_{L^{t,\infty}(\mu)}. \tag{2.5}$$

Lemma 2.3 *Suppose that T is defined by (1.8), $0 < \delta < 1/2$, $1 < p_1, p_2, q < \infty$. If T is bounded from $L^1(\mu) \times L^1(\mu)$ to $L^{1/2,\infty}(\mu)$, then there exists a constant $C > 0$ such that*

$$M_\delta^\sharp T(f_1, f_2)(x) \le CM_{(5)}f_1(x)M_{(5)}f_2(x). \tag{2.6}$$

Proof of Theorem 1.11 Because $L^\infty(\mu)$ with compact support is dense in $L^p(\mu)$ for $1 < p < \infty$, we only consider the situation of $f_1, f_2 \in L^\infty(\mu)$ with compact support. Let $l(B) := \sup_{x,y\in B} d(x, y)$. Next, we consider two cases for proving the result.

Case 1: ***l*(*B*) = *l* ≥ 1**. As in the proof of Theorem 9.1 in [34], to obtain (2.6), it suffices to show that

$$\left(\frac{1}{\mu(6B)}\int_B \left||T(f_1, f_2)(z)|^\delta - |h_B|^\delta\right| d\mu(z)\right)^{1/\delta} \le CM_{(5)}f_1(x)M_{(5)}f_2(x) \tag{2.7}$$

holds for any x and ball B with $x \in B$, and

$$|h_B - h_Q| \le CM_{(5)}f_1(x)M_{(5)}f_2(x) \tag{2.8}$$

for all balls $B \subset Q$ with $x \in B$, where B is an arbitrary ball, and Q is a doubling ball. For any ball B, we denote

$$h_B := m_B\left(T\left(f_1^0, f_2^\infty\right) + T\left(f_1^\infty, f_2^0\right) + T\left(f_1^\infty, f_2^\infty\right)\right),$$

and

$$h_Q := m_Q\left(T\left(f_1^0, f_2^\infty\right) + T\left(f_1^\infty, f_2^0\right) + T\left(f_1^\infty, f_2^\infty\right)\right),$$

where we split each f_i as $f_i = f_i^0 + f_i^\infty$, $f_i^0 = f_i\chi_{\frac{6}{5}B}$ and $f_i^\infty = f_i - f_i^0$, $i = 1, 2$. Since

$$\left|T(f_1, f_2)(y)\right| \le \left|T\left(f_1^0, f_2^0\right)(y)\right| + \left|T\left(f_1^0, f_2^\infty\right)(y)\right| + \left|T\left(f_1^\infty, f_2^0\right)(y)\right| + \left|T\left(f_1^\infty, f_2^\infty\right)(y)\right|,$$

then

$$\begin{aligned}
&\left(\frac{1}{\mu(6B)}\int_B \left||T(f_1, f_2)(z)|^\delta - |h_B|^\delta\right| d\mu(z)\right)^{1/\delta} \\
&\quad\le C\left(\frac{1}{\mu(6B)}\int_B \left|T(f_1, f_2)(z) - h_B\right|^\delta d\mu(z)\right)^{1/\delta} \\
&\quad\le C\left(\frac{1}{\mu(6B)}\int_B \left|T\left(f_1^0, f_2^0\right)(z)\right|^\delta d\mu(z)\right)^{1/\delta} \\
&\qquad + C\left(\frac{1}{\mu(6B)}\int_B \left|T\left(f_1^0, f_2^\infty\right)(z) - T\left(f_1^0, f_2^\infty\right)(y)\right|^\delta d\mu(z)\right)^{1/\delta}
\end{aligned}$$

$$
\begin{aligned}
&+C\left(\frac{1}{\mu(6B)}\int_B\left|T\left(f_1^{\infty},f_2^{0}\right)(z)-T\left(f_1^{\infty},f_2^{0}\right)(y)\right|^{\delta}d\mu(z)\right)^{1/\delta}\\
&+C\left(\frac{1}{\mu(6B)}\int_B\left|T\left(f_1^{\infty},f_2^{\infty}\right)(z)-T\left(f_1^{\infty},f_2^{\infty}\right)(y)\right|^{\delta}d\mu(z)\right)^{1/\delta}\\
&=: I_1+I_2+I_3+I_4.
\end{aligned}
$$

First, we estimate I_1. Applying Lemma 2.2 with $s=\delta$ $(0<\delta<1/2)$ and $t=1/2$, we get

$$
\begin{aligned}
I_1 &\le C\left(\frac{1}{\mu(6B)}\int_B\left|T\left(f_1^{0},f_2^{0}\right)(z)\right|^{\delta}d\mu(z)\right)^{1/\delta}\\
&\le C\left(\frac{\mu(B)}{\mu(6B)}\right)^{1/\delta}\left\|T(f_1\chi_{\frac{6}{5}B},f_2\chi_{\frac{6}{5}B})\right\|_{L^{1/2,\infty}(B,\frac{d\mu}{\mu(B)})}\\
&\le C\left(\frac{\mu(B)}{\mu(6B)}\right)^{1/\delta}\prod_{i=1}^{2}\frac{1}{\mu(B)}\int_{\frac{6}{5}B}\left|f_i(z_i)\right|d\mu(z_i)\\
&\le C\left(\frac{\mu(B)}{\mu(6B)}\right)^{1/\delta}\prod_{i=1}^{2}\left[\frac{\mu(5\times\frac{6}{5}B)}{\mu(B)}\frac{1}{\mu(5\times\frac{6}{5}B)}\int_{\frac{6}{5}B}\left|f_i(z_i)\right|d\mu(z_i)\right]\\
&\le C\left(\frac{\mu(B)}{\mu(6B)}\right)^{1/\delta-2}M_{(5)}f_1(x)M_{(5)}f_2(x)\\
&\le CM_{(5)}f_1(x)M_{(5)}f_2(x).
\end{aligned}
$$

For I_2, let $z,y\in B$, $z_1\in\frac{6}{5}B$ and $z_2\in X\backslash\frac{6}{5}B$, then $\max_{1\le i\le 2}d(z,z_i)\ge d(z,z_2)\ge Cl(B)\ge Cl(B)^{\alpha}\ge \mathrm{C}d(z,y)^{\alpha}$. By Definition 1.7 and the properties of λ, we deduce

$$
\begin{aligned}
&\left|T\left(f_1^{0},f_2^{\infty}\right)(z)-T\left(f_1^{0},f_2^{\infty}\right)(y)\right|\\
&\quad\le C\int_{X\backslash\frac{6}{5}B}\int_{\frac{6}{5}B}\frac{1}{[\sum_{i=1}^{2}\lambda(z,d(z,z_i))]^2}\frac{d(z,y)^{\delta}}{[\sum_{i=1}^{2}d(z,z_i)]^{\delta/\alpha}}\left|f_1(z_1)\right|\left|f_2(z_2)\right|d\mu(z_1)\,d\mu(z_2)\\
&\quad\le C\int_{X\backslash\frac{6}{5}B}\int_{\frac{6}{5}B}\frac{1}{[\sum_{i=1}^{2}\lambda(z,d(z,z_i))]^2}\frac{d(z,y)^{\delta}}{[d(z,z_2)]^{\delta/\alpha}}\left|f_1(z_1)\right|\left|f_2(z_2)\right|d\mu(z_1)\,d\mu(z_2)\\
&\quad\le C\int_{\frac{6}{5}B}\frac{|f_1(z_1)|}{\lambda(z,d(z,z_1))}d\mu(z_1)\sum_{k=1}^{\infty}\int_{6^{k+1}\frac{6}{5}B\backslash 6^{k}\frac{6}{5}B}\frac{1}{\lambda(z,d(z,z_2))}\frac{d(z,y)^{\delta}}{[d(z,z_2)]^{\delta/\alpha}}\left|f_2(z_2)\right|d\mu(z_2)\\
&\quad\le C\frac{\mu(5\times\frac{6}{5}B)}{\lambda(x_B,\frac{6}{5}r_B)}\frac{1}{\mu(5\times\frac{6}{5}B)}\int_{\frac{6}{5}B}\left|f_1(z_1)\right|d\mu(z_1)\\
&\qquad\times\sum_{k=1}^{\infty}6^{-k\delta/\alpha}l^{\delta(1-1/\alpha)}\frac{\mu(5\times 6^{k+1}\frac{6}{5}B)}{\lambda(x_B,6^{k}\frac{6}{5}r_B)}\frac{1}{\mu(5\times 6^{k+1}\frac{6}{5}B)}\int_{6^{k+1}\frac{6}{5}B}\left|f_2(z_2)\right|d\mu(z_2)\\
&\quad\le CM_{(5)}f_1(x)M_{(5)}f_2(x).
\end{aligned}
$$

