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Lyapunov-type inequalities for differential equation with Caputo–Hadamard fractional derivative under multipoint boundary conditions

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Abstract

In this work, we establish Lyapunov-type inequalities for the fractional boundary value problems with Caputo–Hadamard fractional derivative subject to multipoint and integral boundary conditions. As far as we know, there is no literature that has studied these problems.

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1 Introduction

The well-known Lyapunov inequality [1] states that if $u(t)$ is a nontrivial solution of the boundary value problem

$$\begin{aligned} &u''(t) + q(t)u(t) = 0, \quad t \in (a,b), \\ &u(a) = 0 = u(b), \end{aligned} \tag{1.1}$$

where $q(t) \in C([a,b];\mathbb{R})$, then

$$\int_a^b |r(t)|\,dt > \frac{4}{b-a}. \tag{1.2}$$

The Lyapunov inequality (1.2) is a useful tool in various branches of mathematics, including disconjugacy, oscillation theory, and eigenvalue problems. Many improvements and generalizations of inequality (1.2) have appeared in the literature; see [2–13] and references therein.

The study of Lyapunov-type inequalities for fractional differential equations has begun recently. The first result in this direction is due to Ferreira [14]. He obtained a Lyapunov inequality for Riemann–Liouville fractional differential equations; his main result is as follows.

Theorem 1.1 *If the fractional boundary value problem*

$$\left(D_{a^+}^{\alpha}u\right)(t)+q(t)u(t)=0,\quad a<t<b,1<\alpha\leq 2, \tag{1.3}$$

$$u(a)=0=u(b), \tag{1.4}$$

has a nontrivial solution, where q is a real continuous function, then

$$\int_a^b \left|q(s)\right|ds>\Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}, \tag{1.5}$$

where $D_{a^+}^{\alpha}$ is the Riemann–Liouville fractional derivative of order α.

One year later, the same author Ferreira [15] obtained a Lyapunov-type inequality for the Caputo fractional boundary value problem.

Theorem 1.2 *If the fractional boundary value problem*

$$\left({}^{C}D_{a^+}^{\alpha}u\right)(t)+q(t)u(t)=0,\quad a<t<b,1<\alpha\leq 2, \tag{1.6}$$

$$u(a)=0=u(b), \tag{1.7}$$

where q is a real continuous function, has a nontrivial continuous solution, then

$$\int_a^b \left|q(s)\right|ds>\frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}}, \tag{1.8}$$

where ${}^{C}D_{a^+}^{\alpha}$ is the Caputo fractional derivative of order α.

After the publication of [14, 15], the research on Lyapunov inequalities for fractional differential equations has become a hot topic. The results in the literature can be divided into two categories. The first one is using other fractional derivatives instead of the Caputo fractional derivatives or Riemann–Liouville fractional derivatives in equation (1.3) or (1.6). Secondly, the boundary conditions (1.4) or (1.7) are replaced by multipoint boundary conditions or integral boundary conditions. For instance, in [16–18], Lyapunov inequalities for Hadamard fractional differential equations are given. Lyapunov-type inequalities regarding sequential fractional differential equations are obtained in [19–21]. The first paper considering integral boundary conditions is also duo to Ferreira [22]. For the results of multipoint boundary conditions, see [23, 24].

Motivated by the above works, in this paper, we establish Lyapunov-type inequalities for the fractional boundary value problems with Caputo–Hadamard fractional derivative under multipoint boundary condition

$$\left({}_{H}^{C}D_{a^+}^{\alpha}u\right)(t)+q(t)u(t)=0,\quad 0<a<t<b,1<\alpha<2, \tag{1.9}$$

$$u(a)=0,\qquad u(b)=\sum_{i=1}^{m-2}\beta_i u(\xi_i), \tag{1.10}$$

where ${}_{H}^{C}D_{a^+}^{\alpha}$ denotes the Caputo–Hadamard fractional derivative of order α.

In this paper, we assume that $\beta_i \geq 0$ $(i = 1, 2, \ldots, m-2)$, $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, and $0 \leq \sum_{i=1}^{m-2} \beta_i < 1$.

2 Preliminaries

In this section, we recall the concepts of the Riemann–Liouville fractional integral, the Riemann–Liouville fractional derivative, the Caputo fractional derivative of order $\alpha \geq 0$, and the definition of the Caputo–Hadamard fractional derivative.

Definition 2.1 ([25]) Let $\alpha \geq 0$, and let f be a real function on $[a,b]$. The Riemann–Liouville fractional integral of order α is defined by $(I_{a^+}^0 f) \equiv f$ and

$$\left(I_{a^+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,ds, \quad \alpha > 0, t \in [a,b].$$

Definition 2.2 ([25]) The Riemann–Liouville fractional derivative of order $\alpha \geq 0$ is defined by $(D_{a^+}^0 f) \equiv f$ and

$$\left(D_{a^+}^{\alpha} f\right)(t) = \left(D^m I_{a^+}^{m-\alpha} f\right)(t) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dt}\right)^m \int_a^t (t-s)^{m-\alpha-1} f(s)\,ds$$

for $\alpha > 0$, where m is the smallest integer greater than or equal to α.

