Hammad and Abdeljawad *Journal of Inequalities and Applications* (2022) 2022:44
https://doi.org/10.1186/s13660-022-02780-6
Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Quadruple fixed-point techniques for solving integral equations involved with matrices and the Markov process in generalized metric spaces

Hasanen A. Hammad[1] and Thabet Abdeljawad[2,3*]

*Correspondence:
tabdeljawad@psu.edu.sa
[2]Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
[3]Department of Medical Research, China Medical University, Taichung 40402, Taiwan
Full list of author information is available at the end of the article

Abstract

The goal of this manuscript is to establish quadruple fixed-point and coincidence-point consequences in the setting of generalized metric spaces equipped with vector-valued metrics and matrix equations. Moreover, some supportive examples and corollaries are presented here to support the theoretical results. Ultimately, the theoretical results are presented here to discuss some applications to support our study.

MSC: 47H10; 54H25

Keywords: Quadruple fixed points; Matrix equations; Generalized metric spaces; Regularity; Compatibility; Weakly reciprocally continuous; Markov process

1 Introduction

Fixed-point theory is an important pillar of the nonlinear analysis branch, as it plays an active role in many branches of mathematics. The creation of the fixed point of the contractive mappings has become a powerful research center for its multiple applications in ordinary differential, fractional and integral equations.

For contraction mappings, the fundamental result of Banach on ordinary metric spaces endowed with vector-valued metrics is described by Perov [1]. Later, for a self-mapping on generalized metric spaces (Ω, ω), the results of Perov were generalized by Filip and Petrușel [2] and some fixed-point sequences are proved. For more details, see [3–6].

In [7], the notions of mixed-monotone functions and coupled fixed points were initiated and studied. Under partially ordered metric spaces (POMSs) and abstract spaces, some main results in this direction have been developed, for broadening, see [8–12].

Thereafter, a tripled fixed point (TFP) was introduced by Berinde and Borcut [13] in 2011. They initiated it for self-mappings and established some exciting consequences in POMSs. For more topics using this idea, we cite these papers [14–16].

To generalize a TFP, Karapinar initiated the idea of a quadruple fixed point (QFP) and showed some fixed-point results on the topic [17]. Following this study, a QFP is developed and some related fixed-point results are discussed in [18–20].

Similar to this pattern in this manuscript, some QFP results have been obtained in the framework of generalized metric spaces equipped with vector-valued metrics and matrix equations. Also, some old definitions have been circulated and supporting examples have been put forward to strengthen our theoretical results. Finally, to illustrate the importance of fixed-point technology, the theoretical results have been used in three applications. Application I, on the study of the existence and uniqueness of the solution for a four-system of integral equations, application II, on the study of the existence and uniqueness of the solution for a four-system of integral sequences and application III, on finding a unique stationary distribution for the Markov process.

2 Preliminaries

Throughout this paper, the symbols $M_{m,m}(\mathbb{R}^+)$, ϑ and I represent the set of all $m \times m$ matrices with components in $\mathbb{R}^+$, zero and identity matrices, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Matrix convergence is very similar to normal convergence of sequences or vectors, i.e., assume that $\Phi \in M_{m\times m}(\mathbb{R}^+)$, a matrix Φ is called convergent to zero, iff $\lim_{n\to\infty} \Phi^n = \vartheta$. To delve deeper into this trend, we pay close attention to [21, 22].

In the literature, some examples of matrices converging to zero are incorporated as follows:

Example 2.1 The matrix

$$\Phi = \begin{pmatrix} e^1 & e^1 \\ e^2 & e^2 \end{pmatrix} \quad \text{or} \quad \Phi = \begin{pmatrix} e^1 & e^2 \\ e^1 & e^2 \end{pmatrix},$$

in $M_{2,2}(\mathbb{R}^+)$ with $e^1 + e^2 < 1$, converges to zero.

Example 2.2 Any matrix in $M_{2,2}(\mathbb{R}^+)$ in the form of

$$\Phi = \begin{pmatrix} e^1 & e^2 \\ 0 & e^3 \end{pmatrix},$$

converges to zero, provided that $\max\{e^1, e^3\} < 1$.

Example 2.3 The matrix in the form of

$$\Phi = \begin{pmatrix} \Upsilon_1 & 0 & \cdots & 0 \\ 0 & \Upsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Upsilon_m \end{pmatrix}_{m\times m},$$

in $M_{m,m}(\mathbb{R}^+)$, converges to zero, provided that $\max\{\Upsilon_i : 1 \le i \le m\} < 1$.

For a nonconvergence matrix to zero, we have the following example:

Example 2.4 If we consider $e^1 + e^2 \geq 1$ in the matrix below

$$\Phi = \begin{pmatrix} e^1 & e^2 \\ e^3 & e^4 \end{pmatrix},$$

in $M_{2,2}(\mathbb{R}^+)$, then it does not converge to zero.

The definition of addition and multiplication on $\mathbb{R}^m$ is given as follows: Let $\nu, \nu^* \in \mathbb{R}^m$, where $\nu = (\nu_1, \nu_2, \dots, \nu_m)$ and $\nu^* = (\nu_1^*, \nu_2^*, \dots, \nu_m^*)$, then

$$\nu + \nu^* = (\nu_1 + \nu_1^*, \nu_2 + \nu_2^*, \dots, \nu_m + \nu_m^*),$$

and

$$\nu.\nu^* = (\nu_1.\nu_1^*, \nu_2.\nu_2^*, \dots, \nu_m.\nu_m^*).$$

Note that, for $\ell \in \mathbb{R}$, $\nu_i \leq \nu_i^*$ (resp., $\nu_i < \nu_i^*$) for each $1 \leq i \leq m$, also $\nu_i \leq \ell$ (resp., $\nu_i < \ell$) for each $1 \leq i \leq m$, respectively. This topic has been studied extensively in [2, 23, 24].

Definition 2.5 ([2]) A mapping $\omega : \Omega^2 \to \mathbb{R}^m$ (where Ω is a nonempty set) is called a vector-valued metric on Ω, if the assumptions below are satisfied, for each $e^1, e^2, e^3 \in \Omega$,
($\heartsuit_1$) $\omega(e^1, e^2) \geq 0$, $\omega(e^1, e^2) = 0 \Leftrightarrow e^1 = e^2$;
($\heartsuit_2$) $\omega(e^1, e^2) = \omega(e^2, e^1)$;
($\heartsuit_3$) $\omega(e^1, e^2) \leq \omega(e^1, e^3) + \omega(e^3, e^2)$.
If $e^1, e^2 \in \mathbb{R}^m$, where $e^1 = (e_1^1, \dots, e_m^1)$ and $e^2 = (e_1^2, \dots, e_m^2)$, then $e^1 \leq e^2$ iff $e_i^1 \leq e_i^2$ for $1 \leq i \leq m$. Then, the pair (Ω, ω) is called a generalized metric space.

In matrix analysis, the proof of the following equivalent propositions that we need are discussed in [25, 26].

- $\Phi \to \vartheta$;
- $\Phi^n \to \vartheta$ as $n \to \infty$;
- for each $\hbar \in \mathbb{C}$, $|\hbar| < 1$ with $\det(\Phi - \hbar I) = \vartheta$;
- $I - \Phi$ is a nonsingular matrix and

$$(I - \Phi)^{-1} = I + \Phi + \cdots + \Phi^n + \cdots;$$

- the two matrices $\Phi^n l$ and $l\Phi^n$ tend to zero as $n \to \infty$, for each $l \in \mathbb{R}^m$.

Here, ZM refers to the set of all matrices $\Phi \in M_{m,m}(\mathbb{R}^+)$, where $\Phi^n \to \vartheta$. For brevity, we identify the row and column vectors in $\mathbb{R}^m$.

Bhaskar and Lakshmikantham [7] introduced the following two definitions:

Definition 2.6 ([7]) An element $(e^1, e^2) \in \Omega^2$ is called a coupled fixed point (CFP) of the mapping $H : \Omega^2 \to \Omega$ if $H(e^1, e^2) = e^1$ and $H(e^2, e^1) = e^2$.

Definition 2.7 ([7]) The two given mappings $H : \Omega^2 \to \Omega$ and $h : \Omega \to \Omega$ have a CFP $(e^1, e^2) \in \Omega^2$ if $H(e^1, e^2) = he^1$ and $H(e^2, e^1) = he^2$.

A TFP concept was presented by Berinde and Borcut [27] as follows:

Definition 2.8 ([27]) An element $(e^1, e^2, e^3) \in \Omega^3$ is called a TFP of the mapping $H : \Omega^3 \to \Omega$ if

$$H(e^1, e^2, e^3) = e^1, \qquad H(e^2, e^3, e^1) = e^2, \quad \text{and} \quad H(e^3, e^1, e^2) = e^3.$$

Karapinar [18] introduced the two definitions below.

Definition 2.9 Let $\Omega \neq \emptyset$, and $H : \Omega^4 \to \Omega$ be a given mapping. An element $(e^1, e^2, e^3, e^4) \in \Omega^4$ is called a QFP of H if

$$\begin{aligned} &H(e^1, e^2, e^3, e^4) = e^1, \; H(e^2, e^3, e^4, e^1) = e^2, \\ &H(e^3, e^4, e^1, e^2) = e^3, \; H(e^4, e^1, e^2, e^3) = e^4. \end{aligned}$$

Definition 2.10 Let $H : \Omega^4 \to \Omega$ be a mapping defined on a partially ordered set (POS) $(\Omega, \preceq)$. We say that H has the mixed-monotone property (MMP) if for any $e^1, e^2, e^3, e^4 \in \Omega$,

$$\begin{aligned} e^1_1, e^1_2 \in \Omega, e^1_1 \preceq e^1_2 \quad &\Rightarrow \quad H(e^1_1, e^2, e^3, e^4) \preceq H(e^1_2, e^2, e^3, e^4), \\ e^2_1, e^2_2 \in \Omega, e^2_1 \preceq e^2_2 \quad &\Rightarrow \quad H(e^1, e^2_1, e^3, e^4) \succeq H(e^1, e^2_2, e^3, e^4), \\ e^3_1, e^3_2 \in \Omega, e^3_1 \preceq e^3_2 \quad &\Rightarrow \quad H(e^1, e^2, e^3_1, e^4) \preceq H(e^1, e^2, e^3_2, e^4), \\ e^4_1, e^4_2 \in \Omega, e^4_1 \preceq e^4_2 \quad &\Rightarrow \quad H(e^1, e^2, e^3, e^4_1) \succeq H(e^1, e^2, e^3, e^4_2). \end{aligned}$$

The Definition 2.10 is generalized for two mappings as follows:

Definition 2.11 ([1]) Let $H : \Omega^4 \to \Omega$ and $h : \Omega \to \Omega$ be mappings defined on a POS $(\Omega, \preceq)$. We say H has the mixed h-monotone property (MhMP) if for any $e^1, e^2, e^3, e^4 \in \Omega$,

$$\begin{aligned} e^1_1, e^1_2 \in \Omega, he^1_1 \preceq he^1_2 \quad &\Rightarrow \quad H(e^1_1, e^2, e^3, e^4) \preceq H(e^1_2, e^2, e^3, e^4), \\ e^2_1, e^2_2 \in \Omega, he^2_1 \preceq he^2_2 \quad &\Rightarrow \quad H(e^1, e^2_1, e^3, e^4) \succeq H(e^1, e^2_2, e^3, e^4), \\ e^3_1, e^3_2 \in \Omega, he^3_1 \preceq he^3_2 \quad &\Rightarrow \quad H(e^1, e^2, e^3_1, e^4) \preceq H(e^1, e^2, e^3_2, e^4), \\ e^4_1, e^4_2 \in \Omega, he^4_1 \preceq he^4_2 \quad &\Rightarrow \quad H(e^1, e^2, e^3, e^4_1) \succeq H(e^1, e^2, e^3, e^4_2). \end{aligned}$$

Definition 2.12 ([28]) The mapping $\bar{\omega} : \Omega^4 \to \mathbb{R}^m$ defined on a generalized metric space (Ω, ω) equipped with

$$\bar{\omega}\big((e^1, e^2, e^3, e^4), (r^1, r^2, r^3, r^4)\big) = \omega(e^1, r^1) + \omega(e^2, r^2) + \omega(e^3, r^3) + \omega(e^4, r^4),$$

defined a metric on Ω^4 that will be denoted for simplicity by ω, for all $e^1, e^2, e^3, e^4, r^1, r^2, r^3, r^4 \in \Omega$.