Therefore,

$$
I_2\le CM_{(5)}f_1(x)M_{(5)}f_2(x).
$$

Similar to estimate I_2, we immediately obtain

$$I_3 \leq CM_{(5)}f_1(x)M_{(5)}f_2(x).$$

Let us move on to I_4 estimate. By Definition 1.7, we have

$$\begin{aligned}
&\bigl|T\bigl(f_1^\infty,f_2^\infty\bigr)(z)-T\bigl(f_1^\infty,f_2^\infty\bigr)(y)\bigr| \\
&\leq C\int_{X\setminus\frac{6}{5}B}\int_{X\setminus\frac{6}{5}B}\frac{|f_1(z_1)||f_2(z_2)|}{[\sum_{i=1}^2\lambda(z,d(z,z_i))]^2}\frac{d(z,y)^\delta}{[\sum_{i=1}^2 d(z,z_i)]^{\delta/\alpha}}\,d\mu(z_1)\,d\mu(z_2) \\
&\leq C\sum_{k=1}^\infty\int_{6^{k+1}\frac{6}{5}B\setminus 6^k\frac{6}{5}B}\sum_{j=1}^\infty\int_{6^{j+1}\frac{6}{5}B\setminus 6^j\frac{6}{5}B}\frac{|f_1(z_1)||f_2(z_2)|}{[\sum_{i=1}^2\lambda(z,d(z,z_i))]^2} \\
&\quad\times\frac{d(z,y)^\delta}{[\sum_{i=1}^2 d(z,z_i)]^{\delta/\alpha}}\,d\mu(z_1)\,d\mu(z_2) \\
&\leq C\sum_{k=1}^\infty\sum_{j=k}^\infty\int_{6^{k+1}\frac{6}{5}B\setminus 6^k\frac{6}{5}B}\bigl|f_2(z_2)\bigr|\int_{6^{j+1}\frac{6}{5}B\setminus 6^j\frac{6}{5}B}\frac{|f_1(z_1)|}{[\lambda(z,d(z,z_1))]^2} \\
&\quad\times\frac{d(z,y)^\delta}{[d(z,z_1)]^{\delta/\alpha}}\,d\mu(z_1)\,d\mu(z_2) \\
&\quad+C\sum_{k=1}^\infty\sum_{j=1}^{k-1}\int_{6^{k+1}\frac{6}{5}B\setminus 6^k\frac{6}{5}B}\frac{|f_2(z_2)|}{[\lambda(z,d(z,z_2))]^2}\frac{d(z,y)^\delta}{[d(z,z_2)]^{\delta/\alpha}} \\
&\quad\times\int_{6^{j+1}\frac{6}{5}B\setminus 6^j\frac{6}{5}B}f_1(z_1)\,d\mu(z_1)\,d\mu(z_2) \\
&=: I_{41}+I_{42}.
\end{aligned}$$

To estimate I_{41}, we take into account the properties of λ and the fact of $r\geq 1$ and $0<\alpha<1$, so we have

$$\begin{aligned}
I_{41}&\leq C\sum_{j=1}^\infty\biggl(\frac{1}{[\lambda(x_B,6^j\frac{6}{5}r_B)]^2}\int_{6^{j+1}\frac{6}{5}B\setminus 6^j\frac{6}{5}B}\bigl|f_1(z_1)\bigr|\,d\mu(z_1)6^{-j\delta/\alpha}l^{\delta(1-1/\alpha)}\biggr) \\
&\quad\times\sum_{k=1}^{j}\int_{6^{k+1}\frac{6}{5}B\setminus 6^k\frac{6}{5}B}\bigl|f_2(z_2)\bigr|\,d\mu(z_2) \\
&\leq C\sum_{j=1}^\infty\biggl(\frac{1}{[\lambda(x_B,5\times 6^j\frac{6}{5}r_B)]^2}\int_{6^{j+1}\frac{6}{5}B}\bigl|f_1(z_1)\bigr|\,d\mu(z_1)6^{-j\delta/\alpha}l^{\delta(1-1/\alpha)}\biggr) \\
&\quad\times\int_{6^{j+1}\frac{6}{5}B}\bigl|f_2(z_2)\bigr|\,d\mu(z_2) \\
&\leq C\sum_{j=1}^\infty 6^{-j\delta/\alpha}l^{\delta(1-1/\alpha)}\frac{1}{\mu(5\times 6^{j+1}\frac{6}{5}B)}\int_{6^{j+1}\frac{6}{5}B}\bigl|f_1(z_1)\bigr|\,d\mu(z_1) \\
&\quad\times\frac{1}{\mu(5\times 6^{j+1}\frac{6}{5}B)}\int_{6^{j+1}\frac{6}{5}B}\bigl|f_2(z_2)\bigr|\,d\mu(z_2) \\
&\leq CM_{(5)}f_1(x)M_{(5)}f_2(x).
\end{aligned}$$

For I_{42}, by the properties of λ, we have

$$\begin{aligned}
I_{42} &\le C\sum_{k=1}^{\infty}\left(\frac{1}{[\lambda(x_B,6^k\frac{6}{5}r_B)]^2}\int_{6^{k+1}\frac{6}{5}B\setminus 6^k\frac{6}{5}B}\left|f_1(z_2)\right|d\mu(z_2)6^{-k\delta/\alpha}l^{\delta(1-1/\alpha)}\right)\\
&\quad\times\sum_{j=1}^{k-1}\int_{6^{j+1}\frac{6}{5}B\setminus 6^j\frac{6}{5}B}\left|f_1(z_1)\right|d\mu(z_1)\\
&\le C\sum_{k=1}^{\infty}\left(\frac{1}{[\lambda(x_B,5\times 6^k\frac{6}{5}r_B)]^2}\int_{6^{k+1}\frac{6}{5}B}\left|f_2(z_2)\right|d\mu(z_2)6^{-k\delta/\alpha}l^{\delta(1-1/\alpha)}\right)\\
&\quad\times\int_{6^k\frac{6}{5}B}\left|f_1(z_1)\right|d\mu(z_1)\\
&\le C\sum_{k=1}^{\infty}6^{-k\delta/\alpha}l^{\delta(1-1/\alpha)}\frac{1}{\mu(5\times 6^{k+1}\frac{6}{5}B)}\int_{6^{k+1}\frac{6}{5}B}\left|f_2(z_2)\right|d\mu(z_2)\\
&\quad\times\frac{1}{\mu(5\times 6^k\frac{6}{5}B)}\int_{6^k\frac{6}{5}B}\left|f_1(z_1)\right|d\mu(z_1)\\
&\le CM_{(5)}f_1(x)M_{(5)}f_2(x).
\end{aligned}$$

Thus,

$$\left|T\left(f_1^\infty,f_2^\infty\right)(z)-T\left(f_1^\infty,f_2^\infty\right)(y)\right|\le CM_{(5)}f_1(x)M_{(5)}f_2(x).$$

By the above estimate, we obtain (2.7).