Definition 2.3 ([25]) The Caputo fractional derivative of order $\alpha \geq 0$ is defined by $({}^C D_{a^+}^0 f) \equiv f$ and

$$\left({}^C D_{a^+}^{\alpha} f\right)(t) = \left(I_{a^+}^{m-\alpha} D^m f\right)(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-\alpha-1} f^{(m)}(s)\,ds,$$

for $\alpha > 0$, where m is the smallest integer greater than or equal to α.

Definition 2.4 ([25]) The Hadamard fractional integral of order $\alpha \in \mathbb{R}_+$ for a continuous function $f : [a,\infty) \to \mathbb{R}$ is defined by

$$\left({}_H I_{a+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}, \quad \alpha > 0, t \in [a,b].$$

Definition 2.5 ([25]) The Hadamard fractional derivative of order $\alpha \in \mathbb{R}_+$ for a continuous function $f : [a,\infty) \to \mathbb{R}$ is defined by

$$\left({}_H D_{a+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} f(s) \frac{ds}{s}, \quad t \in [a,b],$$

where $n-1 < \alpha < n$, $n = [\alpha] + 1$.

Definition 2.6 ([25]) The Caputo–Hadamard fractional derivative of order $\alpha \in \mathbb{R}_+$ for a function $f \in AC_\delta^n[a,b]$ is defined as

$$\left({}_H^C D_{a^+}^{\alpha} f\right)(t) = \left({}_H I_{a^+}^{n-\alpha} \delta^n f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{n-\alpha-1} \delta^n f(s) \frac{ds}{s},$$

where $n = [\alpha] + 1$, and $f \in AC_\delta^n[a,b] = \{\varphi : [a,b] \to \mathbb{C} : \delta^{(n-1)}\varphi \in AC[a,b], \delta = t\frac{d}{dt}\}$.

Lemma 2.7 ([25]) *Let $\alpha > 0$ and $n = [\alpha] + 1$. If $f \in AC_\delta^n[a,b]$ or $f \in C_\delta^n[a,b]$, then*

$$\left({}_H I_{a+}^{\alpha}\, {}_H^C D_{a+}^{\alpha} f\right)(t) = f(t) - \sum_{k=1}^{n} \frac{\delta^{k-1} f(a)}{(k-1)!} \left(\ln \frac{t}{a}\right)^{k-1}.$$

3 Main results

We begin by writing problem (1.9)–(1.10) in an equivalent integral form.

Lemma 3.1 *A function $u \in C[a,b]$ is a solution to the boundary value problem (1.9)–(1.10) if and only if it satisfies the integral equation*

$$u(t) = \int_a^b G(t,s)q(s)u(s)\,ds + \frac{\ln \frac{t}{a}}{\ln \frac{b}{a} - \sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a}} \int_a^b \sum_{i=1}^{m-2} \beta_i G(\xi_i, s) q(s)u(s)\,ds,$$

where $G(t,s)$ is defined as

$$G(t,s) = \frac{1}{s \ln \frac{b}{a} \Gamma(\alpha)} \begin{cases} \ln \frac{t}{a}(\ln \frac{b}{s})^{\alpha-1} - \ln \frac{b}{a}(\ln \frac{t}{s})^{\alpha-1}, & 0 < a \le s \le t \le b, \\ \ln \frac{t}{a}(\ln \frac{b}{s})^{\alpha-1}, & 0 < a \le t \le s \le b. \end{cases}$$

Proof By Lemma 2.7 $u \in C[a,b]$ is a solution to the boundary value problem (1.9)–(1.10) if and only if

$$u(t) = c_0 + c_1 \left(\ln \frac{t}{a}\right) - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} q(s)u(s) \frac{ds}{s},$$

where c_0 and c_1 are real constants. Since $u(a) = 0$, we immediately get that $c_0 = 0$, and thus

$$u(t) = c_1 \left(\ln \frac{t}{a}\right) - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} q(s)u(s) \frac{ds}{s}.$$

The boundary condition $u(b) = \sum_{i=1}^{m-2} \beta_i u(\xi_i)$ yields

$$\begin{aligned} &c_1 \left(\ln \frac{b}{a}\right) - \frac{1}{\Gamma(\alpha)} \int_a^b \left(\ln \frac{b}{s}\right)^{\alpha-1} q(s)u(s) \frac{ds}{s} \\ &\quad = \sum_{i=1}^{m-2} \beta_i \left[c_1 \left(\ln \frac{\xi_i}{a}\right) - \frac{1}{\Gamma(\alpha)} \int_a^{\xi_i} \left(\ln \frac{\xi_i}{s}\right)^{\alpha-1} q(s)u(s) \frac{ds}{s} \right], \end{aligned}$$

so,

$$\begin{aligned} c_1 &= \frac{\int_a^b (\ln \frac{b}{s})^{\alpha-1} q(s)u(s) \frac{ds}{s} - \sum_{i=1}^{m-2} \beta_i \int_a^{\xi_i} (\ln \frac{\xi_i}{s})^{\alpha-1} q(s)u(s) \frac{ds}{s}}{(\ln \frac{b}{a} - \sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a}) \Gamma(\alpha)} \\ &= \frac{1}{\Gamma(\alpha)} \int_a^b \frac{(\ln \frac{b}{s})^{\alpha-1}}{\ln \frac{b}{a}} q(s)u(s) \frac{ds}{s} - \frac{\sum_{i=1}^{m-2} \beta_i \int_a^{\xi_i} (\ln \frac{\xi_i}{s})^{\alpha-1} q(s)u(s) \frac{ds}{s}}{(\ln \frac{b}{a} - \sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a}) \Gamma(\alpha)} \\ &\quad + \frac{\sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a}}{\ln \frac{b}{a} (\ln \frac{b}{a} - \sum_{i=1}^{m-2} \beta_i \ln \frac{\xi_i}{a}) \Gamma(\alpha)} \int_a^b \left(\ln \frac{b}{s}\right)^{\alpha-1} q(s)u(s) \frac{ds}{s}. \end{aligned}$$