Definition 2.13 ([22]) Assume that $(\Omega, \omega, \preceq)$ is a partially ordered metric space. Ω is called regular if the stipulations below are fulfilled:

(a) for $n \geq 0$, $e^1_n \preceq e^1$ if a nondecreasing sequence $e^1_n \to e^1$,

(b) for $n \geq 0$, $e^2 \preceq e^2_n$ if a nonincreasing sequence $e^2_n \to e^2$.

3 Main results

We begin this part with the following definitions:

Definition 3.1 Let $H:\Omega^4\to\Omega$ and $h:\Omega\to\Omega$ be mappings defined on a metric space (Ω,ω). We say that H and h are called compatible if

$$\lim_{n\to+\infty}\omega\big(h(O_{1234}),H(P_{1234})\big)=0,$$

where $O_{1234}=H(e_n^1,e_n^2,e_n^3,e_n^4)$ and $P_{1234}=(he_n^1,he_n^2,he_n^3,he_n^4)$,

$$\lim_{n\to+\infty}\omega\big(h(O_{2341}),H(P_{2341})\big)=0,$$

where $O_{2341}=H(e_n^2,e_n^3,e_n^4,e_n^1)$ and $P_{2341}=(he_n^2,he_n^3,he_n^4,he_n^1)$,

$$\lim_{n\to+\infty}\omega\big(h(O_{3412}),H(P_{3412})\big)=0,$$

where $O_{3412}=H(e_n^3,e_n^4,e_n^1,e_n^2)$ and $P_{3412}=(he_n^3,he_n^4,he_n^1,he_n^2)$,

$$\lim_{n\to+\infty}\omega\big(h(O_{4123}),H(P_{4123})\big)=0,$$

where $O_{4123}=H(e_n^4,e_n^1,e_n^2,e_n^3)$ and $P_{4123}=(he_n^4,he_n^1,he_n^2,he_n^3)$, whenever $\{e_n^1\}$, $\{e_n^2\}$, $\{e_n^3\}$ and $\{e_n^4\}$ are sequences in Ω so that

$$\lim_{n\to+\infty}O_{1234}=\lim_{n\to+\infty}he_n^1=e^1,\qquad \lim_{n\to+\infty}O_{2341}=\lim_{n\to+\infty}he_n^2=e^2,$$
$$\lim_{n\to+\infty}O_{3412}=\lim_{n\to+\infty}he_n^3=e^3,\qquad \lim_{n\to+\infty}O_{4123}=\lim_{n\to+\infty}he_n^4=e^4,$$

for some $e^1,e^2,e^3,e^4\in\Omega$.

Definition 3.2 The mappings $H:\Omega^4\to\Omega$ and $h:\Omega\to\Omega$ are called reciprocally continuous if for some $e^1,e^2,e^3,e^4\in\Omega$,

$$\lim_{n\to+\infty}h(O_{1234})=he^1\quad\text{and}\quad\lim_{n\to+\infty}H(P_{1234})=H\big(e^1,e^2,e^3,e^4\big),$$
$$\lim_{n\to+\infty}h(O_{2341})=he^2\quad\text{and}\quad\lim_{n\to+\infty}H(P_{2341})=H\big(e^2,e^3,e^4,e^1\big),$$
$$\lim_{n\to+\infty}h(O_{3412})=he^3\quad\text{and}\quad\lim_{n\to+\infty}H(P_{3412})=H\big(e^3,e^4,e^1,e^2\big),$$
$$\lim_{n\to+\infty}h(O_{4123})=he^4\quad\text{and}\quad\lim_{n\to+\infty}H(P_{4123})=H\big(e^4,e^1,e^2,e^3\big),$$

whenever $\{e_n^1\}$, $\{e_n^2\}$, $\{e_n^3\}$, and $\{e_n^4\}$ are sequences in Ω so that

$$\lim_{n\to+\infty}O_{1234}=\lim_{n\to+\infty}he_n^1=e^1,\qquad \lim_{n\to+\infty}O_{2341}=\lim_{n\to+\infty}he_n^2=e^2,$$
$$\lim_{n\to+\infty}O_{3412}=\lim_{n\to+\infty}he_n^3=e^3,\qquad \lim_{n\to+\infty}O_{4123}=\lim_{n\to+\infty}he_n^4=e^4.$$

Definition 3.3 The mappings $H:\Omega^4\to\Omega$ and $h:\Omega\to\Omega$ are called weakly reciprocally continuous if for some $e^1,e^2,e^3,e^4\in\Omega$,

$$\begin{aligned}
&\lim_{n\to+\infty} h(O_{1234}) = he^1 \quad \text{or} \quad \lim_{n\to+\infty} H(P_{1234}) = H\big(e^1,e^2,e^3,e^4\big),\\
&\lim_{n\to+\infty} h(O_{2341}) = he^2 \quad \text{or} \quad \lim_{n\to+\infty} H(P_{2341}) = H\big(e^2,e^3,e^4,e^1\big),\\
&\lim_{n\to+\infty} h(O_{3412}) = he^3 \quad \text{or} \quad \lim_{n\to+\infty} H(P_{3412}) = H\big(e^3,e^4,e^1,e^2\big),\\
&\lim_{n\to+\infty} h(O_{4123}) = he^4 \quad \text{or} \quad \lim_{n\to+\infty} H(P_{4123}) = H\big(e^4,e^1,e^2,e^3\big),
\end{aligned}$$

whenever $\{e_n^1\}$, $\{e_n^2\}$, $\{e_n^3\}$, and $\{e_n^4\}$ are sequences in Ω so that

$$\begin{aligned}
&\lim_{n\to+\infty} O_{1234} = \lim_{n\to+\infty} he_n^1 = e^1, \qquad \lim_{n\to+\infty} O_{2341} = \lim_{n\to+\infty} he_n^2 = e^2,\\
&\lim_{n\to+\infty} O_{3412} = \lim_{n\to+\infty} he_n^3 = e^3, \qquad \lim_{n\to+\infty} O_{4123} = \lim_{n\to+\infty} he_n^4 = e^4.
\end{aligned}$$

Example 3.4 Let $\Omega=[0,1]$ with the usual metric and partial order $\preceq$, $H:\Omega^4\to\Omega$ and $h:\Omega\to\Omega$ be two mappings defined by $H(e^1,e^2,e^3,e^4)=\frac{e^1-e^2+e^3-e^4}{4}$ and $h(e^1)=e^1$. Define sequences $\{e_n^1\}$, $\{e_n^2\}$, $\{e_n^3\}$, and $\{e_n^4\}$ by

$$e_n^1=\frac{1}{n}, \qquad e_n^2=\frac{1}{n+1}, \qquad e_n^3=\frac{1}{\sqrt{n^2+1}}, \quad \text{and} \quad e_n^4=\frac{n}{n^3+1}, \qquad \forall n\in\mathbb{N}.$$

Then, $(\Omega,\preceq)$ is a partially ordered metric space. By simple calculations one can obtain that

$$\begin{aligned}
&\lim_{n\to+\infty} O_{1234} = \lim_{n\to+\infty} he_n^1 = 0, \qquad \lim_{n\to+\infty} O_{2341} = \lim_{n\to+\infty} he_n^2 = 0,\\
&\lim_{n\to+\infty} O_{3412} = \lim_{n\to+\infty} he_n^3 = 0, \qquad \lim_{n\to+\infty} O_{4123} = \lim_{n\to+\infty} he_n^4 = 0,
\end{aligned}$$

for some $e^1=e^2=e^3=e^4=0\in\Omega$. Also, the four limits of Definition 3.1 hold and the eight limits of Definitions 3.2 and 3.3 are fulfilled. Hence, the mappings H and h are compatible, reciprocally continuous and weakly reciprocally continuous.

Definition 3.5 Let the mappings $\Im_n:\Omega^4\to\Omega$ and $h:\Omega\to\Omega$ be given on a metric space (Ω,ω). The sequence $\{\Im_n\}_{n\in\mathbb{N}}$ and h are called compatible if

$$\lim_{n\to+\infty} \omega\big(h\big(O'_{1234}\big),\Im_n\big(P'_{1234}\big)\big)=0,$$

where $O'_{1234}=\Im_n(e_n^1,e_n^2,e_n^3,e_n^4)$ and $P'_{1234}=(he_n^1,he_n^2,he_n^3,he_n^4)$,

$$\lim_{n\to+\infty} \omega\big(h\big(O'_{2341}\big),\Im_n\big(P'_{2341}\big)\big)=0,$$

where $O'_{2341}=\Im_n(e_n^2,e_n^3,e_n^4,e_n^1)$ and $P'_{2341}=(he_n^2,he_n^3,he_n^4,he_n^1)$,

$$\lim_{n\to+\infty} \omega\big(h\big(O'_{3412}\big),\Im_n\big(P'_{3412}\big)\big)=0,$$

where $O'_{3412}=\Im_n(e_n^3,e_n^4,e_n^1,e_n^2)$ and $P'_{3412}=(he_n^3,he_n^4,he_n^1,he_n^2)$,

$$\lim_{n\to+\infty} \omega\big(h\big(O'_{4123}\big),\Im_n\big(P'_{4123}\big)\big)=0,$$

where $O'_{4123} = \Im_n(e_n^4, e_n^1, e_n^2, e_n^3)$ and $P'_{1234} = (he_n^4, he_n^1, he_n^2, he_n^3)$, whenever $\{e_n^1\}$, $\{e_n^2\}$, $\{e_n^3\}$ and $\{e_n^4\}$ are sequences in Ω so that

$$\lim_{n\to+\infty} O'_{1234} = \lim_{n\to+\infty} he^1_{n+1} = e^1, \qquad \lim_{n\to+\infty} O'_{2341} = \lim_{n\to+\infty} he^2_{n+1} = e^2,$$
$$\lim_{n\to+\infty} O'_{3412} = \lim_{n\to+\infty} he^3_{n+1} = e^3, \qquad \lim_{n\to+\infty} O'_{4123} = \lim_{n\to+\infty} he^4_{n+1} = e^4,$$

for some $e^1, e^2, e^3, e^4 \in \Omega$.

Definition 3.6 Let the mappings $\Im_n : \Omega^4 \to \Omega$ and $h : \Omega \to \Omega$ be given on a metric space (Ω, ω). $\{\Im_n\}_{n\in\mathbb{N}}$ and h are called weakly reciprocally continuous if

$$\lim_{n\to+\infty} h(O'_{1234}) = he^1, \qquad \lim_{n\to+\infty} h(O'_{2341}) = he^2$$
$$\lim_{n\to+\infty} h(O'_{3412}) = he^3, \qquad \lim_{n\to+\infty} h(O'_{4123}) = he^4,$$

whenever $\{e_n^1\}$, $\{e_n^2\}$, $\{e_n^3\}$, and $\{e_n^4\}$ are sequences in Ω so that

$$\lim_{n\to+\infty} O'_{1234} = \lim_{n\to+\infty} he^1_{n+1} = e^1, \qquad \lim_{n\to+\infty} O^1_{2341} = \lim_{n\to+\infty} he^2_{n+1} = e^2,$$
$$\lim_{n\to+\infty} O'_{3412} = \lim_{n\to+\infty} he^3_{n+1} = e^3, \qquad \lim_{n\to+\infty} O'_{4123} = \lim_{n\to+\infty} he^4_{n+1} = e^4,$$

for some $e^1, e^2, e^3, e^4 \in \Omega$.