Next, we prove (2.8). Consider two balls $B\subset Q$ with $x\in B$, where B is an arbitrary ball, and Q is a doubling ball. Let $N=N_{B,Q}+1$, then we obtain

$$\begin{aligned}
|h_B-h_Q| &\le \left|m_B\left(T(f_1\chi_{\frac{6}{5}B},f_2\chi_{6^N\frac{6}{5}B\setminus\frac{6}{5}B})+T(f_1\chi_{6^N\frac{6}{5}B\setminus\frac{6}{5}B},f_2\chi_{\frac{6}{5}B})\right)\right|\\
&\quad+\left|m_B\left(T(f_1\chi_{6^N\frac{6}{5}B\setminus\frac{6}{5}B},f_2\chi_{6^N\frac{6}{5}B\setminus\frac{6}{5}B})\right)\right|\\
&\quad+\left|m_B\left(T(f_1\chi_{6^N\frac{6}{5}B},f_2\chi_{6^N\frac{6}{5}B})\right)-m_Q\left(T(f_1\chi_{6^N\frac{6}{5}B},f_2\chi_{6^N\frac{6}{5}B})\right)\right|\\
&\quad+\left|m_B\left(T(f_1\chi_{X\setminus 6^N\frac{6}{5}B},f_2\chi_{6^N\frac{6}{5}B})\right)-m_Q\left(T(f_1\chi_{X\setminus 6^N\frac{6}{5}B},f_2\chi_{6^N\frac{6}{5}B})\right)\right|\\
&\quad+\left|m_B\left(T(f_1\chi_{6^N\frac{6}{5}B},f_2\chi_{X\setminus 6^N\frac{6}{5}B})\right)-m_Q\left(T(f_1\chi_{6^N\frac{6}{5}B},f_2\chi_{X\setminus 6^N\frac{6}{5}B})\right)\right|\\
&\quad+\left|m_Q\left(T(f_1\chi_{\frac{6}{5}Q},f_2\chi_{6^N\frac{6}{5}B\setminus\frac{6}{5}Q})+T(f_1\chi_{6^N\frac{6}{5}B\setminus\frac{6}{5}Q},f_2\chi_{6^N\frac{6}{5}B})\right)\right|\\
&\quad+\left|m_Q\left(T(f_1\chi_{6^N\frac{6}{5}B\setminus\frac{6}{5}Q},f_2\chi_{6^N\frac{6}{5}B\setminus\frac{6}{5}Q})\right)\right|\\
&=: J_1+J_2+J_3+J_4+J_5+J_6+J_7.
\end{aligned}$$

For the estimate J_1, by the condition (1.5) in Definition 1.7 and the properties of λ, we obtain

$$\begin{aligned}
&\left|T(f_1\chi_{\frac{6}{5}B},f_2\chi_{6^N\frac{6}{5}B\setminus\frac{6}{5}B})(z)\right|\\
&\quad\le C\int_{6^N\frac{6}{5}B\setminus\frac{6}{5}B}\int_{\frac{6}{5}B}\frac{|f_1(z_1)||f_2(z_2)|}{[\sum_{i=1}^2\lambda(z,d(z,z_i))]^2}\,d\mu(z_1)\,d\mu(z_2)
\end{aligned}$$

$$
\begin{aligned}
&\le C\int_{\frac{6}{5}B}\frac{|f_1(z_1)|}{\lambda(z,d(z,z_1))}\,d\mu(z_1)\int_{6^N\frac{6}{5}B\setminus\frac{6}{5}B}\frac{|f_2(z_2)|}{\lambda(z,d(z,z_2))}\,d\mu(z_2)\\
&\le C\frac{1}{\mu(5\times\frac{6}{5}B)}\int_{\frac{6}{5}B}\big|f_1(z_1)\big|\,d\mu(z_1)\sum_{k=1}^{N_{B,Q}}\int_{6^{k+1}\frac{6}{5}B\setminus 6^k\frac{6}{5}B}\frac{|f_2(z_2)|}{\lambda(z,d(z,z_2))}\,d\mu(z_2)\\
&\le CM_{(5)}f_1(x)\sum_{k=1}^{N_{B,Q}}\frac{\mu(5\times 6^{k+1}\frac{6}{5}B)}{\lambda(x_B,5\times 6^{k+1}\frac{6}{5}r_B)}\frac{1}{\mu(5\times 6^{k+1}\frac{6}{5}B)}\int_{6^{k+1}\frac{6}{5}B}\big|f_2(z_2)\big|\,d\mu(z_2)\\
&\le CK_{B,Q}M_{(5)}f_1(x)M_{(5)}f_2(x).
\end{aligned}
$$

Also,

$$
\big|T(f_1\chi_{6^N\frac{6}{5}B\setminus\frac{6}{5}B},f_2\chi_{\frac{6}{5}B})(z)\big|\le CK_{B,Q}M_{(5)}f_1(x)M_{(5)}f_2(x).
$$

Then,

$$
J_1\le CK_{B,Q}M_{(5)}f_1(x)M_{(5)}f_2(x).
$$

Let us estimate J_2. By the condition (1.5) in Definition 1.7 and the properties of λ, we have

$$
\begin{aligned}
&\big|T(f_1\chi_{6^N\frac{6}{5}B\setminus\frac{6}{5}B},f_2\chi_{6^N\frac{6}{5}B\setminus\frac{6}{5}B})\big|\\
&\quad\le C\int_{6^N\frac{6}{5}B\setminus\frac{6}{5}B}\int_{6^N\frac{6}{5}B\setminus\frac{6}{5}B}\frac{|f_1(z_1)||f_2(z_2)|}{[\sum_{i=1}^2\lambda(z,d(z,z_i))]^2}\,d\mu(z_1)\,d\mu(z_2)\\
&\quad\le C\sum_{k=1}^{N-1}\sum_{j=1}^{N-1}\int_{6^{k+1}\frac{6}{5}B\setminus 6^k\frac{6}{5}B}\int_{6^{j+1}\frac{6}{5}B\setminus 6^j\frac{6}{5}B}\frac{|f_1(z_1)||f_2(z_2)|}{[\sum_{i=1}^2\lambda(z,d(z,z_i))]^2}\,d\mu(z_1)\,d\mu(z_2)\\
&\qquad+C\int_{6\times\frac{6}{5}B\setminus\frac{6}{5}B}\int_{6\times\frac{6}{5}B\setminus\frac{6}{5}B}\frac{|f_1(z_1)||f_2(z_2)|}{[\sum_{i=1}^2\lambda(z,d(z,z_i))]^2}\,d\mu(z_1)\,d\mu(z_2)\\
&\quad=:J_{21}+J_{22}.
\end{aligned}
$$

For J_{22}, since $z\in B$ and $z_i\in 6\times\frac{6}{5}B\setminus\frac{6}{5}B$, then $\frac{1}{5}r_B\le d(z,z_i)\le\frac{41}{5}r_B$ for $i=1,2$. Therefore,

$$
\begin{aligned}
J_{22}&\le\frac{C}{[\lambda(z,\frac{1}{5}r_B)]^2}\int_{6\times\frac{6}{5}B\setminus\frac{6}{5}B}\int_{6\times\frac{6}{5}B\setminus\frac{6}{5}B}\big|f_1(z_1)\big|\big|f_2(z_2)\big|\mu(z_1)\,d\mu(z_2)\\
&\le\frac{C}{[\lambda(x_B,\frac{1}{5}r_B)]^2}\int_{6\times\frac{6}{5}B\setminus\frac{6}{5}B}\int_{6\times\frac{6}{5}B\setminus\frac{6}{5}B}\big|f_1(z_1)\big|\big|f_2(z_2)\big|\mu(z_1)\mu(z_2)\\
&\le C\prod_{i=1}^2\frac{\mu(30\times\frac{6}{5}B)}{\lambda(x_B,\frac{1}{5}r_B)}\frac{1}{\mu(30\times\frac{6}{5}B)}\int_{6\times\frac{6}{5}B}\big|f_i(z_i)\big|\,d\mu(z_i)\\
&\le CM_{(5)}f_1(x)M_{(5)}f_2(x).
\end{aligned}
$$