Hence

$$
\begin{aligned}
u(t) &= c_1\left(\ln\frac{t}{a}\right) - \frac{1}{\Gamma(\alpha)}\int_a^t \left(\ln\frac{t}{s}\right)^{\alpha-1} q(s)u(s)\frac{ds}{s} \\
&= \frac{\ln\frac{t}{a}}{\Gamma(\alpha)}\int_a^b \frac{(\ln\frac{b}{s})^{\alpha-1}}{\ln\frac{b}{a}} q(s)u(s)\frac{ds}{s} - \frac{(\ln\frac{t}{a})\sum_{i=1}^{m-2}\beta_i\int_a^{\xi_i}(\ln\frac{\xi_i}{s})^{\alpha-1}q(s)u(s)\frac{ds}{s}}{(\ln\frac{b}{a}-\sum_{i=1}^{m-2}\beta_i\ln\frac{\xi_i}{a})\Gamma(\alpha)} \\
&\quad + \frac{(\ln\frac{t}{a})\sum_{i=1}^{m-2}\beta_i\ln\frac{\xi_i}{a}}{\ln\frac{b}{a}(\ln\frac{b}{a}-\sum_{i=1}^{m-2}\beta_i\ln\frac{\xi_i}{a})}\int_a^b\left(\ln\frac{b}{s}\right)^{\alpha-1} q(s)u(s)\frac{ds}{s} \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1} q(s)u(s)\frac{ds}{s} \\
&= \int_a^b G(t,s)q(s)u(s)\,ds + \frac{\ln\frac{t}{a}}{\ln\frac{b}{a}-\sum_{i=1}^{m-2}\beta_i\ln\frac{\xi_i}{a}}\int_a^b\sum_{i=1}^{m-2}\beta_i G(\xi_i,s)q(s)u(s)\,ds,
\end{aligned}
$$

which concludes the proof. □

Lemma 3.2 *Let* $0 < a \le s \le b$ *and* $1 < \alpha < 2$. *Then*

$$0 \le \ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\frac{\alpha-1}{2-\alpha}} \le (2-\alpha)(\alpha-1)^{\frac{\alpha-1}{2-\alpha}}\left(\ln\frac{b}{a}\right)^{\frac{1}{2-\alpha}}.$$

Proof Let

$$f(s) = \ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\frac{\alpha-1}{2-\alpha}}, \quad s \in [a,b].$$

Clearly, $f(a) = f(b) = 0$, and $f(s) > 0$ on (a,b). By Rolle's theorem there exists $s^* \in (a,b)$ such that $f(s^*) = \max f(s)$ on (a,b), that is, $f'(s^*) = 0$. Note that

$$f'(s) = \frac{1}{s}\left(\ln\frac{b}{s}\right)^{\frac{\alpha-1}{2-\alpha}-1}\left[\ln\frac{b}{s} - \frac{\alpha-1}{2-\alpha}\ln\frac{s}{a}\right].$$

Letting $f'(s) = 0$, we obtain $s^* = a^{\alpha-1}b^{2-\alpha}$. It is easy to show that $\frac{s^*}{a} = (\frac{b}{a})^{2-\alpha} > 1$, $\frac{b}{s^*} = (\frac{b}{a})^{\alpha-1} > 1$, and $s^* \in (a,b)$, and thus

$$\max f(s) = f(s^*) = \ln\frac{s^*}{a}\left(\ln\frac{b}{s^*}\right)^{\frac{\alpha-1}{2-\alpha}} = (2-\alpha)(\alpha-1)^{\frac{\alpha-1}{2-\alpha}}\left(\ln\frac{b}{a}\right)^{\frac{1}{2-\alpha}},$$

which concludes the proof. □

Lemma 3.3 *Let* $0 < a \le s \le b$ *and* $1 < \alpha < 2$. *Then*

$$0 \le \frac{1}{s}\ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\alpha-1} \le \frac{1}{a}\ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\alpha-1} \le \frac{1}{a}\cdot\frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\left(\ln\frac{b}{a}\right)^{\alpha}.$$

Proof Let

$$g(s) = \ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\alpha-1}, \quad s \in [a,b].$$

As $g(a) = g(b) = 0$ and $g(s) > 0$ on (a, b). So, there exists $s^* \in (a, b)$ such that $g(s^*) = \max g(s)$ on (a, b), that is, $g'(s^*) = 0$. Note that

$$g'(s) = \frac{1}{s}\left(\ln\frac{b}{s}\right)^{\alpha-2}\left[\ln\frac{b}{s} - (\alpha-1)\ln\frac{s}{a}\right].$$

