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Interchanging a limit and an integral: necessary and sufficient conditions

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Abstract

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of integrable functions on a σ-finite measure space $(\Omega, \mathscr{F}, \mu)$. Suppose that the pointwise limit $\lim_{n\uparrow\infty} f_n$ exists μ-a.e. and is integrable. In this setting we provide necessary and sufficient conditions for the following equality to hold:

$$\lim_{n\uparrow\infty} \int f_n \, d\mu = \int \lim_{n\uparrow\infty} f_n \, d\mu.$$

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1 Introduction

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of integrable functions $f : \Omega \to \overline{\mathbb{R}} \equiv [-\infty, \infty]$ such that the pointwise limit $\lim_{n\uparrow\infty} f_n$ exists μ-a.e. (μ-almost everywhere) and is integrable. One often wishes to show that

$$\lim_{n\uparrow\infty} \int f_n \, d\mu = \int \lim_{n\uparrow\infty} f_n \, d\mu. \tag{1.1}$$

In other words, one often wishes to interchange the limit $\lim_{n\uparrow\infty}$ and the integral $\int$. Various sufficient conditions for (1.1) are known, but only a small number of necessary and sufficient conditions are known in certain special cases. To our knowledge, no nontrivial necessary and sufficient condition for (1.1) applicable to the case that μ is σ-finite is available in the literature.

One of the best known sufficient conditions for (1.1) is that there exists an integrable function g such that $|f_n| \leq g$ for all $n \in \mathbb{N}$. Under this condition, (1.1) holds by the dominated convergence theorem (e.g., [8, p. 89]). The condition is generalized in [9, p. 315] to the condition that there exist sequences $\{\underline{f}_n\}$ and $\{\overline{f}_n\}$ such that $\underline{f}_n \leq f_n \leq \overline{f}_n$ for all $n \in \mathbb{N}$ and such that both $\{\underline{f}_n\}$ and $\{\overline{f}_n\}$ satisfy (1.1); the same result is shown in [5] and [1, p. 134]. The condition is also necessary for (1.1) in a trivial way since it holds with $\underline{f}_n = \overline{f}_n = f_n$ under (1.1).

It is shown in [7] that (1.1) holds if $\Omega \subset \mathbb{R}$, μ is the Lebesgue measure, and the functions f_n and $\lim_{n\uparrow\infty} f_n$ are probability densities on $\mathbb{R}$. More generally, if μ is σ-finite and the functions are nonnegative, a version of the Vitali convergence theorem provides a necessary and sufficient condition for (1.1) (see [8, p. 165]). In particular, uniform integrability of $\{f_n\}$ (defined in Sect. 4) is equivalent to (1.1) provided that μ is σ-finite and that $f_n \geq 0$ for all $n \in \mathbb{N}$. In the absence of this nonnegativity requirement, however, uniform integrability is only sufficient for (1.1), as shown in Sects. 4 and 5.

In the case that μ is finite, some necessary and sufficient conditions for (1.1) are given in [2, 3]. Those conditions are closely related, but not equivalent, to uniform integrability. Nevertheless, they are rarely mentioned in the current literature. We review one of the most general conditions in this case in Sect. 4.

In this paper we provide nontrivial necessary and sufficient conditions for (1.1) only under the assumption that μ is σ-finite. We demonstrate by example that our conditions are strictly weaker than uniform integrability. We express them using the concept of a σ-finite exhausting sequence, which is introduced in [4] to generalize Fatou's lemma. A σ-finite exhausting sequence is an increasing sequence of measurable sets of finite measure such that the complement of its entire union has measure 0. The basic idea of our conditions is to find a σ-finite exhausting sequence such that (1.1) can be verified on each of the sets in the sequence and such that the sequence of integrals outside these sets can be controlled asymptotically.

The rest of the paper is organized as follows. In the next section we present our main results, which we prove in Sect. 3. In Sect. 4 we review some of the known results on (1.1) mentioned above. In Sect. 5 we present a simple example to which the known results do not apply, but our results easily apply.

2 Main results

Throughout the paper, we let $(\Omega, \mathscr{F}, \mu)$ be a measure space. Let $\mathcal{L}$ be the set of measurable functions $f : \Omega \to \overline{\mathbb{R}}$ (with $\overline{\mathbb{R}}$ equipped with the Borel algebra). Let $\mathcal{L}^1$ be the set of integrable functions in $\mathcal{L}$. Let $\mathcal{L}^1_+ = \{f \in \mathcal{L}^1 : f \geq 0\}$.