For J_{21}, by the properties of λ, we have

$$
\begin{aligned}
J_{21} \le{}& C\sum_{k=1}^{N-1}\sum_{j=k}^{N-1}\int_{6^{k+1}\frac{6}{5}B\setminus 6^{k}\frac{6}{5}B}\left|f_1(z_1)\right|\int_{6^{j+1}\frac{6}{5}B\setminus 6^{j}\frac{6}{5}B}\frac{|f_2(z_2)|}{[\lambda(z,d(z,z_2))]^2}\,d\mu(z_2)\,d\mu(z_1)\\
&+C\sum_{k=1}^{N-1}\sum_{j=1}^{k-1}\int_{6^{k+1}\frac{6}{5}B\setminus 6^{k}\frac{6}{5}B}\frac{|f_1(z_1)|}{[\lambda(z,d(z,z_1))]^2}\int_{6^{j+1}\frac{6}{5}B\setminus 6^{j}\frac{6}{5}B}\left|f_2(z_2)\right|d\mu(z_2)\,d\mu(z_1)\\
\le{}& C\sum_{j=1}^{N-1}\frac{1}{[\lambda(x_B,6^{j}\frac{6}{5}r_B)]^2}\int_{6^{j+1}\frac{6}{5}B\setminus 6^{j}\frac{6}{5}B}\left|f_2(z_2)\right|d\mu(z_2)\sum_{k=1}^{j}\int_{6^{j+1}\frac{6}{5}B\setminus 6^{j}\frac{6}{5}B}\left|f_1(z_1)\right|d\mu(z_1)\\
&+C\sum_{k=1}^{N-1}\int_{6^{k+1}\frac{6}{5}B\setminus 6^{k}\frac{6}{5}B}\frac{|f_1(z_1)|}{[\lambda(z,d(z,z_1))]^2}\int_{6^{k}\frac{6}{5}B}\left|f_2(z_2)\right|d\mu(z_2)\,d\mu(z_1)\\
\le{}& C\sum_{j=1}^{N-1}\frac{1}{[\lambda(x_B,6^{j}\frac{6}{5}r_B)]^2}\int_{6^{j+1}\frac{6}{5}B}\left|f_2(z_2)\right|d\mu(z_2)\int_{6^{j+1}\frac{6}{5}B}\left|f_1(z_1)\right|d\mu(z_1)\\
&+C\sum_{j=1}^{N-1}\frac{1}{[\lambda(x_B,6^{j}\frac{6}{5}r_B)]^2}\int_{6^{j+1}\frac{6}{5}B}\left|f_1(z_1)\right|d\mu(z_1)\int_{6^{k}B}\left|f_2(z_2)\right|d\mu(z_2)\\
\le{}& CK_{B,Q}M_{(5)}f_1(x)M_{(5)}f_2(x).
\end{aligned}
$$

For J_3, similar to estimate I_1, we have

$$J_3 \le CM_{(5)}f_1(x)M_{(5)}f_2(x).$$

For J_4 and J_5, similar to estimate I_2, we have

$$J_4 + J_5 \le CM_{(5)}f_1(x)M_{(5)}f_2(x).$$

By a similar method to estimate J_1, we can obtain that

$$J_6 \le CK_{B,Q}M_{(5)}f_1(x)M_{(5)}f_2(x).$$

By a similar method to estimate J_2, we also obtain that

$$J_7 \le CK_{B,Q}M_{(5)}f_1(x)M_{(5)}f_2(x).$$

Hence, (2.8) is proved. Thus, Lemma 2.3, in this case, is proved.

Case 2: $\mathbf{0 < \mathit{l(B)} = \mathit{l} < 1}$. Assume that B_0 and Q_0 are concentric with B and Q, respectively, and $l(B_0) = l(B)^\alpha$, $l(Q_0) = l(Q)^\alpha$. As in the proof of Theorem 9.1 in [34], to obtain (2.6), it suffices to show that

$$\left(\frac{1}{\mu(6B_0)}\int_B\left|\left|T(f_1,f_2)(z)\right|^\delta - |\widetilde{h}_B|^\delta\right|d\mu(z)\right)^{1/\delta} \le CM_{(5)}f_1(x)M_{(5)}f_2(x) \tag{2.9}$$

holds for any x and ball B with $x \in B$, and

$$|\widetilde{h}_B - \widetilde{h}_Q| \le CM_{(5)}f_1(x)M_{(5)}f_2(x) \tag{2.10}$$

for all balls $B \subset Q$ with $x \in B$, where B is an arbitrary ball, Q is a doubling ball. For any ball B, we denote

$$\widetilde{h}_B := m_B\big(T\big(f_1^0,f_2^\infty\big) + T\big(f_1^\infty,f_2^0\big) + T\big(f_1^\infty,f_2^\infty\big)\big),$$

and

$$\widetilde{h}_Q := m_Q\big(T\big(f_1^0,f_2^\infty\big) + T\big(f_1^\infty,f_2^0\big) + T\big(f_1^\infty,f_2^\infty\big)\big),$$

where we split each f_i as $f_i = f_i^0 + f_i^\infty$, $f_i^0 = f_i\chi_{\frac{6}{5}B_0}$ and $f_i^\infty = f_i - f_i^0$, $i = 1, 2$. Since

$$\big|T(f_1,f_2)(y)\big| \le \big|T\big(f_1^0,f_2^0\big)(y)\big| + \big|T\big(f_1^0,f_2^\infty\big)(y)\big| + \big|T\big(f_1^\infty,f_2^0\big)(y)\big| + \big|T\big(f_1^\infty,f_2^\infty\big)(y)\big|,$$

then

$$\begin{aligned}
&\left(\frac{1}{\mu(6B_0)}\int_B\big|\big|T(f_1,f_2)(z)\big|^\delta - |\widetilde{h}_B|^\delta\big|\,d\mu(z)\right)^{1/\delta}\\
&\quad\le C\left(\frac{1}{\mu(6B_0)}\int_B\big|T(f_1,f_2)(z) - \widetilde{h}_B\big|^\delta\,d\mu(z)\right)^{1/\delta}\\
&\quad\le C\left(\frac{1}{\mu(6B_0)}\int_B\big|T\big(f_1^0,f_2^0\big)(z)\big|^\delta\,d\mu(z)\right)^{1/\delta}\\
&\qquad+ C\left(\frac{1}{\mu(6B_0)}\int_B\big|T\big(f_1^0,f_2^\infty\big)(z) - T\big(f_1^0,f_2^\infty\big)(y)\big|^\delta\,d\mu(z)\right)^{1/\delta}\\
&\qquad+ C\left(\frac{1}{\mu(6B_0)}\int_B\big|T\big(f_1^\infty,f_2^0\big)(z) - T\big(f_1^\infty,f_2^0\big)(y)\big|^\delta\,d\mu(z)\right)^{1/\delta}\\
&\qquad+ C\left(\frac{1}{\mu(6B_0)}\int_B\big|T\big(f_1^\infty,f_2^\infty\big)(z) - T\big(f_1^\infty,f_2^\infty\big)(y)\big|^\delta\,d\mu(z)\right)^{1/\delta}\\
&\quad=: L_1 + L_2 + L_3 + L_4.
\end{aligned}$$

We first estimate L_1. Applying Lemma 2.2 with $s = \delta$ $(0 < \delta < 1/2)$ and $t = 1/2$, we get

$$\begin{aligned}
L_1 &\le C\left(\frac{1}{\mu(6B_0)}\int_B\big|T\big(f_1^0,f_2^0\big)(z)\big|^\delta\,d\mu(z)\right)^{1/\delta}\\
&\le C\left(\frac{\mu(B)}{\mu(6B_0)}\right)^{1/\delta}\big\|T(f_1\chi_{\frac{6}{5}B_0},f_2\chi_{\frac{6}{5}B_0})\big\|_{L^{1/2,\infty}(B,\frac{d\mu}{\mu(B)})}\\
&\le C\left(\frac{\mu(B)}{\mu(6B_0)}\right)^{1/\delta}\prod_{i=1}^2\left[\frac{\mu(5\times\frac{6}{5}B_0)}{\mu(B)}\frac{1}{\mu(5\times\frac{6}{5}B_0)}\int_{\frac{6}{5}B_0}\big|f_i(z_i)\big|\,d\mu(z_i)\right]\\
&\le C\left(\frac{\mu(B)}{\mu(6B_0)}\right)^{1/\delta-2}M_{(5)}f_1(x)M_{(5)}f_2(x)\\
&\le CM_{(5)}f_1(x)M_{(5)}f_2(x).
\end{aligned}$$