Letting $g'(s) = 0$, we obtain $s^* = a^{\frac{\alpha-1}{\alpha}} b^{\frac{1}{\alpha}}$, $\frac{s^*}{a} = (\frac{b}{a})^{\frac{1}{\alpha}} > 1$, and $\frac{b}{s^*} = (\frac{b}{a})^{\frac{\alpha-1}{\alpha}} > 1$, which imply that $s^* \in (a, b)$, and thus

$$\max g(s) = g\left(s^*\right) = \ln\frac{s^*}{a}\left(\ln\frac{b}{s^*}\right)^{\alpha-1} = \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\left(\ln\frac{b}{a}\right)^{\alpha},$$

which concludes the proof. □

Lemma 3.4 *Let* $0 < a \le s \le b(a/b)^{\alpha-1}$ *and* $1 < \alpha < 2$. *Then the function*

$$h(s) = (2-\alpha)(\alpha-1)^{\frac{\alpha-1}{2-\alpha}}\left(\ln\frac{b}{a}\right)^{\frac{1}{2-\alpha}}\frac{1}{s}\left(\ln\frac{b}{s}\right)^{\frac{(\alpha-1)^2}{\alpha-2}} - \frac{1}{s}\ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\alpha-1},$$

satisfies

$$\max_{s\in[a,b(a/b)^{\alpha-1}]} h(s) = (2-\alpha)(\alpha-1)^{\frac{\alpha-1}{2-\alpha}}\frac{1}{a}\left(\ln\frac{b}{a}\right)^{\alpha}.$$

Proof For $0 < a \le s \le b(a/b)^{\alpha-1}$, we have $(\alpha-1)\ln\frac{b}{a} < \ln\frac{b}{s} < \ln\frac{b}{a}$, $0 < \ln\frac{s}{a} < (2-\alpha)\ln\frac{b}{a}$. Define the new function

$$\begin{aligned} r(s) = sh(s) &= (2-\alpha)(\alpha-1)^{\frac{\alpha-1}{2-\alpha}}\left(\ln\frac{b}{a}\right)^{\frac{1}{2-\alpha}}\left(\ln\frac{b}{s}\right)^{\frac{(\alpha-1)^2}{\alpha-2}} - \ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\alpha-1} \\ &= \left[(2-\alpha)(\alpha-1)^{\frac{\alpha-1}{2-\alpha}}\left(\ln\frac{b}{a}\right)^{\frac{1}{2-\alpha}} - \ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\frac{\alpha-1}{2-\alpha}}\right]\left(\ln\frac{b}{s}\right)^{\frac{(\alpha-1)^2}{\alpha-2}}. \end{aligned}$$

By Lemma 3.2, $r(s) \ge 0$, and we easily obtain

$$\begin{aligned} r'(s) &= \frac{1}{s}\left[(\alpha-1)^{\frac{3-\alpha}{2-\alpha}}\left(\ln\frac{b}{a}\right)^{\frac{1}{2-\alpha}}\left(\ln\frac{b}{s}\right)^{\frac{(\alpha-1)^2}{\alpha-2}-1} - \left(\ln\frac{b}{s}\right)^{\alpha-1} + (\alpha-1)\ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\alpha-2}\right] \\ &\le \frac{1}{s}\left[(\alpha-1)^{\frac{3-\alpha}{2-\alpha}}\left(\ln\frac{b}{a}\right)^{\frac{1}{2-\alpha}}(\alpha-1)^{\frac{(\alpha-1)^2}{\alpha-2}-1}\left(\ln\frac{b}{a}\right)^{\frac{(\alpha-1)^2}{\alpha-2}-1}\right. \\ &\quad \left. - (\alpha-1)^{\alpha-1}\left(\ln\frac{b}{a}\right)^{\alpha-1} + (\alpha-1)(2-\alpha)\ln\frac{b}{a}(\alpha-1)^{\alpha-2}\left(\ln\frac{b}{a}\right)^{\alpha-2}\right] \\ &= \frac{1}{s}\left[(\alpha-1)^{\alpha}\left(\ln\frac{b}{a}\right)^{\alpha-1} - (\alpha-1)^{\alpha-1}\left(\ln\frac{b}{a}\right)^{\alpha-1} + (2-\alpha)(\alpha-1)^{\alpha-1}\left(\ln\frac{b}{a}\right)^{\alpha-1}\right] \\ &= 0. \end{aligned}$$

So,

$$h'(s) = \left(\frac{r(s)}{s}\right)' = \frac{sr'(s) - r(s)}{s^2} < 0.$$

Therefore

$$\max_{s\in[a,b(\frac{a}{b})^{\alpha-1}]} h(s) = h(a) = (2-\alpha)(\alpha-1)^{\frac{\alpha-1}{2-\alpha}} \frac{1}{a}\left(\ln\frac{b}{a}\right)^{\alpha}.$$

□

Lemma 3.5 *If* $1 < \alpha < 2$, *then*

$$(2-\alpha)(\alpha-1)^{\frac{\alpha-1}{2-\alpha}} \leq \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}.$$