We say that a sequence $\{A_i\}_{i\in\mathbb{N}}$ in $\mathscr{F}$ is *exhausting* if

$$\text{(i)} \quad \forall i \in \mathbb{N}, \quad A_i \subset A_{i+1}, \qquad \text{(ii)} \quad \mu\Bigl(\Omega \setminus \bigcup_{i\in\mathbb{N}} A_i\Bigr) = 0. \tag{2.1}$$

As in [4], we say that a sequence $\{A_i\}_{i\in\mathbb{N}}$ in $\mathscr{F}$ is a *σ-finite exhausting sequence* if it is exhausting and

$$\forall i \in \mathbb{N}, \quad \mu(A_i) < \infty. \tag{2.2}$$

It is easy to see that μ is σ-finite if and only if there exists a σ-finite exhausting sequence. Thus whenever μ is σ-finite, we have at least one σ-finite exhausting sequence.

In what follows, by "$f_n \to f$," we mean "$f_n \to f$ as $n \uparrow \infty$." We are ready to state our main results, which we prove in the next section.

Theorem 2.1 *Suppose that μ is σ-finite. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^1$ such that $f_n \to f$ μ-a.e. for some $f \in \mathcal{L}^1$. Then (1.1) holds if and only if there exists a σ-finite exhausting* *sequence* $\{A_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ *such that*

$$\forall i \in \mathbb{N}, \quad \overline{\lim_{n\uparrow\infty}} \sup_{\omega\in A_i} \left|f_n(\omega) - f(\omega)\right| < \infty, \tag{2.3}$$

$$\lim_{i\uparrow\infty} \overline{\lim_{n\uparrow\infty}} \left| \int_{\Omega\setminus A_i} f_n \, d\mu \right| = 0. \tag{2.4}$$

A simple sufficient condition for (2.3) is uniform convergence on each A_i; see (2.5) below. Condition (2.4) is somewhat similar to some of the conditions (such as uniform integrability) used in the known results reviewed in Sect. 4; see (4.3) and (4.5).

Theorem 2.2 *Under the hypotheses of Theorem* 2.1, (1.1) *holds if and only if any* σ*-finite exhausting sequence* $\{A_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ *satisfying* (2.3) *also satisfies* (2.4).

On the one hand, by Theorem 2.1, (1.1) can be established by constructing only one σ-finite exhausting sequence satisfying both (2.3) and (2.4). On the other hand, by Theorem 2.2, (1.1) can be disproved by constructing only one σ-finite exhausting sequence as well:

Corollary 2.1 *Under the hypotheses of Theorem* 2.1, (1.1) *does not hold if there exists a* σ*-finite exhausting sequence* $\{A_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ *satisfying* (2.3) *but violating* (2.4).

The following result shows that there always exists a σ-finite exhausting sequence satisfying (2.3).

Lemma 2.1 *Under the hypotheses of Theorem* 2.1, *there exists a* σ*-finite exhausting sequence* $\{A_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ *such that*

$$\forall i \in \mathbb{N}, \quad f_n \to f \quad \textit{uniformly on } A_i. \tag{2.5}$$

Hence there exists a σ*-finite exhausting sequence* $\{A_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ *satisfying* (2.3).

Proof Since μ is σ-finite, there is a σ-finite exhausting sequence $\{B_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$. Let $\{\epsilon_i\}_{i\in\mathbb{N}}$ be a sequence in $(0,\infty)$ with $\lim_{i\uparrow\infty} \epsilon_i = 0$. Let $i \in \mathbb{N}$. Recall that $f_n \to f$ μ-a.e.; thus by Egorov's theorem there exists $E_i \in \mathscr{F}$ such that $E_i \subset B_i$, $\mu(B_i \setminus E_i) < \epsilon_i$, and $f_n \to f$ uniformly on E_i.