For L_2, let $z, y \in B$, $z_1 \in \frac{6}{5}B_0$ and $z_2 \in X \setminus \frac{6}{5}B_0$, then $\max_{1\le i\le 2} d(z,z_i) \ge d(z,z_2) \ge Cl(B_0) = Cl(B)^\alpha \ge \mathrm{C}d(z,y)^\alpha$. By Definition 1.7 and the properties of λ, we obtain

$$\begin{aligned}
&\bigl|T\bigl(f_1^0,f_2^\infty\bigr)(z) - T\bigl(f_1^0,f_2^\infty\bigr)(y)\bigr| \\
&\quad \le C \int_{X\setminus\frac{6}{5}B_0} \int_{\frac{6}{5}B_0} \frac{1}{[\sum_{i=1}^2 \lambda(z,d(z,z_i))]^2} \frac{d(z,y)^\delta}{[\sum_{i=1}^2 d(z,z_i)]^{\delta/\alpha}} \bigl|f_1(z_1)\bigr|\bigl|f_2(z_2)\bigr|\,d\mu(z_1)\,d\mu(z_2) \\
&\quad \le C \int_{X\setminus\frac{6}{5}B_0} \int_{\frac{6}{5}B_0} \frac{1}{[\sum_{i=1}^2 \lambda(z,d(z,z_i))]^2} \frac{d(z,y)^\delta}{[d(z,z_2)]^{\delta/\alpha}} \bigl|f_1(z_1)\bigr|\bigl|f_2(z_2)\bigr|\,d\mu(z_1)\,d\mu(z_2) \\
&\quad \le C \int_{\frac{6}{5}B_0} \frac{|f_1(z_1)|}{\lambda(z,d(z,z_1))}\,d\mu(z_1) \\
&\qquad \times \sum_{k=1}^\infty \int_{6^{k+1}\frac{6}{5}B_0\setminus 6^k\frac{6}{5}B_0} \frac{1}{\lambda(z,d(z,z_2))} \frac{d(z,y)^\delta}{[d(z,z_2)]^{\delta/\alpha}} \bigl|f_2(z_2)\bigr|\,d\mu(z_2) \\
&\quad \le C \frac{\mu(5\times\frac{6}{5}B_0)}{\lambda(x_B,\frac{6}{5}r_{B_0})} \frac{1}{\mu(5\times\frac{6}{5}B_0)} \int_{\frac{6}{5}B_0} \bigl|f_1(z_1)\bigr|\,d\mu(z_1) \\
&\qquad \times \sum_{k=1}^\infty 6^{-k\delta/\alpha} \frac{\mu(5\times 6^{k+1}\frac{6}{5}B_0)}{\lambda(x_B,6^k\frac{6}{5}r_{B_0})} \frac{1}{\mu(5\times 6^{k+1}\frac{6}{5}B_0)} \int_{6^{k+1}\frac{6}{5}B_0} \bigl|f_2(z_2)\bigr|\,d\mu(z_2) \\
&\quad \le CM_{(5)}f_1(x)M_{(5)}f_2(x).
\end{aligned}$$

Therefore,

$$L_2 \le CM_{(5)}f_1(x)M_{(5)}f_2(x).$$

Similar to estimate L_2, we also obtain that

$$L_3 \le CM_{(5)}f_1(x)M_{(5)}f_2(x).$$

Let us turn to estimate L_4. Write

$$\begin{aligned}
&\bigl|T\bigl(f_1^\infty,f_2^\infty\bigr)(z) - T\bigl(f_1^\infty,f_2^\infty\bigr)(y)\bigr| \\
&\quad \le C \int_{X\setminus\frac{6}{5}B_0} \int_{X\setminus\frac{6}{5}B_0} \frac{|f_1(z_1)||f_2(z_2)|}{[\sum_{i=1}^2 \lambda(z,d(z,z_i))]^2} \frac{d(z,y)^\delta}{[\sum_{i=1}^2 d(z,z_i)]^{\delta/\alpha}}\,d\mu(z_1)\,d\mu(z_2) \\
&\quad \le C \sum_{k=1}^\infty \int_{6^{k+1}\frac{6}{5}B_0\setminus 6^k\frac{6}{5}B_0} \sum_{j=1}^\infty \int_{6^{j+1}\frac{6}{5}B_0\setminus 6^j\frac{6}{5}B_0} \frac{|f_1(z_1)||f_2(z_2)|}{[\sum_{i=1}^2 \lambda(z,d(z,z_i))]^2} \\
&\qquad \times \frac{d(z,y)^\delta}{[\sum_{i=1}^2 d(z,z_i)]^{\delta/\alpha}}\,d\mu(z_1)\,d\mu(z_2) \\
&\quad \le C \sum_{k=1}^\infty \sum_{j=k}^\infty \int_{6^{k+1}\frac{6}{5}B_0\setminus 6^k\frac{6}{5}B_0} \bigl|f_2(z_2)\bigr| \\
&\qquad \times \int_{6^{j+1}\frac{6}{5}B_0\setminus 6^j\frac{6}{5}B_0} \frac{|f_1(z_1)|}{[\lambda(z,d(z,z_1))]^2} \frac{d(z,y)^\delta}{[d(z,z_1)]^{\delta/\alpha}}\,d\mu(z_1)\,d\mu(z_2) \\
&\qquad + C \sum_{k=1}^\infty \sum_{j=1}^{k-1} \int_{6^{k+1}\frac{6}{5}B_0\setminus 6^k\frac{6}{5}B_0} \frac{|f_2(z_2)|}{[\lambda(z,d(z,z_2))]^2} \frac{d(z,y)^\delta}{[d(z,z_2)]^{\delta/\alpha}}
\end{aligned}$$

$$\times \int_{6^{j+1}\frac{6}{5}B_0\setminus 6^j\frac{6}{5}B_0} f_1(z_1)\,d\mu(z_1)\,d\mu(z_2)$$
$$=: L_{41} + L_{42}.$$

To estimate L_{41}, by the properties of λ, we obtain

$$\begin{aligned}
L_{41} &\le C\sum_{j=1}^{\infty}\left(\frac{1}{[\lambda(x_B,6^j\frac{6}{5}r_{B_0})]^2}\int_{6^{j+1}\frac{6}{5}B_0\setminus 6^j\frac{6}{5}B_0}\left|f_1(z_1)\right|d\mu(z_1)6^{-j\delta/\alpha}\right)\\
&\quad\times\sum_{k=1}^{j}\int_{6^{k+1}\frac{6}{5}B_0\setminus 6^k\frac{6}{5}B_0}\left|f_2(z_2)\right|d\mu(z_2)\\
&\le C\sum_{j=1}^{\infty}\left(\frac{1}{[\lambda(x_B,5\times 6^j\frac{6}{5}r_{B_0})]^2}\int_{6^{j+1}\frac{6}{5}B_0}\left|f_1(z_1)\right|d\mu(z_1)6^{-j\delta/\alpha}\right)\\
&\quad\times\int_{6^{j+1}\frac{6}{5}B_0}\left|f_2(z_2)\right|d\mu(z_2)\\
&\le C\sum_{j=1}^{\infty}6^{-j\delta/\alpha}\frac{1}{\mu(5\times 6^{j+1}\frac{6}{5}B_0)}\int_{6^{j+1}\frac{6}{5}B_0}\left|f_1(z_1)\right|d\mu(z_1)\\
&\quad\times\frac{1}{\mu(5\times 6^{j+1}\frac{6}{5}B_0)}\int_{6^{j+1}\frac{6}{5}B_0}\left|f_2(z_2)\right|d\mu(z_2)\\
&\le CM_{(5)}f_1(x)M_{(5)}f_2(x).
\end{aligned}$$