Proof A proof of this lemma can be found in [15]. Here we give a new proof. Let $0 < a \leq s \leq b$. It is easy to check that

$$\begin{aligned}
&\ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\frac{\alpha-1}{2-\alpha}}\left(\ln\frac{b}{a}\right)^{\frac{(\alpha-1)^2}{\alpha-2}} - \ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\alpha-1} \\
&= \ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\alpha-1}\left[\left(\ln\frac{b}{s}\right)^{\frac{(\alpha-1)^2}{2-\alpha}}\left(\ln\frac{b}{a}\right)^{\frac{(\alpha-1)^2}{\alpha-2}} - 1\right] \\
&= \ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\alpha-1}\left[\left(\frac{\ln\frac{b}{s}}{\ln\frac{b}{a}}\right)^{\frac{(\alpha-1)^2}{2-\alpha}} - 1\right] \\
&\leq 0,
\end{aligned}$$

so,

$$\ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\frac{\alpha-1}{2-\alpha}}\left(\ln\frac{b}{a}\right)^{\frac{(\alpha-1)^2}{\alpha-2}} \leq \ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\alpha-1},$$

and thus

$$\max_{0<a\leq s\leq b} \ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\frac{\alpha-1}{2-\alpha}}\left(\ln\frac{b}{a}\right)^{\frac{(\alpha-1)^2}{\alpha-2}} \leq \max_{0<a\leq s\leq b} \ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\alpha-1}.$$

By Lemmas 3.2 and 3.3 we obtain

$$(2-\alpha)(\alpha-1)^{\frac{\alpha-1}{2-\alpha}}\left(\ln\frac{b}{a}\right)^{\alpha} \leq \frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}\left(\ln\frac{b}{a}\right)^{\alpha}.$$

Thus the proof is completed. □

Lemma 3.6 *The function G defined in Lemma* 3.1 *satisfies the following property*:

$$\left|G(t,s)\right| \leq \frac{1}{a}\cdot\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}\Gamma(\alpha)}\left(\ln\frac{b}{a}\right)^{\alpha-1}.$$

Proof The Green's function $G(t,s)$ can be rewritten as the following form:

$$\left(\ln\frac{b}{a}\right)\Gamma(\alpha)G(t,s)=\begin{cases}\frac{1}{s}\ln\frac{t}{a}(\ln\frac{b}{s})^{\alpha-1}-\frac{1}{s}\ln\frac{b}{a}(\ln\frac{t}{s})^{\alpha-1}, & a\le s\le t\le b,\\ \frac{1}{s}\ln\frac{t}{a}(\ln\frac{b}{s})^{\alpha-1}, & a\le t\le s\le b.\end{cases}$$

Define two functions

$$g_1(t,s)=\frac{1}{s}\ln\frac{t}{a}\left(\ln\frac{b}{s}\right)^{\alpha-1}-\frac{1}{s}\ln\frac{b}{a}\left(\ln\frac{t}{s}\right)^{\alpha-1},\quad a\le s\le t\le b,$$
$$g_2(t,s)=\frac{1}{s}\ln\frac{t}{a}\left(\ln\frac{b}{s}\right)^{\alpha-1},\quad a\le t\le s\le b.$$

Obviously, $g_2(t,s)$ is an increasing function in t, and $0\le g_2(t,s)\le g_2(s,s)$. By Lemma 3.3 we obtain

$$g_2(t,s)\le g_2(s,s)\le\frac{1}{a}\cdot\frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\left(\ln\frac{b}{a}\right)^\alpha.$$

Now we turn our attention to the function $g_1(t,s)$. We start by fixing an arbitrary $s\in[a,b)$. Differentiating $g_1(t,s)$ with respect to t, we get

$$\frac{\partial g_1(t,s)}{\partial t}=\frac{1}{st}\left[\left(\ln\frac{b}{s}\right)^{\alpha-1}-(\alpha-1)\ln\frac{b}{a}\left(\ln\frac{t}{s}\right)^{\alpha-2}\right].$$

It follows that $\frac{\partial g_1(t^*,s)}{\partial t}=0$ if and only if $t^*=se^{[\frac{(\alpha-1)\ln\frac{b}{a}}{(\ln\frac{b}{s})^{\alpha-1}}]^{\frac{1}{2-\alpha}}}$, provided that $t^*\le b$, that is, as long as $s\le b(a/b)^{\alpha-1}$. So, if $s>b(a/b)^{\alpha-1}$, then $t^*>b$ and $t<t^*=se^{[\frac{(\alpha-1)\ln\frac{b}{a}}{(\ln\frac{b}{s})^{\alpha-1}}]^{\frac{1}{2-\alpha}}}$, and therefore $\frac{\partial g_1(t,s)}{\partial t}<0$, $g_1(t,s)$ is strictly decreasing with respect to t, and thus we have

$$0=g_1(b,s)\le g_1(t,s)\le g_1(s,s)=g_2(s,s).$$

From this we conclude that

$$\left|g_1(t,s)\right|\le g_2(s,s)\le\frac{1}{a}\cdot\frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\left(\ln\frac{b}{a}\right)^\alpha,\quad s\in\left(b(a/b)^{\alpha-1},b\right].$$

It remains to verify the result for $s\le b(a/b)^{\alpha-1}$, that is, for $t^*\le b$. It is easy to check that $\frac{\partial g_1(t,s)}{\partial t}<0$ for $t<t^*$ and $\frac{\partial g_1(t,s)}{\partial t}\ge 0$ for $t\ge t^*$. This, together with the fact that $g_1(b,s)=0$, implies that $g_1(t^*,s)\le 0$, and we only have to show that

$$\left|g_1\left(t^*,s\right)\right|\le\frac{1}{a}\cdot\frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\left(\ln\frac{b}{a}\right)^\alpha,\quad s\in\left[a,b(a/b)^{\alpha-1}\right].$$