For each $i \in \mathbb{N}$, define

$$A_i = \bigcup_{j=1}^{i} E_j \subset B_i. \tag{2.6}$$

Then $f_n \to f$ uniformly on A_i. Since this is true for any $i \in \mathbb{N}$, we obtain (2.5). By construction, we have

$$\forall i \in \mathbb{N}, \quad A_i \subset A_{i+1}, \qquad \mu(A_i) < \infty, \quad A_i \subset B_i, \tag{2.7}$$

$$\lim_{i\uparrow\infty} \mu(B_i \setminus A_i) = 0. \tag{2.8}$$

Hence it follows by [4, Lemma 8.4] that $\{A_i\}$ is a σ-finite exhausting sequence. Finally, the second conclusion holds since (2.5) implies (2.3). □

Lemma 2.1 is similar to Lusin's version of Egorov's theorem [6, p. 19], which shows that there exists a sequence $\{A_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ such that $\mu(\Omega \setminus \bigcup_{i\in\mathbb{N}} A_i) = 0$ and such that $f_n \to f$ uniformly on A_i for each $i \in \mathbb{N}$. Our result differs in that it explicitly shows that $\mu(A_i) < \infty$ for each $i \in \mathbb{N}$.

By Lemma 2.1, there exists a σ-finite exhausting sequence satisfying (2.3). Given such a sequence, (1.1) is equivalent to (2.4):

Corollary 2.2 *Under the hypotheses of Theorem* 2.1, *let* $\{A_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ *be a* σ-*finite exhausting sequence satisfying* (2.3) (*which exists by Lemma* 2.1). *Then* (1.1) *is equivalent to* (2.4).

Proof That (2.4) implies (1.1) is immediate from Theorem 2.1. The reverse implication follows from Theorem 2.2. □

3 Proof of Theorems 2.1 and 2.2

To simplify our arguments, we define

$$\Phi = \left\{\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{L}^1 : \exists f \in \mathcal{L}^1, f_n \to f \ \mu\text{-a.e.}\right\}. \tag{3.1}$$

Whenever a sequence $\{f_n\}$ from Φ is given, we let f be the μ-a.e. limit of $\{f_n\}$ given in (3.1). With this convention, (1.1) can be written as

$$\lim_{n\uparrow\infty} \int f_n \, d\mu = \int f \, d\mu. \tag{3.2}$$

We also define

$$\Psi = \left\{\{f_n\}_{n\in\mathbb{N}} \in \Phi : \{f_n\} \text{ satisfies } (3.2)\right\}. \tag{3.3}$$

We prove Theorems 2.1 and 2.2 by showing that the following four conditions are equivalent.

Condition 3.1 $\{f_n\} \in \Psi$.

Condition 3.2 There exists an exhausting sequence $\{A_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ satisfying (2.4) such that

$$\forall i \in \mathbb{N}, \quad \lim_{n\uparrow\infty} \int_{A_i} f_n \, d\mu = \int_{A_i} f \, d\mu. \tag{3.4}$$

Condition 3.3 There exists a σ-finite exhausting sequence $\{A_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ satisfying (2.3) and (2.4).

Condition 3.4 Any σ-finite exhausting sequence $\{A_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ satisfying (2.3) also satisfies (2.4).

Note that Conditions 3.3 and 3.4 are the "if and only if" conditions in Theorems 2.1 and 2.2, respectively. The following result, which does not assume that μ is σ-finite, shows that

Conditions 3.1 and 3.2 are equivalent, but Condition 3.2 is necessary for Condition 3.1 only in a trivial way.

Lemma 3.1 *Conditions 3.1 and 3.2 are equivalent.*

Proof Condition 3.1 implies Condition 3.2 with $A_i = \Omega$ for all $i \in \mathbb{N}$. Conversely, assume Condition 3.2. Let $\{A_i\}_{i\in\mathbb{N}}$ be an exhausting sequence in $\mathscr{F}$ satisfying (2.4) and (3.4). For any $i, n \in \mathbb{N}$, we have

$$\int f_n \, d\mu = \int_{A_i} f_n \, d\mu + \int_{\Omega\setminus A_i} f_n \, d\mu \tag{3.5}$$

$$\leq \int_{A_i} f_n \, d\mu + \left| \int_{\Omega\setminus A_i} f_n \, d\mu \right|. \tag{3.6}$$

Applying $\overline{\lim}_{n\uparrow\infty}$ and using (3.4), we have

$$\varlimsup_{n\uparrow\infty} \int f_n \, d\mu \leq \int_{A_i} f \, d\mu + \varlimsup_{n\uparrow\infty} \left| \int_{\Omega\setminus A_i} f_n \, d\mu \right|. \tag{3.7}$$

Applying $\underline{\lim}_{i\uparrow\infty}$ to the right-hand side and recalling that f is integrable, we obtain

$$\varlimsup_{n\uparrow\infty} \int f_n \, d\mu \leq \lim_{i\uparrow\infty} \int_{A_i} f \, d\mu + \varliminf_{i\uparrow\infty} \varlimsup_{n\uparrow\infty} \left| \int_{\Omega\setminus A_i} f_n \, d\mu \right| = \int f \, d\mu, \tag{3.8}$$

where the equality holds by integrability of f and (2.4).