Example 3.7 Let $\Omega = [0,1]$ with the distance $\omega(e^1, e^2) = |e^1 - e^2|$, $\Im_n : \Omega^4 \to \Omega$ and $h : \Omega \to \Omega$ be two mappings described as $\Im_n(e^1, e^2, e^3, e^4) = \frac{1}{2^n} - \frac{e^1e^2e^3e^4}{2}$ and $he^1 = e^1$. Define four sequences $\{e_n^1\}$, $\{e_n^2\}$, $\{e_n^3\}$, and $\{e_n^4\}$ as

$$e_n^1 = \frac{n}{n^2+1}, \qquad e_n^2 = \frac{1}{\sqrt{n^2+1}}, \qquad e_n^3 = \frac{1}{e^n+1}, \quad \text{and} \quad e_n^4 = \frac{1}{n^3+1}, \qquad \forall n \in \mathbb{N}.$$

Then, (Ω, ω) is a metric space. By routine calculations, one can write

$$\lim_{n\to+\infty} O'_{1234} = \lim_{n\to+\infty} he^1_{n+1} = e^1 = 0, \qquad \lim_{n\to+\infty} O'_{2341} = \lim_{n\to+\infty} he^2_{n+1} = e^2 = 0,$$
$$\lim_{n\to+\infty} O'_{3412} = \lim_{n\to+\infty} he^3_{n+1} = e^3 = 0, \qquad \lim_{n\to+\infty} O'_{4123} = \lim_{n\to+\infty} he^4_{n+1} = e^4 = 0,$$

for some $e^1 = e^2 = e^3 = e^4 = 0 \in \Omega$. In addition:

$$\lim_{n\to+\infty} \omega\bigl(h(O'_{1234}), \Im_n(P'_{1234})\bigr) = 0, \qquad \lim_{n\to+\infty} \omega\bigl(h(O'_{2341}), \Im_n(P'_{2341})\bigr) = 0,$$
$$\lim_{n\to+\infty} \omega\bigl(h(O'_{3412}), \Im_n(P'_{3412})\bigr) = 0, \qquad \lim_{n\to+\infty} \omega\bigl(h(O'_{4123}), \Im_n(P'_{4123})\bigr) = 0.$$

Also,

$$\lim_{n\to+\infty} h(O'_{1234}) = he^1 = 0, \qquad \lim_{n\to+\infty} h(O'_{2341}) = he^2 = 0,$$
$$\lim_{n\to+\infty} h(O'_{3412}) = he^3 = 0, \qquad \lim_{n\to+\infty} h(O'_{4123}) = he^4 = 0,$$

whenever $\{e_n^1\}$, $\{e_n^2\}$, $\{e_n^3\}$, and $\{e_n^4\}$ are sequences in Ω. Therefore, according to Definitions 3.5 and 3.6 the mappings $\{\Im_n\}_{n\in\mathbb{N}}$ and h are compatible and weakly reciprocally continuous.

Motivated by Definition 2.11, we state the definition below:

Definition 3.8 Let the mappings $\Im_n : \Omega^4 \to \Omega$ and $h : \Omega \to \Omega$ be given on a POS $(\Omega, \preceq)$. We say that $\{\Im_n\}_{n\in\mathbb{N}_0}$ have the MhMP if for any $e^1, e^2, e^3, e^4, e'^1, e'^2, e'^3, e'^4 \in \Omega$,

$$\begin{aligned} he^1 \preceq he'^1 \quad &\Rightarrow \quad \Im_n(e^1,e^2,e^3,e^4) \preceq \Im_{n+1}(e'^1,e'^2,e'^3,e'^4),\\ he^2 \succeq he'^2 \quad &\Rightarrow \quad \Im_n(e^2,e^3,e^4,e^1) \succeq \Im_{n+1}(e'^2,e'^3,e'^4,e'^1),\\ he^3 \preceq he'^3 \quad &\Rightarrow \quad \Im_n(e^3,e^4,e^1,e^2) \preceq \Im_{n+1}(e'^3,e'^4,e'^1,e'^2),\\ he^4 \succeq he'^4 \quad &\Rightarrow \quad \Im_n(e^4,e^1,e^2,e^3) \succeq \Im_{n+1}(e'^4,e'^1,e'^2,e'^3). \end{aligned}$$

Definition 3.9 For given mappings $\Im_i : \Omega^4 \to \Omega$ and $h : \Omega \to \Omega$. We say $\{\Im_i\}_{i\in\mathbb{N}_0}$ and h satisfy the condition (O) if

$$\begin{aligned} &\omega\big(\Im_i(e^1,e^2,e^3,e^4), \Im_j(r^1,r^2,r^3,r^4)\big)\\ &\quad \preceq \Phi\big[\omega\big(h(e^1), \Im_i(e^1,e^2,e^3,e^4)\big) + \omega\big(h(r^1), \Im_j(r^1,r^2,r^3,r^4)\big)\big]\\ &\qquad + \Psi\big(\omega\big(h(e^1), h(r^1)\big)\big), \end{aligned} \tag{3.1}$$

for $e^1,e^2,e^3,e^4,r^1,r^2,r^3,r^4 \in \Omega$ with $h(e^1) \preceq h(r^1)$, $h(r^2) \preceq h(e^2)$, $h(e^3) \preceq h(r^3)$, $h(r^4) \preceq h(e^4)$ or $h(e^1) \succeq h(r^1)$, $h(r^2) \succeq h(e^2)$, $h(e^3) \succeq h(r^3)$, $h(r^4) \succeq h(e^4)$, $I \neq \Phi = (\phi_{ij})$, $I \neq \Psi = (\psi_{ij}) \in M_{m,m}(\mathbb{R}^+)$, $(\Phi + \Psi)(\Phi - I)^{-1} \in ZM$.

Example 3.10 (a) $\Phi = \frac{1}{5}\begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}$ and $\Psi = \frac{1}{7}\begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}$ are matrices in ZM. We can easily prove that $(\Phi + \Psi)(\Phi - I)^{-1} \in ZM$.

(b) Suppose that $\Phi = \beta I$ and $\Psi = ((I - \beta)^3 - \beta)I$ are matrices in ZM. It is obvious that $(\Phi + \Psi)(\Phi - I)^{-1} \in ZM$ for $\beta = \frac{1}{4}, \frac{1}{5}, \frac{1}{7}, \frac{1}{8}$.

(c) $\Phi = \begin{pmatrix} \frac{1}{4} & 0\\ 0 & \frac{1}{4}\end{pmatrix}$ and $\Psi = \begin{pmatrix} 0 & \frac{1}{4}\\ \frac{1}{4} & 0\end{pmatrix}$ are matrices in ZM. We can easily show that $(\Phi + \Psi)(\Phi - I)^{-1} \in ZM$.

Definition 3.11 We say $\Im_0$ and h have a mixed quadruple transcendence point (MQTP), if there exists $e_\circ^1, e_\circ^2, e_\circ^3, e_\circ^4 \in \Omega$ so that

$$\begin{aligned} &\Im_\circ(e_\circ^1,e_\circ^2,e_\circ^3,e_\circ^4) \succeq h(e_\circ^1), \qquad \Im_\circ(e_\circ^2,e_\circ^3,e_\circ^4,e_\circ^1) \preceq h(e_\circ^2),\\ &\Im_\circ(e_\circ^3,e_\circ^4,e_\circ^1,e_\circ^2,) \succeq h(e_\circ^3), \quad \text{and} \quad \Im_\circ(e_\circ^4,e_\circ^1,e_\circ^2,e_\circ^3) \preceq h(e_\circ^4), \end{aligned} \tag{3.2}$$

provided that $\Im_\circ$ and h have a nondecreasing transcendence point in $e_\circ^1$, $e_\circ^3$ and a nonincreasing transcendence point in $e_\circ^2$, $e_\circ^4$.

Lemma 3.12 *Suppose that $\Im_i : \Omega^4 \to \Omega$ and $h : \Omega \to \Omega$ are two mappings on a partially ordered complete generalized metric space (POCGMS) $(\Omega, \omega, \preceq)$. Assume also $\{\Im_i\}_{n\in\mathbb{N}_0}$ have MhMP with $\Im_i(\Omega^4) \subseteq h(\Omega)$. If $\Im_\circ$ and h have a MQTP, then*

(i) *there are sequences* $\{e_n^1\}$, $\{e_n^2\}$, $\{e_n^3\}$, *and* $\{e_n^4\}$ *in* Ω *so that*

$$he_n^1 = \Im_{n-1}\left(e_{n-1}^1, e_{n-1}^2, e_{n-1}^3, e_{n-1}^4\right), \qquad he_n^2 = \Im_{n-1}\left(e_{n-1}^2, e_{n-1}^3, e_{n-1}^4, e_{n-1}^1\right),$$
$$he_n^3 = \Im_{n-1}\left(e_{n-1}^3, e_{n-1}^4, e_{n-1}^1, e_{n-1}^2\right), \quad \text{and} \quad he_n^4 = \Im_{n-1}\left(e_{n-1}^4, e_{n-1}^1, e_{n-1}^2, e_{n-1}^3\right).$$

(ii) $\{he_n^1\}$, $\{he_n^3\}$ *are nondecreasing sequences and* $\{he_n^2\}$, $\{he_n^4\}$ *are nonincreasing sequences.*

Proof (i) Suppose that for $e_\circ^1, e_\circ^2, e_\circ^3, e_\circ^4 \in \Omega$, the stipulation (3.2) is fulfilled. Since $\Im_\circ(\Omega^4) \subseteq h(\Omega)$, we can construct $e_1^1, e_1^2, e_1^3, e_1^4 \in \Omega$ so that

$$he_1^1 = \Im_\circ\left(e_\circ^1, e_\circ^2, e_\circ^3, e_\circ^4\right), \qquad he_1^2 = \Im_\circ\left(e_\circ^2, e_\circ^3, e_\circ^4, e_\circ^1\right),$$
$$he_1^3 = \Im_\circ\left(e_\circ^3, e_\circ^4, e_\circ^1, e_\circ^2\right), \quad \text{and} \quad he_1^4 = \Im_\circ\left(e_\circ^4, e_\circ^1, e_\circ^2, e_\circ^3\right). \tag{3.3}$$

Again, since $\Im_\circ(\Omega^4) \subseteq h(\Omega)$, there are $e_2^1, e_2^2, e_2^3, e_2^4 \in \Omega$ so that

$$he_2^1 = \Im_1\left(e_1^1, e_1^2, e_1^3, e_1^4\right), \qquad he_2^2 = \Im_1\left(e_1^2, e_1^3, e_1^4, e_1^1\right),$$
$$he_2^3 = \Im_1\left(e_1^3, e_1^4, e_1^1, e_1^2\right), \quad \text{and} \quad he_2^4 = \Im_\circ\left(e_1^4, e_1^1, e_1^2, e_1^3\right).$$

It follows with the same scenario that

$$\begin{cases} he_n^1 = \Im_{n-1}(e_{n-1}^1, e_{n-1}^2, e_{n-1}^3, e_{n-1}^4), \\ he_n^2 = \Im_{n-1}(e_{n-1}^2, e_{n-1}^3, e_{n-1}^4, e_{n-1}^1), \\ he_n^3 = \Im_{n-1}(e_{n-1}^3, e_{n-1}^4, e_{n-1}^1, e_{n-1}^2), \\ he_n^4 = \Im_{n-1}(e_{n-1}^4, e_{n-1}^1, e_{n-1}^2, e_{n-1}^3). \end{cases} \tag{3.4}$$

(ii) Now, we prove by mathematical induction that for all $n > 0$,

$$he_n^1 \preceq he_{n+1}^1, \qquad he_n^3 \preceq he_{n+1}^3, \qquad he_n^2 \succeq he_{n+1}^2, \quad \text{and} \quad he_n^4 \succeq he_{n+1}^4. \tag{3.5}$$