Indeed, by Lemmas 3.4 and 3.5 we obtain

$$\begin{aligned}\left|g_1\left(t^*,s\right)\right|&=\left|\frac{1}{s}\ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\alpha-1}-(2-\alpha)(\alpha-1)^{\frac{\alpha-1}{2-\alpha}}\left(\ln\frac{b}{a}\right)^{\frac{1}{2-\alpha}}\frac{1}{s}\left(\ln\frac{b}{s}\right)^{\frac{(\alpha-1)^2}{\alpha-2}}\right|\\&=(2-\alpha)(\alpha-1)^{\frac{\alpha-1}{2-\alpha}}\left(\ln\frac{b}{a}\right)^{\frac{1}{2-\alpha}}\frac{1}{s}\left(\ln\frac{b}{s}\right)^{\frac{(\alpha-1)^2}{\alpha-2}}-\frac{1}{s}\ln\frac{s}{a}\left(\ln\frac{b}{s}\right)^{\alpha-1}\end{aligned}$$

$$\begin{aligned}
&\le (2-\alpha)(\alpha-1)^{\frac{\alpha-1}{2-\alpha}} \frac{1}{a}\left(\ln\frac{b}{a}\right)^{\alpha} \\
&\le \frac{1}{a}\cdot\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}\left(\ln\frac{b}{a}\right)^{\alpha}.
\end{aligned}$$

The proof is completed. □

Now we are ready to prove our Lyapunov-type inequality.

Theorem 3.7 *If a nontrivial continuous solution of the Caputo–Hadamard fractional boundary value problem*

$$\begin{aligned}
&\left({}_{H}^{C}D_{a^+}^{\alpha}u\right)(t)+q(t)u(t)=0, \quad 0<a<t<b, 1<\alpha<2, \\
&u(a)=0, \qquad u(b)=\sum_{i=1}^{m-2}\beta_i u(\xi_i),
\end{aligned}$$

exists, where $\beta_i \ge 0$ $(i=1,2,\dots,m-2)$, $a<\xi_1<\xi_2<\cdots<\xi_{m-2}<b$, $0\le\sum_{i=1}^{m-2}\beta_i<1$, *and* q *is a real continuous function on* $[a,b]$, *then*

$$\int_a^b \left|q(s)\right|ds \ge a\cdot\frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(\ln b-\ln a)]^{\alpha-1}}\cdot\frac{\ln\frac{b}{a}-\sum_{i=1}^{m-2}\beta_i\ln\frac{\xi_i}{a}}{\ln\frac{b}{a}+\sum_{i=1}^{m-2}\beta_i\ln\frac{b}{\xi_i}}. \tag{3.1}$$

Proof Let $B=C[a,b]$ be the Banach space endowed with norm $\|u\|=\sup_{t\in[a,b]}|u(t)|$. It follows from Lemma 3.1 that a solution u to the boundary value problem satisfies the integral equation

$$u(t)=\int_a^b G(t,s)q(s)u(s)\,ds+\frac{\ln\frac{t}{a}}{\ln\frac{b}{a}-\sum_{i=1}^{m-2}\beta_i\ln\frac{\xi_i}{a}}\int_a^b\sum_{i=1}^{m-2}\beta_i G(\xi_i,s)q(s)u(s)\,ds.$$

Now an application of Lemma 3.6 yields

$$\begin{aligned}
\|u\| &\le \frac{1}{a}\cdot\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}\Gamma(\alpha)}\left(\ln\frac{b}{a}\right)^{\alpha-1}\left(1+\frac{\ln\frac{b}{a}\sum_{i=1}^{m-2}\beta_i}{\ln\frac{b}{a}-\sum_{i=1}^{m-2}\beta_i\ln\frac{\xi_i}{a}}\right)\int_a^b\left|q(s)\right|ds\|u\| \\
&=\frac{1}{a}\cdot\frac{(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}\Gamma(\alpha)}\left(\ln\frac{b}{a}\right)^{\alpha-1}\frac{\ln\frac{b}{a}+\sum_{i=1}^{m-2}\beta_i\ln\frac{b}{\xi_i}}{\ln\frac{b}{a}-\sum_{i=1}^{m-2}\beta_i\ln\frac{\xi_i}{a}}\int_a^b\left|q(s)\right|ds\|u\|,
\end{aligned}$$

which implies that (3.1) holds. □

Letting $\beta_i=0$ $(i=1,2,\dots,m-2)$ in Theorem 3.7, we have the following result.

Corollary 3.8 *If a nontrivial continuous solution of the Caputo-Hadamard fractional boundary value problem*

$$\begin{aligned}
&\left({}_{H}^{C}D_{a^+}^{\alpha}u\right)(t)+q(t)u(t)=0, \quad 0<a<t<b, 1<\alpha<2, \\
&u(a)=0, \qquad u(b)=0,
\end{aligned}$$

exists, where q is a real continuous function in $[a,b]$, then

$$\int_a^b \left|q(s)\right| ds \geq a \cdot \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(\ln b - \ln a)]^{\alpha-1}}. \tag{3.2}$$

4 Remarks

Applying the Green's approach, we can also obtain Lyapunov-type inequalities for Caputo–Hadamard fractional differential equations under integral boundary conditions,

$$\left({}_H^C D_{a^+}^\alpha u\right)(t) + q(t)u(t) = 0, \quad 0 < a < t < b, 1 < \alpha < 2, \tag{4.1}$$

$$u(a) = 0, \qquad u(b) = \lambda \int_a^b h(s)u(s)\,ds, \quad \lambda \geq 0. \tag{4.2}$$

where $h : [a,b] \to [0,\infty)$ with $h \in L^1(a,b)$.