Note from (3.5) that

$$\int f_n \, d\mu \geq \int_{A_i} f_n \, d\mu - \left| \int_{\Omega\setminus A_i} f_n \, d\mu \right|. \tag{3.9}$$

Applying $\underline{\lim}_{n\uparrow\infty}$ and then $\overline{\lim}_{i\uparrow\infty}$, and using (2.4), we obtain

$$\varliminf_{n\uparrow\infty} \int f_n \, d\mu \geq \int f \, d\mu. \tag{3.10}$$

This together with (3.8) implies Condition 3.1. □

It is easy to see that Theorems 2.1 and 2.2 follow from the following result.

Theorem 3.1 *Suppose that μ is σ-finite. Then Conditions 3.1–3.4 are equivalent.*

Proof We show this result by verifying the following chain of implications:

$$\begin{aligned} \text{Cond. 3.1} \quad &\Leftarrow \quad \text{Cond. 3.2} \quad \Leftarrow \quad \text{Cond. 3.3} \\ &\Leftarrow \quad \text{Cond. 3.4} \quad \Leftarrow \quad \text{Cond. 3.1}. \end{aligned} \tag{3.11}$$

Lemma 3.1 shows that Condition 3.2 implies Condition 3.1. To see that Condition 3.3 implies Condition 3.2, fix $i \in \mathbb{N}$ for the moment. Then (2.3) implies that there exists $\eta > 0$

such that

$$\forall \omega \in A_i, \quad f(\omega) - \eta < \varliminf_{n\uparrow\infty} f_n(\omega) \le \varlimsup_{n\uparrow\infty} f_n(\omega) < f(\omega) + \eta. \tag{3.12}$$

Thus by the dominated convergence theorem, we obtain

$$(2.2) \quad \text{and} \quad (2.3) \quad \Rightarrow \quad (3.4). \tag{3.13}$$

Hence Condition 3.3 implies Condition 3.2. By Lemma 2.1, there exists a σ-finite exhausting sequence $\{A_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ satisfying (2.3). Thus Condition 3.4 implies Condition 3.3. We have verified the first three implications in (3.11). It remains to show the last implication.

To this end, assume Condition 3.1, which implies (3.2). Let $\{A_i\}_{i\in\mathbb{N}}$ be any σ-finite exhausting sequence satisfying (2.3). It suffices to verify (2.4). Applying $\lim_{n\uparrow\infty}$ to (3.5) and recalling (3.13), we see that

$$\lim_{n\uparrow\infty} \int f_n \, d\mu = \int_{A_i} f \, d\mu + \lim_{n\uparrow\infty} \int_{\Omega\setminus A_i} f_n \, d\mu, \tag{3.14}$$

where the limit on the right-hand side exists since the limit on the left-hand side exists by (3.2). Since the left-hand side of (3.14) does not depend on i, we can apply $\lim_{i\uparrow\infty}$ to the right-hand side and use the integrability of f to get

$$\lim_{n\uparrow\infty} \int f_n \, d\mu = \int f \, d\mu + \lim_{i\uparrow\infty} \lim_{n\uparrow\infty} \int_{\Omega\setminus A_i} f_n \, d\mu. \tag{3.15}$$

Hence by (3.2) we have

$$\lim_{i\uparrow\infty} \lim_{n\uparrow\infty} \int_{\Omega\setminus A_i} f_n \, d\mu = 0. \tag{3.16}$$

Therefore

$$0 = \left| \lim_{i\uparrow\infty} \lim_{n\uparrow\infty} \int_{\Omega\setminus A_i} f_n \, d\mu \right| = \lim_{i\uparrow\infty} \lim_{n\uparrow\infty} \left| \int_{\Omega\setminus A_i} f_n \, d\mu \right|. \tag{3.17}$$

Thus (2.4) follows. This completes the proof. □

4 Known results

In this section we review some of the known results on (1.1) mentioned in Sect. 1. Throughout this section we take a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{L}$ as given and assume the following.

Assumption 4.1 $\{f_n\}_{n\in\mathbb{N}} \in \Phi$.