In the light of (3.3) and since (3.2) holds, we obtain

$$he_\circ^1 \preceq he_1^1, \qquad he_\circ^3 \preceq he_1^3, \qquad he_\circ^2 \succeq he_1^2, \quad \text{and} \quad he_\circ^4 \succeq he_1^4,$$

that is for $n = 0$, (3.5) holds. Let (3.5) be fulfilled for $n > 0$. Now, by (3.4) and (3.5), the result is completed. □

Below, our main theorem of this section is stated as follows:

Theorem 3.13 *Let the stipulations of Lemma 3.12 hold, suppose that* $\{\Im_i\}_{i\in\mathbb{N}_0}$ *and* h *are compatible, weakly reciprocally continuous, where* h *is continuous, satisfies the condition* (O), *is monotonic nondecreasing, and* $h(\Omega) \subseteq \Omega$ *is complete. Then,* $\{\Im_i\}_{i\in\mathbb{N}_0}$ *and* h *have a quadruple coincidence point* (*QCP*) *provided that* $h(\Omega)$ *is regular and* Φ, Ψ *are nonzero matrices in ZM.*

Proof Let $\{e_n^1\}$, $\{e_n^2\}$, $\{e_n^3\}$, and $\{e_n^4\}$ be sequences constructed by Lemma 3.12. It follows from (3.1) that

$$\begin{aligned}
\omega(he_n^1, he_{n+1}^1) &= \omega(\Im_{n-1}(e_{n-1}^1, e_{n-1}^2, e_{n-1}^3, e_{n-1}^4), \Im_n(e_n^1, e_n^2, e_n^3, e_n^4)) \\
&\leq \Phi[\omega(he_{n-1}^1, \Im_{n-1}(e_{n-1}^1, e_{n-1}^2, e_{n-1}^3, e_{n-1}^4)) + \omega(he_n^1, \Im_n(e_n^1, e_n^2, e_n^3, e_n^4))] \\
&\quad + \Psi(\omega(he_{n-1}^1, he_n^1)) \\
&= \Phi[\omega(he_{n-1}^1, he_n^1) + \omega(h(e_n^1), he_{n+1}^1)] + \Psi(\omega(he_{n-1}^1, he_n^1)).
\end{aligned}$$

This leads to,

$$\omega(he_n^1, he_{n+1}^1) \leq (\Phi + \Psi)(I - \Phi)^{-1}\omega(he_{n-1}^1, he_n^1). \tag{3.6}$$

Similarly, we can write

$$\omega(he_n^2, he_{n+1}^2) \leq (\Phi + \Psi)(I - \Phi)^{-1}\omega(he_{n-1}^2, he_n^2), \tag{3.7}$$

$$\omega(he_n^3, he_{n+1}^3) \leq (\Phi + \Psi)(I - \Phi)^{-1}\omega(he_{n-1}^3, he_n^3), \tag{3.8}$$

and

$$\omega(he_n^4, he_{n+1}^4) \leq (\Phi + \Psi)(I - \Phi)^{-1}\omega(he_{n-1}^4, he_n^4). \tag{3.9}$$

Adding (3.6)–(3.9), one can obtain

$$\begin{aligned}
\eta_n &= \omega(he_n^1, he_{n+1}^1) + \omega(he_n^2, he_{n+1}^2) + \omega(he_n^3, he_{n+1}^3) + \omega(he_n^4, he_{n+1}^4) \\
&= (\Phi + \Psi)(I - \Phi)^{-1}[\omega(he_{n-1}^1, he_n^1) + \omega(he_{n-1}^2, he_n^2) + \omega(he_{n-1}^3, he_n^3) + \omega(he_{n-1}^4, he_n^4)] \\
&= ((\Phi + \Psi)(I - \Phi)^{-1})\eta_{n-1}.
\end{aligned}$$

Put $(\Phi + \Psi)(I - \Phi)^{-1} = \Xi$, then for $n \in \mathbb{N}$, we have

$$\vartheta \leq \eta_n \leq \Xi\eta_{n-1} \leq \Xi^2\eta_{n-2} \leq \cdots \leq \Xi^n\eta_\circ.$$

By using the triangle inequality, for $\ell > 0$, we have

$$\begin{aligned}
&\omega(he_n^1, he_{n+\ell}^1) + \omega(he_n^2, he_{n+\ell}^2) + \omega(he_n^3, he_{n+\ell}^3) + \omega(he_n^4, he_{n+\ell}^4) \\
&\quad \leq \omega(he_n^1, he_{n+1}^1) + \omega(he_n^2, he_{n+1}^2) + \omega(he_n^3, he_{n+1}^3) + \omega(he_n^4, he_{n+1}^4) \\
&\qquad + \omega(he_{n+1}^1, he_{n+2}^1) + \omega(he_{n+1}^2, he_{n+2}^2) + \omega(he_{n+1}^3, he_{n+2}^3) + \omega(he_{n+1}^4, he_{n+2}^4) + \cdots \\
&\qquad + \omega(he_{n+\ell-1}^1, he_{n+\ell}^1) + \omega(he_{n+\ell-1}^2, he_{n+\ell}^2) + \omega(he_{n+\ell-1}^3, he_{n+\ell}^3) + \omega(he_{n+\ell-1}^4, he_{n+\ell}^4) \\
&\quad = \eta_n + \eta_{n+1} + \cdots + \eta_{n+\ell-1} \\
&\quad \leq (\Xi^n + \Xi^{n+1} + \cdots + \Xi^{n+\ell-1})\eta_\circ \\
&\quad = \Xi^n(I + \Xi + \cdots + \Xi^{\ell-1} + \cdots)\eta_\circ \\
&\quad = \Xi^n(I - \Xi)^{-1}\eta_\circ.
\end{aligned}$$

This implies that

$$\omega\big(he^1_n, he^1_{n+\ell}\big) + \omega\big(he^2_n, he^2_{n+\ell}\big) + \omega\big(he^3_n, he^3_{n+\ell}\big) + \omega\big(he^4_n, he^4_{n+\ell}\big) \\ \leq \big((\Phi+\Psi)(I-\Phi)^{-1}\big)^n\big(I - \big((\Phi+\Psi)(I-\Phi)^{-1}\big)\big)^{-1}\eta_\circ.$$

Passing the limit as $n \to +\infty$, we have

$$\lim_{n\to+\infty} \omega\big(he^1_n, he^1_{n+\ell}\big) + \omega\big(he^2_n, he^2_{n+\ell}\big) + \omega\big(he^3_n, he^3_{n+\ell}\big) + \omega\big(he^4_n, he^4_{n+\ell}\big) = 0,$$

yielding

$$\lim_{n\to+\infty} \omega\big(he^1_n, he^1_{n+\ell}\big) = \lim_{n\to+\infty} \omega\big(he^2_n, he^2_{n+\ell}\big) = \lim_{n\to+\infty} \omega\big(he^3_n, he^3_{n+\ell}\big) \\ = \lim_{n\to+\infty} \omega\big(he^4_n, he^4_{n+\ell}\big) = 0.$$

This illustrate that $\{he^1_n\}$, $\{he^2_n\}$, $\{he^3_n\}$, and $\{he^4_n\}$ are Cauchy sequences in Ω. The completeness of $h(\Omega)$ leads to $(e'^1, e'^2, e'^3, e'^4) \in \Omega^4$, so that

$$\lim_{n\to+\infty} he^1_n = he'^1 = e^1, \qquad \lim_{n\to+\infty} he^2_n = he'^2 = e^2 \\ \lim_{n\to+\infty} he^3_n = he'^3 = e^3, \quad \text{and} \quad \lim_{n\to+\infty} he^4_n = he'^4 = e^4.$$

By construction, we obtain

$$\lim_{n\to+\infty} he^1_{n+1} = \lim_{n\to+\infty} \Im_n\big(e^1_n, e^2_n, e^3_n, e^4_n\big), \qquad \lim_{n\to+\infty} he^2_{n+1} = \lim_{n\to+\infty} \Im_n\big(e^2_n, e^3_n, e^4_n, e^1_n\big), \\ \lim_{n\to+\infty} he^3_{n+1} = \lim_{n\to+\infty} \Im_n\big(e^3_n, e^4_n, e^1_n, e^2_n\big), \quad \text{and} \quad \lim_{n\to+\infty} he^4_{n+1} = \lim_{n\to+\infty} \Im_n\big(e^4_n, e^1_n, e^2_n, e^3_n\big).$$

The weakly reciprocally continuous and compatibility of $\{\Im_i\}_{i\in\mathbb{N}_0}$ and $h(\Omega)$ leads to

$$\lim_{n\to+\infty} \Im_n\big(he^1_n, he^2_n, he^3_n, he^4_n\big) = he^1, \qquad \lim_{n\to+\infty} \Im_n\big(he^2_n, he^3_n, he^4_n, he^1_n\big) = he^2, \\ \lim_{n\to+\infty} \Im_n\big(he^3_n, he^4_n, he^1_n, he^2_n\big) = he^3, \quad \text{and} \quad \lim_{n\to+\infty} \Im_n\big(he^4_n, he^1_n, he^2_n, he^3_n\big) = he^4.$$

Since $\{he^1_n\}$, $\{he^3_n\}$ are nondecreasing and $\{he^2_n\}$, $\{he^4_n\}$ are nonincreasing, using the regularity of Ω, one can obtain for all $n \geq 0$, $he^1_n \preceq e^1$, $e^2 \preceq he^2_n$, $he^3_n \preceq e^3$, and $e^4 \preceq he^4_n$. Hence, by (3.1), we can write

$$\omega\big(\Im_i\big(e^1, e^2, e^3, e^4\big), \Im_n\big(he^1_n, he^2_n, he^3_n, he^4_n\big)\big) \\ \leq \Phi\big[\omega\big(he^1, \Im_i\big(e^1, e^2, e^3, e^4\big)\big) + \omega\big(he^1_n, \Im_n\big(he^1_n, he^2_n, he^3_n, he^4_n\big)\big)\big] + \Psi\big(\omega\big(he^1, he^1_n\big)\big).$$

Passing the limit as $n \to \infty$, we obtain $\Im_i(e^1, e^2, e^3, e^4) = he^1$. Similarly, we can conclude that $\Im_i(e^2, e^3, e^4, e^1) = he^2$, $\Im_i(e^3, e^4, e^1, e^2) = he^3$, and $\Im_i(e^4, e^1, e^2, e^3) = he^4$. Hence, (e^1, e^2, e^3, e^4) is a QCP of $\{\Im_i\}_{i\in\mathbb{N}_0}$ and h. □

The result below is given from Theorem 3.13 by taking $h = \mathrm{Id}$, where Id is the identity map.

Corollary 3.14 *Let* $(\Omega,\omega,\preceq)$ *be a POCGMS and* $\{\Im_i\}_{i\in\mathbb{N}\cup\{0\}}:\Omega^4\to\Omega$ *be a mixed-monotone sequence of mappings, where* $\{\Im_i\}_{i\in\mathbb{N}\cup\{0\}}$ *and* $\mathrm{Id}:\Omega\to\Omega$ *satisfy the condition* (*O*). *Also,* $\Im_0$ *and* Id *have a MQTP and* $\mathrm{Id}(\Omega)$ *is regular. Then, there is* $(e^1,e^2,e^3,e^4)\in\Omega^4$ *so that* $\Im_i(e^1,e^2,e^3,e^4)=e^1$, $\Im_i(e^2,e^3,e^4,e^1)=e^2$, $\Im_i(e^3,e^4,e^1,e^2)=e^3$, *and* $\Im_i(e^4,e^1,e^2,e^3)=e^4$ *for* $i\in\mathbb{N}_0$.

If we omit some conditions of Corollary 3.14 and take Φ as a zero matrix and expand the distance $\omega((e^1),(r^1))$, we obtain the important result below.

Corollary 3.15 *Let* $(\Omega,\omega,\preceq)$ *be a POCGMS and* $\urcorner:\Omega^4\to\Omega$ *be a mixed-monotone mapping, such that*

$$\omega\big(\Im_i(e^1,e^2,e^3,e^4),\Im_j(r^1,r^2,r^3,r^4)\big)\le\Psi\big(\omega\big((e^1,e^2,e^3,e^4),(r^1,r^2,r^3,r^4)\big)\big).$$

If $\urcorner$ *has a MQTP, then* $\urcorner$ *has a QFP in* Ω.