Lemma 4.1 *A function $u \in C[a,b]$ is a solution to the boundary value problem (4.1)–(4.2) if and only if it satisfies the integral equation*

$$u(t) = \int_a^b G(t,s)q(s)u(s)\,ds + \frac{\lambda \ln \frac{t}{a}}{\ln \frac{b}{a} - \lambda\sigma} \int_a^b \left(\int_a^b G(t,s)h(t)\,dt\right) q(s)u(s)\,ds,$$

where $h : [a,b] \to [0,\infty)$ with $h \in L^1(a,b)$, $\sigma = \int_a^b h(t) \ln \frac{t}{a}\,dt$, $0 \leq \lambda\sigma < \ln \frac{b}{a}$, and $G(t,s)$ is defined in Lemma 3.1.

Proof By Lemma 2.7 $u \in C[a,b]$ is a solution to the boundary value problem (4.1)–(4.2) if and only if

$$u(t) = c_0 + c_1\left(\ln \frac{t}{a}\right) - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} q(s)u(s)\frac{ds}{s},$$

where c_0 and c_1 are real constants. Since $u(a) = 0$, we immediately get that $c_0 = 0$, and thus

$$u(t) = c_1\left(\ln \frac{t}{a}\right) - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\ln \frac{t}{s}\right)^{\alpha-1} q(s)u(s)\frac{ds}{s}.$$

The boundary condition $u(b) = \lambda \int_a^b h(s)u(s)\,ds$ yields

$$c_1\left(\ln \frac{b}{a}\right) - \frac{1}{\Gamma(\alpha)} \int_a^b \left(\ln \frac{b}{s}\right)^{\alpha-1} q(s)u(s)\frac{ds}{s} = \lambda \int_a^b h(t)u(t)\,dt,$$

so,

$$c_1 = \frac{1}{(\ln \frac{b}{a})\Gamma(\alpha)} \int_a^b \left(\ln \frac{b}{s}\right)^{\alpha-1} q(s)u(s)\frac{ds}{s} + \frac{\lambda}{\ln \frac{b}{a}} \int_a^b h(t)u(t)\,dt,$$

and therefore the solution of the boundary value problem (4.1)–(4.2) is

$$\begin{aligned}
u(t) &= c_1\left(\ln\frac{t}{a}\right) - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1} g(s)\frac{ds}{s} \\
&= \frac{(\ln\frac{t}{a})}{(\ln\frac{b}{a})\Gamma(\alpha)}\int_a^b\left(\ln\frac{b}{s}\right)^{\alpha-1} q(s)u(s)\frac{ds}{s} + \frac{\lambda(\ln\frac{t}{a})}{(\ln\frac{b}{a})}\int_a^b h(t)u(t)\,dt \\
&\quad - \frac{1}{\Gamma(\alpha)}\int_a^t\left(\ln\frac{t}{s}\right)^{\alpha-1} q(s)u(s)\frac{ds}{s} \\
&= \int_a^b G(t,s)q(s)u(s)\,ds + \frac{\lambda\ln\frac{t}{a}}{\ln\frac{b}{a}}\int_a^b h(t)u(t)\,dt.
\end{aligned}$$

Multiplying both sides of this equality by $h(t)$ and integrating from a to b, we obtain

$$\int_a^b h(t)u(t)\,dt = \int_a^b\left(\int_a^b G(t,s)q(s)u(s)\,ds\right)h(t)\,dt + \frac{\lambda\sigma}{\ln\frac{b}{a}}\int_a^b h(t)u(t)\,dt$$

and

$$\int_a^b h(t)u(t)\,dt = \frac{\ln\frac{b}{a}}{\ln\frac{b}{a} - \lambda\sigma}\int_a^b\left(\int_a^b G(t,s)q(s)u(s)\,ds\right)h(t)\,dt,$$

and thus

$$u(t) = \int_a^b G(t,s)q(s)u(s)\,ds + \frac{\lambda\ln\frac{t}{a}}{\ln\frac{b}{a} - \lambda\sigma}\int_a^b\left(\int_a^b G(t,s)q(s)u(s)\,ds\right)h(t)\,dt,$$

which concludes the proof. □

Theorem 4.2 *If a nontrivial continuous solution of the Caputo–Hadamard fractional boundary value problem*

$$\begin{aligned}
&\left({}^{C}_{H}D^{\alpha}_{a^+}u\right)(t) + q(t)u(t) = 0, \quad 0 < a < t < b, 1 < \alpha < 2, \\
&u(a) = 0, \qquad u(b) = \lambda\int_a^b h(s)u(s)\,ds, \quad \lambda \ge 0,
\end{aligned}$$

exists, where $q:[a,b]\to\mathbb{R}$ *is a continuous function,* $h:[a,b]\to[0,\infty)$ *with* $h\in L^1(a,b)$, $\sigma = \int_a^b h(t)\ln\frac{t}{a}\,dt$, *and* $0 \le \lambda\sigma < \ln\frac{b}{a}$, *then we have*

$$\int_a^b |q(s)|\,ds \ge \frac{a[\ln\frac{b}{a} - \lambda\sigma]}{\ln\frac{b}{a} + \lambda[(\ln\frac{b}{a})\int_a^b h(t)\,dt - \sigma]}\cdot\frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(\ln b - \ln a)]^{\alpha-1}}. \tag{4.3}$$