For L_{42}, by the properties of λ, we have

$$\begin{aligned}
L_{42} &\le C\sum_{k=1}^{\infty}\left(\frac{1}{[\lambda(x_B,6^k\frac{6}{5}r_{B_0})]^2}\int_{6^{k+1}\frac{6}{5}B_0\setminus 6^k\frac{6}{5}B_0}\left|f_1(z_2)\right|d\mu(z_2)6^{-k\delta/\alpha}\right)\\
&\quad\times\sum_{j=1}^{k-1}\int_{6^{j+1}\frac{6}{5}B_0\setminus 6^j\frac{6}{5}B_0}\left|f_1(z_1)\right|d\mu(z_1)\\
&\le C\sum_{k=1}^{\infty}\left(\frac{1}{[\lambda(x_B,5\times 6^k\frac{6}{5}r_{B_0})]^2}\int_{6^{k+1}\frac{6}{5}B_0}\left|f_2(z_2)\right|d\mu(z_2)6^{-k\delta/\alpha}\right)\\
&\quad\times\int_{6^k\frac{6}{5}B_0}\left|f_1(z_1)\right|d\mu(z_1)\\
&\le C\sum_{k=1}^{\infty}6^{-k\delta/\alpha}\frac{1}{\mu(5\times 6^{k+1}\frac{6}{5}B_0)}\int_{6^{k+1}\frac{6}{5}B_0}\left|f_2(z_2)\right|d\mu(z_2)\\
&\quad\times\frac{1}{\mu(5\times 6^k\frac{6}{5}B_0)}\int_{6^k\frac{6}{5}B_0}\left|f_1(z_1)\right|d\mu(z_1)\\
&\le CM_{(5)}f_1(x)M_{(5)}f_2(x).
\end{aligned}$$

Thus,

$$\left|T\left(f_1^\infty,f_2^\infty\right)(z)-T\left(f_1^\infty,f_2^\infty\right)(y)\right|\le CM_{(5)}f_1(x)M_{(5)}f_2(x).$$

By the above estimate, we obtain (2.9).

Next, we prove (2.10). Consider two balls $B \subset Q$ with $x \in B$, where B is an arbitrary ball, and Q is a doubling ball. Denote $N = N_{B,Q} + 1$. Recall that B_0 and Q_0 are concentric with B and Q, respectively, and $l(B_0) = l(B)^\alpha$, $l(Q_0) = l(Q)^\alpha$. Then,

$$\begin{aligned}
|\widetilde{h}_B - \widetilde{h}_Q| \le{}& \big|m_B\big(T(f_1\chi_{\frac{6}{5}B_0}, f_2\chi_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}B_0}) + T(f_1\chi_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}B_0}, f_2\chi_{\frac{6}{5}B_0})\big)\big| \\
&+ \big|m_B\big(T(f_1\chi_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}B_0}, f_2\chi_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}B_0})\big)\big| \\
&+ \big|m_B\big(T(f_1\chi_{6^N\frac{6}{5}B_0}, f_2\chi_{6^N\frac{6}{5}B_0})\big) - m_Q\big(T(f_1\chi_{6^N\frac{6}{5}B_0}, f_2\chi_{6^N\frac{6}{5}B_0})\big)\big| \\
&+ \big|m_B\big(T(f_1\chi_{X\setminus 6^N\frac{6}{5}B_0}, f_2\chi_{6^N\frac{6}{5}B_0})\big) - m_Q\big(T(f_1\chi_{X\setminus 6^N\frac{6}{5}B_0}, f_2\chi_{6^N\frac{6}{5}B_0})\big)\big| \\
&+ \big|m_B\big(T(f_1\chi_{6^N\frac{6}{5}B_0}, f_2\chi_{X\setminus 6^N\frac{6}{5}B_0})\big) - m_Q\big(T(f_1\chi_{6^N\frac{6}{5}B_0}, f_2\chi_{X\setminus 6^N\frac{6}{5}B_0})\big)\big| \\
&+ \big|m_Q\big(T(f_1\chi_{\frac{6}{5}Q_0}, f_2\chi_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}Q_0}) + T(f_1\chi_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}Q_0}, f_2\chi_{6^N\frac{6}{5}B_0})\big)\big| \\
&+ \big|m_Q\big(T(f_1\chi_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}Q_0}, f_2\chi_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}Q_0})\big)\big| \\
=:{}& M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + M_7.
\end{aligned}$$

To estimate M_1, by the condition (1.5) in Definition 1.7 and the properties of λ, we obtain

$$\begin{aligned}
&\big|T(f_1\chi_{\frac{6}{5}B_0}, f_2\chi_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}B_0})(z)\big| \\
&\quad\le C\int_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}B_0}\int_{\frac{6}{5}B_0}\frac{|f_1(z_1)||f_2(z_2)|}{[\sum_{i=1}^2\lambda(z,d(z,z_i))]^2}\,d\mu(z_1)\,d\mu(z_2) \\
&\quad\le C\int_{\frac{6}{5}B_0}\frac{|f_1(z_1)|}{\lambda(z,d(z,z_1))}\,d\mu(z_1)\int_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}B_0}\frac{|f_2(z_2)|}{\lambda(z,d(z,z_2))}\,d\mu(z_2) \\
&\quad\le C\frac{1}{\mu(5\times\frac{6}{5}B_0)}\int_{\frac{6}{5}B_0}\big|f_1(z_1)\big|\,d\mu(z_1)\sum_{k=1}^{N_{B,Q}}\int_{6^{k+1}\frac{6}{5}B_0\setminus 6^k\frac{6}{5}B_0}\frac{|f_2(z_2)|}{\lambda(z,d(z,z_2))}\,d\mu(z_2) \\
&\quad\le CM_{(5)}f_1(x)\sum_{k=1}^{N_{B,Q}}\frac{\mu(5\times 6^{k+1}\frac{6}{5}B_0)}{\lambda(x_B,5\times 6^{k+1}\frac{6}{5}r_{B_0})}\frac{1}{\mu(5\times 6^{k+1}\frac{6}{5}B_0)}\int_{6^{k+1}\frac{6}{5}B_0}\big|f_2(z_2)\big|\,d\mu(z_2) \\
&\quad\le CK_{B,Q}M_{(5)}f_1(x)M_{(5)}f_2(x).
\end{aligned}$$

Also,

$$\big|T(f_1\chi_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}B_0}, f_2\chi_{\frac{6}{5}B_0})(z)\big| \le CK_{B,Q}M_{(5)}f_1(x)M_{(5)}f_2(x).$$

Then,

$$M_1 \le CK_{B,Q}M_{(5)}f_1(x)M_{(5)}f_2(x).$$

Let us estimate M_2. By the condition (1.5) in Definition 1.7 and the properties of λ, we obtain

$$\begin{aligned}
&\big|T(f_1\chi_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}B_0}, f_2\chi_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}B_0})\big| \\
&\quad\le C\int_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}B_0}\int_{6^N\frac{6}{5}B_0\setminus\frac{6}{5}B_0}\frac{|f_1(z_1)||f_2(z_2)|}{[\sum_{i=1}^2\lambda(z,d(z,z_i))]^2}\,d\mu(z_1)\,d\mu(z_2)
\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{k=1}^{N-1}\sum_{j=1}^{N-1}\int_{6^{k+1}\frac{6}{5}B_0\backslash 6^k\frac{6}{5}B_0}\int_{6^{j+1}\frac{6}{5}B_0\backslash 6^j\frac{6}{5}B_0}\frac{|f_1(z_1)||f_2(z_2)|}{[\sum_{i=1}^2\lambda(z,d(z,z_i))]^2}\,d\mu(z_1)\,d\mu(z_2)\\
&\quad + C\int_{6\times\frac{6}{5}B_0\backslash\frac{6}{5}B_0}\int_{6\times\frac{6}{5}B_0\backslash\frac{6}{5}B_0}\frac{|f_1(z_1)||f_2(z_2)|}{[\sum_{i=1}^2\lambda(z,d(z,z_i))]^2}\,d\mu(z_1)\,d\mu(z_2)\\
&=: M_{21}+M_{22}.
\end{aligned}
$$