Definition 3.16 We say that (e^1,e^2,e^3,e^4) is a quadruple comparable (QC) with $(\zeta^1,\zeta^2,\zeta^3,\zeta^4)$ iff

$$\begin{aligned}
&e^1\succeq\zeta^1,\quad e^2\preceq\zeta^2,\quad e^3\succeq\zeta^3,\quad e^4\preceq\zeta^4\quad\text{or}\\
&e^1\preceq\zeta^1,\quad e^2\succeq\zeta^2,\quad e^3\preceq\zeta^3,\quad e^4\succeq\zeta^4\quad\text{or}\\
&e^1\succeq\zeta^2,\quad e^2\preceq\zeta^3,\quad e^3\succeq\zeta^4\quad e^4\preceq\zeta^1\quad\text{or}\\
&e^1\preceq\zeta^2,\quad e^2\succeq\zeta^3,\quad e^3\preceq\zeta^4,\quad e^4\succeq\zeta^1\quad\text{or}\\
&e^1\succeq\zeta^3,\quad e^2\preceq\zeta^4,\quad e^3\succeq\zeta^1\quad e^4\preceq\zeta^2\quad\text{or}\\
&e^1\preceq\zeta^3,\quad e^2\succeq\zeta^4,\quad e^3\preceq\zeta^1,\quad e^4\succeq\zeta^2\quad\text{or}\\
&e^1\succeq\zeta^4,\quad e^2\preceq\zeta^1,\quad e^3\succeq\zeta^2\quad e^4\preceq\zeta^3\quad\text{or}\\
&e^1\preceq\zeta^4,\quad e^2\succeq\zeta^1,\quad e^3\preceq\zeta^2,\quad e^4\succeq\zeta^3.
\end{aligned}$$

If we replace (e^1,e^2,e^3,e^4) and $(\zeta^1,\zeta^2,\zeta^3,\zeta^4)$ with (he^1,he^2,he^3,he^4) and $(h\zeta^1,h\zeta^2,h\zeta^3,h\zeta^4)$ in Definition 3.16, we call (e^1,e^2,e^3,e^4) a QC with $(\zeta^1,\zeta^2,\zeta^3,\zeta^4)$ with respect to (w.r.t.) h.

Theorem 3.17 *Let* $(\Omega,\omega,\preceq)$ *be a POCGMS. Assume that* $\{\Im_i\}_{i\in\mathbb{N}_0}:\Omega^4\to\Omega$ *is a sequence of mappings and* $h:\Omega\to\Omega$, *where* $\{\Im_i\}_{i\in\mathbb{N}_0}$ *and h satisfy the condition* (*O*) *and have QCPs with quadruple comparable* (*w.r.t.*) *h. Then,* $\{\Im_i\}_{i\in\mathbb{N}_0}$ *and h have a unique common QFP.*

Proof It follows from Theorem 3.13 that the set of QCPs is nonempty. Now, begin with proving that (e^1,e^2,e^3,e^4) and $(\zeta^1,\zeta^2,\zeta^3,\zeta^4)$ are QCPs, with meaning, if

$$\begin{aligned}
&he^1=\Im_i\big(e^1,e^2,e^3,e^4\big),\qquad he^2=\Im_i\big(e^2,e^3,e^4,e^1\big),\\
&he^3=\Im_i\big(e^3,e^4,e^1,e^2\big),\qquad he^4=\Im_i\big(e^4,e^1,e^2,e^3\big),\\
&h\zeta^1=\Im_i\big(\zeta^1,\zeta^2,\zeta^3,\zeta^4\big),\qquad h\zeta^2=\Im_i\big(\zeta^2,\zeta^3,\zeta^4,\zeta^1\big),\\
&h\zeta^3=\Im_i\big(\zeta^3,\zeta^4,\zeta^1,\zeta^2\big),\qquad h\zeta^4=\Im_i\big(\zeta^4,\zeta^1,\zeta^2,\zeta^3\big),
\end{aligned}$$

then $he^1 = h\zeta^1$, $he^2 = h\zeta^2$, $he^3 = h\zeta^3$, and $he^4 = h\zeta^4$. Since the set of QCPs is a QC, then by (3.1), we obtain

$$\begin{aligned}\omega(he^1, h\zeta^1) &= \omega(\Im_i(e^1,e^2,e^3,e^4), \Im_i(\zeta^1,\zeta^2,\zeta^3,\zeta^4)) \\ &\leq \Phi[\omega(he^1, \Im_i(e^1,e^2,e^3,e^4)) + \omega(h\zeta^1, \Im_i(\zeta^1,\zeta^2,\zeta^3,\zeta^4))] \\ &\quad + \Psi(\omega(he^1, h\zeta^1)).\end{aligned}$$

As $I \neq \Psi \in ZM$, $\omega(he^1, h\zeta^1) = \vartheta$, hence $he^1 = h\zeta^1$. Similarly, we can prove that $he^2 = h\zeta^2$, $he^3 = h\zeta^3$, and $he^4 = h\zeta^4$. Hence, $he^1 = he^2 = he^3 = he^4 = h\zeta^1 = h\zeta^2 = h\zeta^3 = h\zeta^4$. This tells us (he^1, he^2, he^3, he^4) is a unique QCP of $\{\Im_i\}_{i\in\mathbb{N}_0}$ and h. Since $\{\Im_i\}_{i\in\mathbb{N}_0}$ and h are weakly compatible and since two compatible mappings commute at their coincidence points, thus, clearly (e^1, e^2, e^3, e^4) is a unique common QFP of $\{\Im_i\}_{i\in\mathbb{N}_0}$ and h. □

Example 3.18 Suppose that $\Omega = [0,\infty)$. Define

$$\omega(e^1, e^2) = \begin{pmatrix} |e^1 - e^2| \\ |e^1 - e^2| \end{pmatrix},$$

and

$$\Phi = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix}.$$

It is clear that the pair (Ω, ω) is a POCGMS and $\Phi, \Psi \in ZM$. Define two mappings $\Im_i : \Omega^4 \to \Omega$ and $h : \Omega \to \Omega$ with

$$\Im_i(e^1,e^2,e^3,e^4) = \left(\frac{e^1+e^2+e^3+e^4}{4^i}\right) \quad \text{and} \quad h(e^1) = 16e^1,$$

by mathematical induction, we can fulfill the condition (3.1) for all $e^1, e^2, e^3, e^4 \in \Omega$, that is, we find that the first side has its greatest value when $i = 1$, and $j \to \infty$ as below

$$\begin{aligned}&\begin{pmatrix} |\frac{e^1+e^2+e^3+e^4}{4} - \frac{\nu^1+\nu^2+\nu^3+\nu^4}{4^j}| \\ |\frac{e^1+e^2+e^3+e^4}{4} - \frac{\nu^1+\nu^2+\nu^3+\nu^4}{4^j}| \end{pmatrix} \\ &\quad \leq \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} |16e^1 - \frac{e^1+e^2+e^3+e^4}{4}| + |16\nu^1 - \frac{\nu^1+\nu^2+\nu^3+\nu^4}{4^j}| \\ |16e^1 - \frac{e^1+e^2+e^3+e^4}{4}| + |16\nu^1 - \frac{\nu^1+\nu^2+\nu^3+\nu^4}{4^j}| \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix} \begin{pmatrix} |16(e^1 - \nu^1)| \\ |16(e^1 - \nu^1)| \end{pmatrix}.\end{aligned}$$

Now, for $j = j + 1$, we obtain

$$\begin{aligned}\kappa &= \begin{pmatrix} |\frac{e^1+e^2+e^3+e^4}{4} - \frac{1}{4}\frac{\nu^1+\nu^2+\nu^3+\nu^4}{4^j}| \\ |\frac{e^1+e^2+e^3+e^4}{4} - \frac{1}{4}\frac{\nu^1+\nu^2+\nu^3+\nu^4}{4^j}| \end{pmatrix} \\ &\leq \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} |16e^1 - \frac{e^1+e^2+e^3+e^4}{4}| + |16\nu^1 - \frac{1}{4}\frac{\nu^1+\nu^2+\nu^3+\nu^4}{4^j}| \\ |16e^1 - \frac{e^1+e^2+e^3+e^4}{4}| + |16\nu^1 - \frac{1}{4}\frac{\nu^1+\nu^2+\nu^3+\nu^4}{4^j}| \end{pmatrix} + \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} |e^1 - \nu^1| \\ |e^1 - \nu^1| \end{pmatrix} \\ &= \tau.\end{aligned}$$

Hence,

$$\begin{aligned}\kappa &\leq \frac{1}{4}\begin{pmatrix}|\frac{e^1+e^2+e^3+e^4}{4}-\frac{\frac{\nu^1+\nu^2+\nu^3+\nu^4}{4^j}}{}|\\|\frac{e^1+e^2+e^3+e^4}{4}-\frac{\nu^1+\nu^2+\nu^3+\nu^4}{4^j}|\end{pmatrix}+\frac{3}{4}\begin{pmatrix}|\frac{e^1+e^2+e^3+e^4}{4}|\\|\frac{e^1+e^2+e^3+e^4}{4}|\end{pmatrix}\\ &\leq \frac{1}{4}\begin{pmatrix}\frac{1}{4}&0\\0&\frac{1}{4}\end{pmatrix}\begin{pmatrix}|16e^1-\frac{e^1+e^2+e^3+e^4}{4}|+|16\nu^1-\frac{1}{4}\frac{\nu^1+\nu^2+\nu^3+\nu^4}{4^j}|\\|16e^1-\frac{e^1+e^2+e^3+e^4}{4}|+|16\nu^1-\frac{1}{4}\frac{\nu^1+\nu^2+\nu^3+\nu^4}{4^j}|\end{pmatrix}\\ &\quad+\frac{1}{4}\begin{pmatrix}0&4\\4&0\end{pmatrix}\begin{pmatrix}|e^1-\nu^1|\\|e^1-\nu^1|\end{pmatrix}+\frac{3}{4}\begin{pmatrix}|\frac{e^1+e^2+e^3+e^4}{4}|\\|\frac{e^1+e^2+e^3+e^4}{4}|\end{pmatrix}\\ &\leq \tau.\end{aligned}$$

Therefore, all stipulations of Theorem 3.13 are fulfilled, and $(0,0,0,0)$ is a QCP of $\{\Im_i\}_{i\in\mathbb{N}_0}$ and h, also, this point is a unique common QFP of the same mappings according to Theorem 3.17.

The definition below is very important for the applications.

Definition 3.19 Assume that $\Phi=(\phi_{ij})$ and $\Psi=(\psi_{ij})$ are two matrices in ZM. Then,

$$\begin{aligned}&\Phi\leq\Psi \quad \text{if and only if} \quad \phi_{ij}\leq\psi_{ij}, 1\leq i,j\leq m,\\ &\max\{\Phi,\Psi\}=\Theta=(\theta_{ij}),\end{aligned}$$

where $\theta_{ij}=\max\{\phi_{ij},\psi_{ij}\}$. It is obvious that, if $\Phi\leq\Psi$ then $\max\{\Phi,\Psi\}=\Psi$.