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References

1. Lyapunov, A.M.: Problème général de la stabilité du mouvement (French Translation of a Russian paper dated 1893). Ann. Fac. Sci. Univ. Toulouse, **2**, 27–247 (Reprinted as Ann. Math. Studies, No. 17, Princeton Univ. Press, Princeton, NJ, USA, 1947)
2. Brown, R.C., Hinton, D.B.: Lyapunov inequalities and their applications. In: Rassias, T.M. (ed.) Survey on Classical Inequalities, pp. 1–25. Kluwer Academic, Dordrecht (2000)
3. Cheng, S.: Lyapunov inequalities for differential and difference equations. Fasc. Math. **23**, 25–41 (1991)
4. Hilfer, R.: Fractional calculus and regular variation in thermodynamics. In: Hilfer, R. (ed.) Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
5. Hilfer, R., Luchko, Y., Tomovski, Z.: Operational method for the solution of fractional differential equations with generalized Riemann–Liouville fractional derivatives. Fract. Calc. Appl. Anal. **3**(12), 299–318 (2009)
6. Ntouyas, S.K., Ahmad, B., Horikis, T.P.: Recent developments of Lyapunov-type inequalities for fractional differential equations. In: Differential and Integral Inequalities, pp. 619–686. Springer, Cham (2019)
7. Dhar, S., Kong, Q.: Lyapunov-type inequalities for third-order half-linear equations and applications to boundary value problems. Nonlinear Anal. **110**, 170–181 (2014)
8. Dhar, S., Kong, Q.: Lyapunov-type inequalities for higher order half-linear differential equations. Appl. Math. Comput. **273**, 114–124 (2016)
9. Jleli, M., Samet, B.: Lyapunov-type inequalities for a fractional differential equation with mixed boundary conditions. Math. Inequal. Appl. **18**(2), 443–451 (2015)
10. Jleli, M., Ragoub, L., Samet, B.: Lyapunov-type inequality for a fractional differential equation under a Robin boundary condition. J. Funct. Spaces **2015**, Article ID 468536 (2015)
11. Tiryaki, A.: Recent development of Lyapunov-type inequalities. Adv. Dyn. Syst. Appl. **5**(2), 231–248 (2010)
12. Wang, Y., Liang, S., Xia, C.: A Lyapunov-type inequality for a fractional differential equation under Sturm–Liouville boundary conditions. Math. Inequal. Appl. **20**(1), 139–148 (2017)
13. Wang, Y., Wang, Q.: Lyapunov-type inequalities for nonlinear differential equation with Hilfer fractional derivative operator. J. Math. Inequal. **12**, 709–717 (2018)
14. Ferreira, R.A.C.: A Lyapunov-type inequality for a fractional boundary value problem. Fract. Calc. Appl. Anal. **16**(4), 978–984 (2013)
15. Ferreira, R.A.C.: On a Lyapunov-type inequality and the zeros of a certain Mittag-Leffler function. J. Math. Anal. Appl. **412**(2), 1058–1063 (2014)
16. Dhar, S.: On linear and nonlinear fractional Hadamard boundary value problems. Differ. Equ. Appl. **10**, 329–339 (2018)
17. Jleli, M., Kirane, M., Samet, B.: Hartman–Wintner-type inequality for a fractional boundary value problem via a fractional derivative with respect to another function. Discrete Dyn. Nat. Soc. **2017**, Article ID 5123240 (2017)
18. Laadjal, Z., Adjeroud, N., Ma, Q.: Lyapunov-type inequality for the Hadamard fractional boundary value problem on a general interval $[a,b]$. J. Math. Inequal. **13**, 789–799 (2019)
19. Ferreira, R.A.C.: Lyapunov-type inequalities for some sequential fractional boundary value problems. Adv. Dyn. Syst. Appl. **11**, 33–43 (2016)
20. Ferreira, R.A.C.: Novel Lyapunov-type inequalities for sequential fractional boundary value problems. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **113**, 171–179 (2019)
21. Zhang, W., Liu, W.: Lyapunov-type inequalities for sequential fractional boundary value problems using Hilfer's fractional derivative. J. Inequal. Appl. **2019**, 98 (2019)
22. Ferreira, R.A.C.: Lyapunov inequalities for some differential equations with integral-type boundary conditions. In: Advances in Mathematical Inequalities and Applications. Trends Math., pp. 59–70. Birkhäuser/Springer, Singapore (2018)
23. Wang, Y., Wang, Q.: Lyapunov-type inequalities for fractional differential equations under multi-point boundary conditions. J. Math. Inequal. **13**, 611–619 (2019)
24. Wang, Y., Wang, Q.: Lyapunov-type inequalities for nonlinear fractional differential equation with Hilfer fractional derivative under multi-point boundary conditions. Fract. Calc. Appl. Anal. **21**, 833–843 (2018)
25. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006)