For M_{22}, since $z\in B_0$ and $z_i\in 6\times\frac{6}{5}B_0\backslash\frac{6}{5}B_0$, then $\frac{1}{5}r_{B_0}\leq d(z,z_i)\leq\frac{41}{5}r_{B_0}$ for $i=1,2$. Therefore,

$$
\begin{aligned}
M_{22}&\leq\frac{C}{[\lambda(z,\frac{1}{5}r_{B_0})]^2}\int_{6\times\frac{6}{5}B_0\backslash\frac{6}{5}B_0}\int_{6\times\frac{6}{5}B_0\backslash\frac{6}{5}B_0}\frac{|f_1(z_1)||f_2(z_2)|}{d}\mu(z_1)\,d\mu(z_2)\\
&\leq\frac{C}{[\lambda(x_B,\frac{1}{5}r_{B_0})]^2}\int_{6\times\frac{6}{5}B_0\backslash\frac{6}{5}B_0}\int_{6\times\frac{6}{5}B_0\backslash\frac{6}{5}B_0}\frac{|f_1(z_1)||f_2(z_2)|}{d}\mu(z_1)\,d\mu(z_2)\\
&\leq C\prod_{i=1}^2\frac{\mu(30\times\frac{6}{5}B_0)}{\lambda(x_B,\frac{1}{5}r_{B_0})}\frac{1}{\mu(30\times\frac{6}{5}B_0)}\int_{6\times\frac{6}{5}B_0}\left|f_i(z_i)\right|d\mu(z_i)\\
&\leq CM_{(5)}f_1(x)M_{(5)}f_2(x).
\end{aligned}
$$

For M_{21}, by the properties of λ, we obtain

$$
\begin{aligned}
M_{21}&\leq C\sum_{k=1}^{N-1}\sum_{j=k}^{N-1}\int_{6^{k+1}\frac{6}{5}B_0\backslash 6^k\frac{6}{5}B_0}\left|f_1(z_1)\right|\int_{6^{j+1}\frac{6}{5}B_0\backslash 6^j\frac{6}{5}B_0}\frac{|f_2(z_2)|}{[\lambda(z,d(z,z_2))]^2}\,d\mu(z_2)\,d\mu(z_1)\\
&\quad + C\sum_{k=1}^{N-1}\sum_{j=1}^{k-1}\int_{6^{k+1}\frac{6}{5}B_0\backslash 6^k\frac{6}{5}B_0}\frac{|f_1(z_1)|}{[\lambda(z,d(z,z_1))]^2}\int_{6^{j+1}\frac{6}{5}B_0\backslash 6^j\frac{6}{5}B_0}\left|f_2(z_2)\right|d\mu(z_2)\,d\mu(z_1)\\
&\leq C\sum_{j=1}^{N-1}\frac{1}{[\lambda(x_B,6^j\frac{6}{5}r_{B_0})]^2}\int_{6^{j+1}\frac{6}{5}B_0\backslash 6^j\frac{6}{5}B_0}\left|f_2(z_2)\right|d\mu(z_2)\\
&\quad\times\sum_{k=1}^{j}\int_{6^{j+1}\frac{6}{5}B_0\backslash 6^j\frac{6}{5}B_0}\left|f_1(z_1)\right|d\mu(z_1)\\
&\quad + C\sum_{k=1}^{N-1}\int_{6^{k+1}\frac{6}{5}B_0\backslash 6^k\frac{6}{5}B_0}\frac{|f_1(z_1)|}{[\lambda(z,d(z,z_1))]^2}\int_{6^k\frac{6}{5}B_0}\left|f_2(z_2)\right|d\mu(z_2)\,d\mu(z_1)\\
&\leq C\sum_{j=1}^{N-1}\frac{1}{[\lambda(x_B,6^j\frac{6}{5}r_{B_0})]^2}\int_{6^{j+1}\frac{6}{5}B_0}\left|f_2(z_2)\right|d\mu(z_2)\int_{6^{j+1}\frac{6}{5}B_0}\left|f_1(z_1)\right|d\mu(z_1)\\
&\quad + C\sum_{j=1}^{N-1}\frac{1}{[\lambda(x_B,6^j\frac{6}{5}r_{B_0})]^2}\int_{6^{j+1}\frac{6}{5}B_0}\left|f_1(z_1)\right|d\mu(z_1)\int_{6^kB_0}\left|f_2(z_2)\right|d\mu(z_2)\\
&\leq CK_{B,Q}M_{(5)}f_1(x)M_{(5)}f_2(x).
\end{aligned}
$$

For M_3, similar to estimate L_1, we have

$$
M_3\leq CM_{(5)}f_1(x)M_{(5)}f_2(x).
$$

For M_4 and M_5, similar to estimate L_2, we have

$$M_4 + M_5 \le CM_{(5)}f_1(x)M_{(5)}f_2(x).$$

By a similar method to estimate M_1, we also obtain that

$$M_6 \le CK_{B,Q}M_{(5)}f_1(x)M_{(5)}f_2(x).$$

By a similar method to estimate M_2, we also obtain that

$$M_7 \le CK_{B,Q}M_{(5)}f_1(x)M_{(5)}f_2(x).$$

Hence, (2.10) is proved. Thus, the proof of Lemma 2.3 is completed. □

Proof of Theorem 1.11 Let $0 < \delta < 1/2$, $1 < p_1, p_2, q < \infty$, $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$, $f_1 \in L^{p_1}(\mu)$ and $f_2 \in L^{p_2}(\mu)$. By $|f(x)| \le N_\delta f(x)$, Lemma 2.1, Lemma 2.3, Hörder's inequality and the boundedness of $M_{(\rho)}$ for $\rho \ge 5$, we obtain

$$\begin{aligned}\|T(f_1,f_2)\|_{L^q(\mu)} &\le \|N_\delta(T(f_1,f_2))\|_{L^q(\mu)}\\ &\le C\|M_\delta^\sharp(T(f_1,f_2))\|_{L^q(\mu)} \le C\|M_{(5)}(f_1)M_{(5)}(f_2)\|_{L^q(\mu)}\\ &\le C\|M_{(5)}(f_1)\|_{L^{p_1}(\mu)}\|M_{(5)}(f_1)\|_{L^{p_2}(\mu)} \le C\|f_1\|_{L^{p_1}(\mu)}\|f_2\|_{L^{p_2}(\mu)}.\end{aligned}$$

Thus, the proof of Theorem 1.11 is finished. □

Next, let us prove Theorem 1.12. We first prove the following lemma.

Lemma 2.4 *Assume that λ satisfies the ϵ-weak reverse doubling condition. Let $1 < q \le p < \infty$. If $f \in M_q^p(\mu)$, then for $x \in B(x_B, r_B)$, we have*

$$\int_{X\setminus 2B} \frac{|f(y)|}{\lambda(x,d(x,y))}\,d\mu(y) \le C\big(\lambda(x_B,r_B)\big)^{-\frac{1}{p}}\|f\|_{M_q^p(\mu)}.$$

Proof By Hölder's inequality, we have

$$\begin{aligned}&\int_{X\setminus 2B} \frac{|f(y)|}{\lambda(x,d(x,y))}\,d\mu(y)\\ &\quad= \sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB} \frac{|f(y)|}{\lambda(x,d(x,y))}\,d\mu(y)\\ &\quad\le C\sum_{j=1}^{\infty}\frac{1}{\lambda(x_B,2^jr_B)}\int_{2^{j+1}B\setminus 2^jB}|f(y)|\,d\mu(y)\\ &\quad\le C\sum_{j=1}^{\infty}\frac{1}{\lambda(x_B,2^jr_B)}\left[\int_{2^{j+1}B}|f(y)|^q\,d\mu(y)\right]^{1/q}\mu\big(2^{j+1}B\big)^{1/q'}\\ &\quad\le C\sum_{j=1}^{\infty}\frac{1}{\lambda(x_B,2^jr_B)}\|f\|_{L^q(\mu,2^{j+1}B)}\big[\mu\big(2^{j+2}B\big)\big]^{1/p-1/q}\big[\mu\big(2^{j+2}B\big)\big]^{1/q-1/p}\mu\big(2^{j+1}B\big)^{1/q'}\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{j=1}^{\infty}\frac{1}{\lambda(x_B,2^j r_B)}\|f\|_{M_q^p(\mu)}\bigl[\mu\bigl(2^{j+2}B\bigr)\bigr]^{1-1/p}\\
&\le C\|f\|_{M_q^p(\mu)}\sum_{j=1}^{\infty}\bigl[\lambda\bigl(x_B,2^j r_B\bigr)\bigr]^{-1/p}\\
&\le C\|f\|_{M_q^p(\mu)}\sum_{j=1}^{\infty}\frac{1}{[C(2^j)]^{1/p}}\bigl[\lambda(x_B,r_B)\bigr]^{-1/p}\\
&\le C\|f\|_{M_q^p(\mu)}\bigl[\lambda(x_B,r_B)\bigr]^{-1/p}.
\end{aligned}
$$

Thus, the proof of Lemma 2.4 has been completed. □

Now we give the proof of Theorem 1.12.