4 Application (I)

By using the results of Sect. 3, in this part, the existence and uniqueness of the solutions of the integral equations system are extracted. Consider the system below:

$$\begin{cases} e^1(\delta)=\int_0^{\varrho}(w(\delta,\sigma,e^1(\sigma))+x(\delta,\sigma,e^2(\sigma))+y(\delta,\sigma,e^3(\sigma))+z(\delta,\sigma,e^4(\sigma)))\,d\sigma\\ \qquad +b(\delta),\\ e^2(\delta)=\int_0^{\varrho}(w(\delta,\sigma,e^2(\sigma))+x(\delta,\sigma,e^3(\sigma))+y(\delta,\sigma,e^4(\sigma))+z(\delta,\sigma,e^1(\sigma)))\,d\sigma\\ \qquad +b(\delta),\\ e^3(\delta)=\int_0^{\varrho}(w(\delta,\sigma,e^3(\sigma))+x(\delta,\sigma,e^4(\sigma))+y(\delta,\sigma,e^1(\sigma))+z(\delta,\sigma,e^2(\sigma)))\,d\sigma\\ \qquad +b(\delta),\\ e^4(\delta)=\int_0^{\varrho}(w(\delta,\sigma,e^4(\sigma))+x(\delta,\sigma,e^1(\sigma))+y(\delta,\sigma,e^2(\sigma))+z(\delta,\sigma,e^3(\sigma)))\,d\sigma\\ \qquad +b(\delta),\end{cases} \tag{4.1}$$

for all $\delta,\sigma\in[0,\varrho]$, for some $\varrho>0$.

As usual, we consider $\Omega=C([0,\varrho],\mathbb{R})$ is a continuous real function created on $[0,\varrho]$ and equipped with a metric

$$\omega(e^1,e^2)=\begin{pmatrix}\max_{0\leq\delta\leq\varrho}|e^1(\delta)-e^2(\delta)|\\ \max_{0\leq\delta\leq\varrho}|e^1(\delta)-e^2(\delta)|\end{pmatrix}.$$

Define a partial order “⪯]] on Ω as follows:

$$\text{for } e^1, e^2 \in \Omega,\ \text{for any } \delta \in [0,\varrho],\ e^1 \preceq e^2 \text{ iff } e^1(\delta) \preceq e^2(\delta).$$

Thus, $(\Omega, \omega, \preceq)$ is a POCGMS.

Now, system (4.1) will be taken under the hypotheses below:

$(\dagger_i)$ the functions $w, x, y, z : [0,\varrho] \times [0,\varrho] \times \mathbb{R} \to \mathbb{R}^2$ and $b : [0,\varrho] \to \mathbb{R}$ are continuous;

$(\dagger_{ii})$ for all $e^1, e^2 \in \Omega$, there is $\mu : [0,\varrho] \to M_{2\times 2}([0,\varrho])$, so that

$$\begin{cases} 0 \le |w(\delta,\sigma,e^1(\sigma)) - w(\delta,\sigma,e^2(\sigma))| \le \mu_1(\delta)\omega(e^1,e^2), \\ 0 \le |x(\delta,\sigma,e^2(\sigma)) - x(\delta,\sigma,e^1(\sigma))| \le \mu_2(\delta)\omega(e^1,e^2), \\ 0 \le |y(\delta,\sigma,e^1(\sigma)) - y(\delta,\sigma,e^2(\sigma))| \le \mu_3(\delta)\omega(e^1,e^2), \\ 0 \le |z(\delta,\sigma,e^2(\sigma)) - z(\delta,\sigma,e^1(\sigma))| \le \mu_4(\delta)\omega(e^1,e^2), \end{cases} \tag{4.2}$$

for all $\delta, \sigma \in [0,\varrho]$ with $\mu(\delta) \le \Phi = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$ and $\mu(\delta) \le \Psi = \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix}$. This holds because $\Phi, \Psi \in ZM$;

$(\dagger_{iii})$ consider $\mu_1(\delta) + \mu_2(\delta) + \mu_3(\delta) + \mu_4(\delta) < 1$ and

$$\mu(\delta) = \max\{\mu_1(\delta), \mu_2(\delta), \mu_3(\delta), \mu_4(\delta)\};$$

$(\dagger_{iv})$ there exist continuous functions $\lambda, \rho, \upsilon, \varkappa : [0,\varrho] \to \mathbb{R}$ so that

$$\begin{cases} \lambda \le \int_0^\varrho (w(\delta,\sigma,\lambda(\sigma)) + x(\delta,\sigma,\rho(\sigma)) + y(\delta,\sigma,\upsilon(\sigma)) + z(\delta,\sigma,\varkappa(\sigma)))\,d\sigma + b(\delta), \\ \rho \ge \int_0^\varrho (w(\delta,\sigma,\rho(\sigma)) + x(\delta,\sigma,\upsilon(\sigma)) + y(\delta,\sigma,\varkappa(\sigma)) + z(\delta,\sigma,\lambda(\sigma)))\,d\sigma + b(\delta), \\ \upsilon \le \int_0^\varrho (w(\delta,\sigma,\upsilon(\sigma)) + x(\delta,\sigma,\varkappa(\sigma)) + y(\delta,\sigma,\lambda(\sigma)) + z(\delta,\sigma,\rho(\sigma)))\,d\sigma + b(\delta), \\ \varkappa \ge \int_0^\varrho (w(\delta,\sigma,\varkappa(\sigma)) + x(\delta,\sigma,\lambda(\sigma)) + y(\delta,\sigma,\rho(\sigma)) + z(\delta,\sigma,\upsilon(\sigma)))\,d\sigma + b(\delta). \end{cases}$$

Theorem 4.1 *The problem* (4.1) *has a unique solution in* Ω *if the four assumptions* $(\dagger_i)$ – $(\dagger_{vi})$ *are fulfilled.*

Proof Define an operator $\Im_i : \Omega^4 \to \Omega$ by

$$\begin{aligned} &\Im_i(e^1,e^2,e^3,e^4) \\ &\quad = b(\delta) + \int_0^\varrho \left(w(\delta,\sigma,e^1(\sigma)) + x(\delta,\sigma,e^2(\sigma)) + y(\delta,\sigma,e^3(\sigma)) + z(\delta,\sigma,e^4(\sigma))\right) d\sigma, \end{aligned}$$

for any $e^1, e^2, e^3, e^4 \in \Omega$ and $\delta, \sigma \in [0,\varrho]$.

To obtain the required result, we will show that the operator $\{\Im_i\}_{i\in\mathbb{N}}$ fulfills the stipulations of Corollary 3.14.

Initially, we discuss the MMP of $\{\Im_i\}_{i\in\mathbb{N}}$. Suppose that $e^1, \nu^1 \in \Omega$ with $e^1 \preceq \nu^1$, then for $\delta, \sigma \in [0,\varrho]$, we have

$$\Im_i(e^1,e^2,e^3,e^4)(\delta) - \Im_i(\nu^1,e^2,e^3,e^4)(\delta) = \int_0^\varrho \left(w(\delta,\sigma,e^1(\sigma)) - w(\delta,\sigma,\nu^1(\sigma))\right) d\sigma.$$

For any $\delta \in [0,\varrho]$, given that $e^1(\delta) \le \nu^1(\delta)$ and according to our hypothesis (4.2), we can write

$$\Im_i(\nu^1,e^2,e^3,e^4)(\delta) - \Im_i(e^1,e^2,e^3,e^4)(\delta) \le 0,$$

hence $\Im_i(e^1,e^2,e^3,e^4)(\delta) \ge \Im_i(\nu^1,e^2,e^3,e^4)(\delta)$. Let $e^2,\nu^2 \in \Omega$ with $e^2 \le \nu^2$, then we obtain

$$\Im_i(e^1,\nu^2,e^3,e^4)(\delta) - \Im_i(e^1,e^2,e^3,e^4)(\delta) = \int_0^\varrho \bigl(g(\delta,\sigma,\nu^2(\sigma)) - g(\delta,\sigma,e^2(\sigma))\bigr)\,d\sigma.$$

Given that $e^2(\delta) \le \nu^2(\delta)$ for any $\delta \in [0,\varrho]$, and according to our hypothesis (4.2), we obtain

$$\Im_i(e^1,\nu^2,e^3,e^4)(\delta) - \Im_i(e^1,e^2,e^3,e^4)(\delta) \ge 0,$$

that is, $\Im_i(e^1,e^2,e^3,e^4)(\delta) \le \Im_i(e^1,\nu^2,e^3,e^4)(\delta)$. Similarly, we can conclude that $\Im_i(e^1,e^2,e^3,e^4)(\delta) \ge \Im_i(e^1,e^2,\nu^3,e^4)(\delta)$ and $\Im_i(e^1,e^2,e^3,e^4)(\delta) \le \Im_i(e^1,e^2,e^3,\nu^4)(\delta)$. Therefore, $\{\Im_i\}_{i\in\mathbb{N}}$ has a MMP.

Next, we shall estimate $\omega(\Im_i(e^1,e^2,e^3,e^4),\Im_j(\nu^1,\nu^2,\nu^3,\nu^4))$ for $e^1 \preceq \nu^1$, $\nu^2 \preceq e^2$, $e^3 \preceq \nu^3$, $\nu^4 \preceq e^4$ or $e^1 \succeq \nu^1$, $\nu^2 \succeq e^2$, $e^3 \succeq \nu^3$, $\nu^4 \succeq e^4$, and with $\{\Im_i\}_{i\in\mathbb{N}}$ having a MMP, we have

$$\begin{aligned}
&\omega\bigl(\Im_i(e^1,e^2,e^3,e^4),\Im_j(\nu^1,\nu^2,\nu^3,\nu^4)\bigr)\\
&\quad= \begin{pmatrix} \max_{0\le\delta\le\varrho} |\Im_i(e^1,e^2,e^3,e^4)(\delta) - \Im_j(\nu^1,\nu^2,\nu^3,\nu^4)(\delta)| \\ \max_{0\le\delta\le\varrho} |\Im_i(e^1,e^2,e^3,e^4)(\delta) - \Im_j(\nu^1,\nu^2,\nu^3,\nu^4)(\delta)| \end{pmatrix}.
\end{aligned}$$

Now, take into account (4.2), for all $\delta \in [0,\varrho]$, we can write

$$\begin{aligned}
&\bigl|\Im_i(e^1,e^2,e^3,e^4)(\delta) - \Im_j(\nu^1,\nu^2,\nu^3,\nu^4)(\delta)\bigr|\\
&\quad= \biggl|\int_0^\varrho \bigl(w(\delta,\sigma,e^1(\sigma)) + x(\delta,\sigma,e^2(\sigma)) + y(\delta,\sigma,e^3(\sigma)) + z(\delta,\sigma,e^4(\sigma))\bigr)\,d\sigma\\
&\qquad - \int_0^\varrho \bigl(w(\delta,\sigma,\nu^1(\sigma)) + x(\delta,\sigma,\nu^2(\sigma)) + y(\delta,\sigma,\nu^3(\sigma)) + z(\delta,\sigma,\nu^4(\sigma))\bigr)\,d\sigma\biggr|\\
&\quad\le \int_0^\varrho \bigl|w(\delta,\sigma,e^1(\sigma)) - w(\delta,\sigma,\nu^1(\sigma))\bigr|\,d\sigma + \int_0^\varrho \bigl|x(\delta,\sigma,e^2(\sigma)) - x(\delta,\sigma,\nu^2(\sigma))\bigr|\,d\sigma\\
&\qquad + \int_0^\varrho \bigl|y(\delta,\sigma,e^3(\sigma)) - y(\delta,\sigma,\nu^3(\sigma))\bigr|\,d\sigma + \int_0^\varrho \bigl|z(\delta,\sigma,e^4(\sigma)) - z(\delta,\sigma,\nu^4(\sigma))\bigr|\,d\sigma\\
&\quad\le \mu_1(\delta)\omega(e^1,\nu^1) + \mu_2(\delta)\omega(e^2,\nu^2) + \mu_3(\delta)\omega(e^3,\nu^3) + \mu_4(\delta)\omega(e^4,\nu^4)\\
&\quad\le \mu(\delta)\bigl(\omega(e^1,\nu^1) + \omega(e^2,\nu^2) + \omega(e^3,\nu^3) + \omega(e^4,\nu^4)\bigr)\\
&\quad\le \Psi\bigl(\omega(e^1,\nu^1) + \omega(e^2,\nu^2) + \omega(e^3,\nu^3) + \omega(e^4,\nu^4)\bigr).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\omega\bigl(\Im_i(e^1,e^2,e^3,e^4),\Im_j(\nu^1,\nu^2,\nu^3,\nu^4)\bigr)\\
&\quad\le \Psi\begin{pmatrix} \omega(e^1,\nu^1) + \omega(e^2,\nu^2) + \omega(e^3,\nu^3) + \omega(e^4,\nu^4) \\ \omega(e^1,\nu^1) + \omega(e^2,\nu^2) + \omega(e^3,\nu^3) + \omega(e^4,\nu^4) \end{pmatrix}
\end{aligned}$$

$$\leq \Phi\big[\omega\big(e^1, \Im_i\big(e^1, e^2, e^3, e^4\big)\big) + \omega\big(\nu^1, \Im_j\big(\nu^1, \nu^2, \nu^3, \nu^4\big)\big)\big] + \Psi\omega\big(e^1, \nu^1\big).$$

Consider λ, ρ, υ, $\varkappa$ are the same as hypothesis $(\dagger_{iv})$, then, we obtain

$$\lambda \leq \Im_i(\lambda, \rho, \upsilon, \varkappa), \qquad \rho \geq \Im_i(\rho, \upsilon, \varkappa, \lambda),$$
$$\upsilon \leq \Im_i(\upsilon, \varkappa, \lambda, \rho), \quad \text{and} \quad \varkappa \geq \Im_i(\varkappa, \lambda, \rho, \upsilon).$$

Put $e_0^1 = \lambda$, $e_0^2 = \rho$, $e_0^3 = \upsilon$, and $e_0^4 = \varkappa$, then all requirement stipulations of Corollary 3.14 hold. Hence, the point (e^1, e^2, e^3, e^4) is a QFP of $\{\Im_i\}_{i\in\mathbb{N}}$, which is a unique solution of the problem (4.1). □

5 Application (II)

This part is a generalization of Sect. 4, where we introduce the sequence of the integral equations system as the following:

$$\begin{cases} e^1(\delta) = \int_0^{\varrho} (w_i(\delta, \sigma, e^1(\sigma)) + x_i(\delta, \sigma, e^2(\sigma)) + y_i(\delta, \sigma, e^3(\sigma)) + z_i(\delta, \sigma, e^4(\sigma)))\, d\sigma \\ \qquad + b(\delta), \\ e^2(\delta) = \int_0^{\varrho} (w_i(\delta, \sigma, e^2(\sigma)) + x_i(\delta, \sigma, e^3(\sigma)) + y_i(\delta, \sigma, e^4(\sigma)) + z_i(\delta, \sigma, e^1(\sigma)))\, d\sigma \\ \qquad + b(\delta), \\ e^3(\delta) = \int_0^{\varrho} (w_i(\delta, \sigma, e^3(\sigma)) + x_i(\delta, \sigma, e^4(\sigma)) + y_i(\delta, \sigma, e^1(\sigma)) + z_i(\delta, \sigma, e^2(\sigma)))\, d\sigma \\ \qquad + b(\delta), \\ e^4(\delta) = \int_0^{\varrho} (w_i(\delta, \sigma, e^4(\sigma)) + x_i(\delta, \sigma, e^1(\sigma)) + y_i(\delta, \sigma, e^2(\sigma)) + z_i(\delta, \sigma, e^3(\sigma)))\, d\sigma \\ \qquad + b(\delta), \end{cases} \tag{5.1}$$

for all $\delta, \sigma \in [0, \varrho]$, for some $\varrho > 0$. Similar to Theorem 4.1 the problem (5.1) has a simultaneous solution with the stipulations below by considering all requirements of Sect. 3 in terms of the definition of the partial order "$\preceq$" on Ω, the distance "ω" and a partially ordered complete generalized metric space $(\Omega, \omega, \preceq)$.

System (5.1) will be considered under the following assumptions:

$(\ddagger_i)$ the functions $w_i, x_i, y_i, z_i : [0, \varrho] \times [0, \varrho] \times \mathbb{R} \to \mathbb{R}^2$ and $b : [0, \varrho] \to \mathbb{R}$ are continuous;

$(\ddagger_{ii})$ for all $e^1, e^2 \in \Omega$, there is $\mu : [0, \varrho] \to M_{2\times 2}([0, \varrho])$, so that

$$\begin{cases} 0 \leq |w_i(\delta, \sigma, e^1(\sigma)) - w_i(\delta, \sigma, e^2(\sigma))| \leq \mu_1(\delta)\omega(e^1, e^2), \\ 0 \leq |x_i(\delta, \sigma, e^2(\sigma)) - x_i(\delta, \sigma, e^1(\sigma))| \leq \mu_2(\delta)\omega(e^1, e^2), \\ 0 \leq |y_i(\delta, \sigma, e^1(\sigma)) - y_i(\delta, \sigma, e^2(\sigma))| \leq \mu_3(\delta)\omega(e^1, e^2), \\ 0 \leq |z_i(\delta, \sigma, e^2(\sigma)) - z_i(\delta, \sigma, e^1(\sigma))| \leq \mu_4(\delta)\omega(e^1, e^2), \end{cases}$$

for all $\delta, \sigma \in [0, \varrho]$ with $\mu(\delta) \leq \Phi = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$ and $\mu(\delta) \leq \Psi = \begin{pmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & 0 \end{pmatrix}$. This holds because $\Phi, \Psi \in ZM$;

$(\ddagger_{iii})$ consider $\mu_1(\delta) + \mu_2(\delta) + \mu_3(\delta) + \mu_4(\delta) < 1$ and

$$\mu(\delta) = \max\big\{\mu_1(\delta), \mu_2(\delta), \mu_3(\delta), \mu_4(\delta)\big\};$$

($\ddagger_{iv}$) there is continuous functions $\lambda, \rho, \upsilon, \varkappa : [0, \varrho] \to \mathbb{R}$ so that

$$\begin{cases} \lambda \leq \int_0^{\varrho} (w_i(\delta, \sigma, \lambda(\sigma)) + x_i(\delta, \sigma, \rho(\sigma)) + y_i(\delta, \sigma, \upsilon(\sigma)) + z_i(\delta, \sigma, \varkappa(\sigma)))\, d\sigma + b(\delta), \\ \rho \geq \int_0^{\varrho} (w_i(\delta, \sigma, \rho(\sigma)) + x_i(\delta, \sigma, \upsilon(\sigma)) + y_i(\delta, \sigma, \varkappa(\sigma)) + z_i(\delta, \sigma, \lambda(\sigma)))\, d\sigma + b(\delta), \\ \upsilon \leq \int_0^{\varrho} (w_i(\delta, \sigma, \upsilon(\sigma)) + x_i(\delta, \sigma, \varkappa(\sigma)) + y_i(\delta, \sigma, \lambda(\sigma)) + z_i(\delta, \sigma, \rho(\sigma)))\, d\sigma + b(\delta), \\ \varkappa \geq \int_0^{\varrho} (w_i(\delta, \sigma, \varkappa(\sigma)) + x_i(\delta, \sigma, \lambda(\sigma)) + y_i(\delta, \sigma, \rho(\sigma)) + z_i(\delta, \sigma, \upsilon(\sigma)))\, d\sigma + b(\delta). \end{cases}$$

6 Application (III)

In this part, a unique stationary distribution for the Markov process is discussed. Suppose that $\mathbb{R}_+^n = \{(e^1, e^2, \dots, e^n) = e : e_i \geq 0, i \geq 1\}$ and

$$\Lambda_{n-1}^4 = \Bigg\{ \wp = (e^1, e^2, e^3, e^4) \in \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n \times \mathbb{R}_+^n : \\ \sum_{i=1}^n \wp_i = \sum_{i=1}^n (e_i^1 + e_i^2 + e_i^3 + e_i^4) = 1 \Bigg\},$$

refer to a $4(n-1)$-dimensional unit simplex and $\wp \in \Lambda_{n-1}^4$ could be considered a possibility over $4n$ possible statuses. Here, the Markov process is a stochastic process that verifies that $4n$ statuses are realized in each period $\hbar = 1, 2, \dots$ with the probability contingent on the current achieved status. Assume that a_{ij} refer to the probability contingent that status i is achieved in the subsequent period beginning in status j. Hence, in period $\hbar$ and $\hbar + 1$, the prior probability vector $\wp^{\hbar}$ and the posterior probability $\wp^{\hbar+1}$ given by $\wp_i^{\hbar+1} = \sum_j a_{ij} \wp_j^{\hbar}$ for each $j \geq 1$. To put this in a matrix form, let $\wp^{\hbar}$ be a column vector, then $\wp^{\hbar+1} = \daleth \wp^{\hbar}$. Take into account, for conditional probability, $a_{ij} \geq 0$ and $\sum_{i=1}^n a_{ij} = 1$ are required. At any period $\wp^{\hbar}$ is called a stationary distribution of the Markov process, the problem of finding a stationary distribution is equivalent to the fixed-point problem $\daleth \wp^{\hbar} = \wp^{\hbar}$, whenever $\wp^{\hbar} = \wp^{\hbar+1}$.

For all i, consider $\varepsilon_i = \min_j a_{ij}$ such that $\varepsilon = \sum_{i=1}^n \varepsilon_i$.

Now, the basic theory for this part is formulated as follows:

Theorem 6.1 *Via the hypothesis $a_{ij} \geq 0$, there exists a unique stationary distribution for the Markov process.*

Proof Define a distance $\omega : \Lambda_{n-1}^4 \times \Lambda_{n-1}^4 \to \mathbb{R}^2$ by

$$\begin{aligned} \omega(D, E) &= \xi\big((e^1, e^2, e^3, e^4), (r^1, r^2, r^3, r^4)\big) \\ &= \Bigg(\sum_{i=1}^n \big(|e_i^1 - r_i^1| + |e_i^2 - r_i^2| + |e_i^3 - r_i^3| + |e_i^4 - r_i^4|\big), \\ &\quad \sum_{i=1}^n \big(|e_i^1 - r_i^1| + |e_i^2 - r_i^2| + |e_i^3 - r_i^3| + |e_i^4 - r_i^4|\big) \Bigg), \end{aligned}$$

for each $D, E \in \Lambda^4_{n-1}$. Clearly, for each $D, E \in \Lambda^4_{n-1}$, $\omega(D,E) \geq (0,0)$. If $\omega(D,E) = (0,0)$, yields

$$\begin{aligned}&\Bigg(\sum_{i=1}^{n}\big(|e_i^1 - r_i^1| + |e_i^2 - r_i^2| + |e_i^3 - r_i^3| + |e_i^4 - r_i^4|\big),\\ &\quad \sum_{i=1}^{n}\big(|e_i^1 - r_i^1| + |e_i^2 - r_i^2| + |e_i^3 - r_i^3| + |e_i^4 - r_i^4|\big)\Bigg)\\ &= (0,0),\end{aligned}$$

which leads to, $|e_i^1 - r_i^1| + |e_i^2 - r_i^2| + |e_i^3 - r_i^3| + |e_i^4 - r_i^4| = 0$ for all i, this implies that $e^1 = r^1$, $e^2 = r^2$, $e^3 = r^3$, and $e^4 = r^4$, i.e., $D = E$. Conversely, take $D = E$, then for all i, $e_i^1 = r_i^1$, $e_i^2 = r_i^2$, $e_i^3 = r_i^3$, and $e_i^4 = r_i^4$, $\Rightarrow |e_i^1 - k_i^1| = |e_i^2 - k_i^2| = |e_i^3 - k_i^3| = |e_i^4 - k_i^4| = 0$, yields,

$$\begin{aligned}&\Bigg(\sum_{i=1}^{n}\big(|e_i^1 - r_i^1| + |e_i^2 - r_i^2| + |e_i^3 - r_i^3| + |e_i^4 - r_i^4|\big),\\ &\quad \sum_{i=1}^{n}\big(|e_i^1 - r_i^1| + |e_i^2 - r_i^2| + |e_i^3 - r_i^3| + |e_i^4 - r_i^4|\big)\Bigg)\\ &= (0,0) \quad \Rightarrow \quad \omega(D,E) = (0,0).\end{aligned}$$