Proof of Theorem 1.12 Fixing a ball $B\in X$, we have

$$
\begin{aligned}
\bigl|T(f_1,f_2)(z)\bigr| &\le \bigl|T\bigl(f_1^0,f_2^0\bigr)(z)\bigr| + \bigl|T\bigl(f_1^0,f_2^\infty\bigr)(z)\bigr| + \bigl|T\bigl(f_1^\infty,f_2^0\bigr)(z)\bigr| + \bigl|T\bigl(f_1^\infty,f_2^0\bigr)(z)\bigr|\\
&:= G_1+G_2+G_3+G_4,
\end{aligned}
$$

where $f_i^0=f_i\chi_{2B}$ and $f_i^\infty=f_i-f_i^0$ for $i=1,2$.

For G_1, by the result of Theorem 1.11, we have

$$
\begin{aligned}
\bigl\|T\bigl(f_1^0,f_2^0\bigr)\bigr\|_{M_q^p}(\mu) &\le \sup_{B\in X}\mu(6B)^{\frac{1}{p}-\frac{1}{q}}\biggl(\int_B\bigl|T\bigl(f_1^0,f_2^0\bigr)(z)\bigr|^q\,d\mu(y)\biggr)^{\frac{1}{q}}\\
&\le \sup_{B\in X}\mu(6B)^{\frac{1}{p}-\frac{1}{q}}\bigl\|T\bigl(f_1^0,f_2^0\bigr)\bigr\|_{L^q(\mu)}\\
&\le C\sup_{B\in X}\mu(6B)^{\frac{1}{p}-\frac{1}{q}}\prod_{i=1}^{2}\bigl\|f_i^0\bigr\|_{L^{q_i}(\mu)}\\
&\le C\sup_{B\in X}\mu(6B)^{\frac{1}{p}-\frac{1}{q}}\prod_{i=1}^{2}\biggl(\int_{2B}|f_i|^{q_i}\,d\mu(y)\biggr)^{\frac{1}{q_i}}\\
&\le C\sup_{B\in X}\prod_{i=1}^{2}\mu(6B)^{\frac{1}{p_i}-\frac{1}{q_i}}\biggl(\int_{2B}|f_i|^{q_i}\,d\mu(y)\biggr)^{\frac{1}{q_i}}\\
&\le C\prod_{i=1}^{2}\|f_i\|_{M_{q_i}^{p_i}}(\mu).
\end{aligned}
$$

For G_2, by condition (1.5) in Definition 1.7 and Lemma 2.4, we have

$$
\begin{aligned}
\bigl|T\bigl(f_1^0,f_2^\infty\bigr)(z)\bigr| &\le C\int_{2B}\bigl|f_1(z_1)\bigr|\int_{X\setminus 2B}\frac{|f_2(z_2)|}{[\lambda(x,d(x,z_1))+\lambda(x,d(x,z_2))]^2}\,d\mu(z_2)\,d\mu(z_1)\\
&\le C\int_{2B}\frac{|f_1(z_1)|}{\lambda(x,d(x,z_1))}\,d\mu(z_1)\int_{X\setminus 2B}\frac{|f_2(z_2)|}{\lambda(x,d(x,z_2))}\,d\mu(z_2)\\
&\le C\frac{1}{\mu(2B)}\int_{2B}\bigl|f_1(z_1)\bigr|\,d\mu(z_1)\int_{X\setminus 2B}\frac{|f_2(z_2)|}{\lambda(x,d(x,z_2))}\,d\mu(z_2)
\end{aligned}
$$

$$\begin{aligned}&\leq C\frac{1}{\mu(2B)}\big(\mu(2B)\big)^{1-1/q_1}\|f_1\|_{L^{q_1}(\mu,2B)}\big(\lambda(x_B,r_B)\big)^{-1/p_2}\|f_2\|_{M^{p_2}_{q_2}(\mu)}\\&\leq C\mu(2B)^{-1/p_1}\big(\lambda(x_B,r_B)\big)^{-1/p_2}\prod_{i=1}^{2}\|f_i\|_{M^{p_i}_{q_i}}(\mu).\end{aligned}$$

Therefore,

$$\begin{aligned}\big\|T\big(\big(f_1^0,f_2^\infty\big)\big)\big\|_{M^p_q(\mu)}&\leq\sup_{B\in X}\mu(6B)^{1/p-1/q}\bigg(\int_B\big|T\big(\big(f_1^0,f_2^\infty\big)\big)(z)\big|^q\,d\mu(z)\bigg)^{1/q}\\&\leq C\mu(6B)^{1/p-1/q}\mu(2B)^{-1/p_1}\big(\lambda(x,r)\big)^{-1/p_2}\mu(B)^{1/q}\prod_{i=1}^{2}\|f_i\|_{M^{p_i}_{q_i}}(\mu)\\&\leq C\mu(2B)^{1/p-1/q}\mu(2B)^{-1/p_1}\big(\mu(2B)\big)^{-1/p_2}\big(\mu(B)\big)^{1/q}\prod_{i=1}^{2}\|f_i\|_{M^{p_i}_{q_i}}(\mu)\\&\leq C\bigg[\frac{\mu(B)}{\mu(2B)}\bigg]^{1/q}\prod_{i=1}^{2}\|f_i\|_{M^{p_i}_{q_i}}(\mu)\\&\leq C\prod_{i=1}^{2}\|f_i\|_{M^{p_i}_{q_i}}(\mu).\end{aligned}$$

For G_3, similar to G_2, we have

$$\big\|T\big(\big(f_1^\infty,f_2^0\big)\big)\big\|_{M^p_q(\mu)}\leq\prod_{i=1}^{2}\|f_i\|_{M^{p_i}_{q_i}}(\mu).$$

Let us move on to estimate G_4. For $y\in B$, by Lemma 2.4, we have

$$\begin{aligned}\big|T\big(f_1^\infty,f_2^\infty\big)(z)\big|&\leq C\int_{X\setminus 2B}\int_{X\setminus 2B}\frac{|f_1(z_1)f_2(z_2)|}{[\lambda(x,d(x,z_1))+\lambda(x,d(x,z_2))]^2}\,d\mu(z_2)\,d\mu(z_1)\\&\leq C\prod_{i=1}^{2}\int_{X\setminus 2B}\frac{|f_i(z_i)|}{\lambda(x,d(x,z_i))}\,d\mu(z_i)\\&\leq C\big(\lambda(x_B,r_B)\big)^{-\frac{1}{p}}\|f\|_{M^{p_i}_{q_i}(\mu)}.\end{aligned}$$

Thus,

$$\begin{aligned}\big\|T\big(f_1^\infty,f_2^\infty\big)\big\|_{M^p_q(\mu)}&\leq\sup_{B\in X}\mu(6B)^{1/p-1/q}\bigg(\int_B\big|T\big(\big(f_1^\infty,f_2^\infty\big)\big)(z)\big|^q\,d\mu(z)\bigg)^{1/q}\\&\leq C\mu(6B)^{1/p-1/q}\big(\lambda(x,r)\big)^{-1/p}\mu(B)^{1/q}\prod_{i=1}^{2}\|f_i\|_{M^{p_i}_{q_i}}(\mu)\\&\leq C\mu(2B)^{1/p-1/q}\big(\mu(2B)\big)^{-1/p}\big(\mu(B)\big)^{1/q}\prod_{i=1}^{2}\|f_i\|_{M^{p_i}_{q_i}}(\mu)\end{aligned}$$

$$\leq C\left[\frac{\mu(B)}{\mu(2B)}\right]^{1/q}\prod_{i=1}^{2}\|f_i\|_{M_{q_i}^{p_i}}(\mu)$$
$$\leq C\prod_{i=1}^{2}\|f_i\|_{M_{q_i}^{p_i}}(\mu).$$

The proof of Theorem 1.12 is finished. □

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Availability of data and materials
No data have been used in this study.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HW analyzed the problems and proved Theorem 1.11. RX proposed the problems and proved Theorem 1.12. All authors read and approved the final manuscript.

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