Also, we obtain

$$\begin{aligned}\omega(D,E) &= \Bigg(\sum_{i=1}^{n}\big(|e_i^1 - r_i^1| + |e_i^2 - r_i^2| + |e_i^3 - r_i^3| + |e_i^4 - r_i^4|\big),\\ &\quad \sum_{i=1}^{n}\big(|e_i^1 - r_i^1| + |e_i^2 - r_i^2| + |e_i^3 - r_i^3| + |e_i^4 - r_i^4|\big)\Bigg)\\ &= \Bigg(\sum_{i=1}^{n}\big(|r_i^1 - e_i^1| + |r_i^2 - e_i^2| + |r_i^3 - e_i^3| + |r_i^4 - e_i^4|\big),\\ &\quad \sum_{i=1}^{n}\big(|r_i^1 - e_i^1| + |r_i^2 - e_i^2| + |r_i^3 - e_i^3| + |r_i^4 - e_i^4|\big)\Bigg)\\ &= \omega(E,D).\end{aligned}$$

Now,

$$\begin{aligned}\omega(D,E) &= \Bigg(\sum_{i=1}^{n}\big(|e_i^1 - r_i^1| + |e_i^2 - r_i^2| + |e_i^3 - r_i^3| + |e_i^4 - r_i^4|\big),\\ &\quad \sum_{i=1}^{n}\big(|e_i^1 - r_i^1| + |e_i^2 - r_i^2| + |e_i^3 - r_i^3| + |e_i^4 - r_i^4|\big)\Bigg)\\ &= \left(\sum_{i=1}^{n}\begin{pmatrix}|(e_i^1 - l_i^1) + (l_i^1 - r_i^1)| + |(e_i^2 - l_i^2) + (l_i^2 - r_i^2)|\\ +|(e_i^3 - l_i^3) + (l_i^3 - r_i^3)| + |(e_i^4 - l_i^4) + (l_i^4 - r_i^4)|\end{pmatrix},\right.\\ &\quad \left.\begin{pmatrix}|(e_i^1 - l_i^1) + (l_i^1 - r_i^1)| + |(e_i^2 - l_i^2) + (l_i^2 - r_i^2)|\\ +|(e_i^3 - l_i^3) + (l_i^3 - r_i^3)| + |(e_i^4 - l_i^4) + (l_i^4 - r_i^4)|\end{pmatrix}\right)\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\sum_{i=1}^{n} \begin{pmatrix} |e_i^1 - l_i^1| + |l_i^1 - r_i^1| + |e_i^2 - l_i^2| + |l_i^2 - r_i^2| \\ +|e_i^3 - l_i^3| + |l_i^3 - r_i^3| + |e_i^4 - l_i^4| + |l_i^4 - r_i^4| \end{pmatrix}, \right. \\
&\quad \left. \sum_{i=1}^{n} \begin{pmatrix} |e_i^1 - l_i^1| + |l_i^1 - r_i^1| + |e_i^2 - l_i^2| + |l_i^2 - r_i^2| \\ +|e_i^3 - l_i^3| + |l_i^3 - r_i^3| + |e_i^4 - l_i^4| + |l_i^4 - r_i^4| \end{pmatrix} \right) \\
&= \left(\sum_{i=1}^{n} \left(\left|e_i^1 - l_i^1\right| + \left|e_i^2 - l_i^2\right| + \left|e_i^3 - l_i^3\right| + \left|e_i^4 - l_i^4\right| \right), \right. \\
&\quad \left. \sum_{i=1}^{n} \left(\left|e_i^1 - l_i^1\right| + \left|e_i^2 - l_i^2\right| + \left|e_i^3 - l_i^3\right| + \left|e_i^4 - l_i^4\right| \right) \right) \\
&\quad + \left(\sum_{i=1}^{n} \left(\left|l_i^1 - r_i^1\right| + \left|l_i^2 - r_i^2\right| + \left|l_i^3 - r_i^3\right| + \left|l_i^4 - r_i^4\right| \right), \right. \\
&\quad \left. \sum_{i=1}^{n} \left(\left|l_i^1 - r_i^1\right| + \left|l_i^2 - r_i^2\right| + \left|l_i^3 - r_i^3\right| + \left|l_i^4 - r_i^4\right| \right) \right) \\
&= \omega(D,B) + \omega(B,E),
\end{aligned}
$$

where $D = (l^1, l^2, l^3, l^4) \in \Lambda_{n-1}^4$. Thus, $(\Lambda_{n-1}^4, \omega)$ is a generalized metric space. One can prove easily the completeness and if we define on Λ_{n-1}^4 the following order, for (e^1, e^2, e^3, e^4), $(r^1, r^2, r^3, r^4) \in \Lambda_{n-1}^4$,

$$
\left(r^1, r^2, r^3, r^4\right) \preceq \left(e^1, e^2, e^3, e^4\right) \quad \Leftrightarrow \quad e^1 \succeq r^1, e^2 \preceq r^2, e^3 \succeq r^3, \text{ and } e^4 \preceq r^4,
$$

then the triple $(\Lambda_{n-1}^4, \omega, \preceq)$ is a POCGMS.

Let $\hbar = \urcorner\wp$ for all $\wp \in \Lambda_{n-1}$, then each $\sqsupset_j = \sum_{j=1}^n a_{ij}\wp_j \geq 0$ moreover, since each $\sum_{j=1}^n a_{ij} = 1$, we can write

$$
\sum_{j=1}^{n} \sqsupset_j = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij}\wp_j = \sum_{j=1}^{n} a_{ij} \sum_{j=1}^{n} \left(e_j^1 + e_j^2 + e_j^3 + e_j^4\right) = \sum_{j=1}^{n} \left(e_j^1 + e_j^2 + e_j^3 + e_j^4\right) = 1,
$$

which implies that $\hbar \in \Lambda_{n-1}^4$. Thus, we have $\urcorner : \Lambda_{n-1}^4 \to \Lambda_{n-1}^4$. Now, we shall show that $\urcorner$ is a contraction. Consider $\urcorner_i$ refers to the ith row of $\urcorner$. Hence, for any $(e^1, e^2, e^3, e^4), (r^1, r^2, r^3, r^4) \in \Lambda_{n-1}^4$, we obtain

$$
\begin{aligned}
&\omega\left(\urcorner\left(e^1, e^2, e^3, e^4\right), \urcorner\left(r^1, r^2, r^3, r^4\right)\right) \\
&= \left(\sum_{i=1}^{n} \left(\left| \sum_{j=1}^{n} a_{ij}\left(e_j^1 + e_j^2 + e_j^3 + e_j^4\right) - a_{ij}\left(r_j^1 + r_j^2 + r_j^3 + r_j^4\right) \right| \right), \right. \\
&\quad \left. \sum_{i=1}^{n} \left(\left| \sum_{j=1}^{n} a_{ij}\left(e_j^1 + e_j^2 + e_j^3 + e_j^4\right) - a_{ij}\left(r_j^1 + r_j^2 + r_j^3 + r_j^4\right) \right| \right) \right) \\
&= \left(\sum_{i=1}^{n} \begin{pmatrix} \left| \sum_{j=1}^{n} (a_{ij} - \varepsilon_i)[(e_j^1 + e_j^2 + e_j^3 + e_j^4) - (r_j^1 + r_j^2 + r_j^3 + r_j^4)] \right. \\ \left. + \varepsilon_i(e_j^1 + e_j^2 + e_j^3 + e_j^4) - (r_j^1 + r_j^2 + r_j^3 + r_j^4) \right| \end{pmatrix}, \right. \\
&\quad \left. \sum_{i=1}^{n} \begin{pmatrix} | \sum_{j=1}^{n} (a_{ij} - \varepsilon_i)[(e_j^1 + e_j^2 + e_j^3 + e_j^4) - (r_j^1 + r_j^2 + r_j^3 + r_j^4)] \\ + \varepsilon_i(e_j^1 + e_j^2 + e_j^3 + e_j^4) - (r_j^1 + r_j^2 + r_j^3 + r_j^4)| \end{pmatrix} \right)
\end{aligned}
$$

$$\begin{aligned}
&\leq \left(\sum_{i=1}^{n}\left(\left|\sum_{j=1}^{n}(a_{ij}-\varepsilon_i)\left[\left(e_j^1+e_j^2+e_j^3+e_j^4\right)-\left(r_j^1+r_j^2+r_j^3+r_j^4\right)\right]\right|\right)\right.\\
&\quad+\left(\left|\sum_{j=1}^{n}\varepsilon_i\left(e_j^1+e_j^2+e_j^3+e_j^4\right)-\left(r_j^1+r_j^2+r_j^3+r_j^4\right)\right|\right),\\
&\quad\sum_{i=1}^{n}\left(\left|\sum_{j=1}^{n}(a_{ij}-\varepsilon_i)\left[\left(e_j^1+e_j^2+e_j^3+e_j^4\right)-\left(r_j^1+r_j^2+r_j^3+r_j^4\right)\right]\right|\right)\\
&\quad\left.+\left(\left|\sum_{j=1}^{n}\varepsilon_i\left(e_j^1+e_j^2+e_j^3+e_j^4\right)-\left(r_j^1+r_j^2+r_j^3+r_j^4\right)\right|\right)\right)\\
&\leq \begin{pmatrix}\sum_{i=1}^{n}\sum_{j=1}^{n}(|a_{ij}-\varepsilon_i|)(|e_j^1-r_j^1|+|e_j^2-r_j^2|+|e_j^3-r_j^3|+|e_j^4-r_j^4|),\\ \sum_{i=1}^{n}\sum_{j=1}^{n}(|a_{ij}-\varepsilon_i|)(|e_j^1-r_j^1|+|e_j^2-r_j^2|+|e_j^3-r_j^3|+|e_j^4-r_j^4|)\end{pmatrix}\\
&=\begin{pmatrix}\sum_{j=1}^{n}(|e_j^1-r_j^1|+|e_j^2-r_j^2|+|e_j^3-r_j^3|+|e_j^4-r_j^4|)\times\sum_{i=1}^{n}|a_{ij}-\varepsilon_i|,\\ \sum_{j=1}^{n}(|e_j^1-r_j^1|+|e_j^2-r_j^2|+|e_j^3-r_j^3|+|e_j^4-r_j^4|)\times\sum_{i=1}^{n}|a_{ij}-\varepsilon_i|\end{pmatrix}\\
&=(I-\varepsilon)\begin{pmatrix}\sum_{j=1}^{n}(|e_j^1-r_j^1|+|e_j^2-r_j^2|+|e_j^3-r_j^3|+|e_j^4-r_j^4|),\\ \sum_{j=1}^{n}(|e_j^1-r_j^1|+|e_j^2-r_j^2|+|e_j^3-r_j^3|+|e_j^4-r_j^4|)\end{pmatrix}\\
&=\Psi\omega\left(\left(e^1,e^2,e^3,e^4\right),\left(r^1,r^2,r^3,r^4\right)\right).
\end{aligned}$$

Hence, with $\Psi = (I - \varepsilon)$ all stipulations of Corollary 3.15 are fulfilled. Hence, there is a unique QFP of the mapping $\urcorner$ that is a unique stationary distribution for the Markov process. Also, for any $\wp^\dagger \in \Lambda_{n-1}$, the sequence $\{\urcorner^n \wp^\dagger\}$ converges to a unique stationary distribution. □

Acknowledgements
The author T. Abdeljawad would like to thank Prince Sultan University for the support through the TAS research lab.

Funding
There is no sponsor.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors wrote and revised the whole paper. The author HAH is a major contributor. He wrote most of the first draft, analyzed the results and validated them. The author TA participated in the first draft, edited the last version, analyzed and validated the results. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Faculty of Science, Sohag University, Sohag 82524, Egypt. [2]Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia. [3]Department of Medical Research, China Medical University, Taichung 40402, Taiwan.

Publisher's Note

Received: 21 May 2021 Accepted: 5 April 2022 Published online: 18 April 2022

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