Essentially following [3], for $i \in \mathbb{N}$ and $\eta > 0$, define

$$S(i,\eta) = \bigl\{\omega \in \Omega : \forall n \ge i, \bigl|f_n(\omega) - f(\omega)\bigr| \le \eta\bigr\}, \tag{4.1}$$

$$C(i,\eta) = \Omega \setminus S(i,\eta). \tag{4.2}$$

In [3, p. 434] the following result is shown apparently under the implicit assumption that Ω is a subset of $\mathbb{R}$ or $\mathbb{R}^n$ of finite Lebesgue measure.

Theorem 4.1 *Suppose that μ is finite. Then $\{f_n\} \in \Psi$ if and only if*

$$\forall \eta > 0, \quad \lim_{i\uparrow\infty} \sup_{n\geq i} \left| \int_{C(i,\eta)} f_n \, d\mu \right| = 0. \tag{4.3}$$

Proof See [3, I]. □

The following result is a generalization of the dominated convergence theorem due to [9, p. 315].

Theorem 4.2 *$\{f_n\} \in \Psi$ if and only if there exist $\{\underline{f}_n\}_{n\in\mathbb{N}}, \{\overline{f}_n\}_{n\in\mathbb{N}} \subset \Psi$ such that*

$$\forall n \in \mathbb{N}, \quad \underline{f}_n \leq f_n \leq \overline{f}_n \quad \mu\text{-a.e.} \tag{4.4}$$

Proof The "only if" part trivially holds by setting $\underline{f}_n = \overline{f}_n = f$ for all $n \in \mathbb{N}$. For the "if" part, see [5] or [1, p. 134]. □

A sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{L}$ is called *uniformly integrable* ([8, p. 163]) if

$$\inf_{h\in\mathcal{L}^1_+} \sup_{n\in\mathbb{N}} \int_{|f_n|\geq h} |f_n| \, d\mu = 0. \tag{4.5}$$

See [8, Theorem 16.8] for various conditions equivalent to (4.5). The following result is an immediate implication of a version of the Vitali convergence theorem [8, Theorem 16.4].

Theorem 4.3 *Suppose that μ is σ-finite, and that $\{f_n\} \subset \mathcal{L}^1_+$. Then $\{f_n\} \in \Psi$ if and only if $\{f_n\}$ is uniformly integrable.*

Proof This follows from [8, Lemma 16.4, Theorem 16.6] with $p = 1$. □

Without the assumption that $\{f_n\} \subset \mathcal{L}^1_+$, uniform integrability is still sufficient for (1.1), as shown in the next result. We prove it here for completeness; we do not claim originality.

Theorem 4.4 *Suppose that μ is σ-finite. If $\{f_n\}$ is uniformly integrable, then $\{f_n\} \in \Psi$.*

Proof Suppose that $\{f_n\}$ is uniformly integrable. Then $\{|f_n|\}$ is uniformly integrable, and $\{|f_n|\} \in \Psi$ by Theorem 4.3. Since $-|f_n| \leq f_n \leq |f_n|$ for all $n \in \mathbb{N}$, it follows by Theorem 4.2 that $\{f_n\} \in \Psi$. □

In the next section we show by example that uniform integrability is not necessary for (1.1).

5 A simple example

In this section we present a simple example that satisfies (1.1) but violates (4.3) as well as uniform integrability. Let $\Omega = \mathbb{R}_+$. Let $\mathscr{F}$ be the σ-algebra of Lebesgue measurable subsets of Ω. Let μ be the Lebesgue measure restricted to $\mathscr{F}$. For $n \in \mathbb{N}$, define $f_n : \Omega \to \mathbb{R}$ by

$$f_n(\omega) = \begin{cases} -1/n & \text{if } \omega \in [n, n+n^2), \\ n & \text{if } \omega \in [n+n^2, n+n^2+1), \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Note that $f_n \to 0$ pointwise and that $\int f_n \, d\mu = 0$ for all $n \in \mathbb{N}$. Thus (1.1) holds.

Since μ is not finite here, Theorem 4.1 simply does not apply. Furthermore, the "if and only if" condition in the theorem, (4.3), does not hold either, as we demonstrate here. For this purpose, let $\eta > 0$, and let $i \in \mathbb{N}$ be such that $1/i < \eta < i$. Then

$$C(i,\eta) = \bigcup_{j=i}^{\infty} \bigl[j + j^2, j + j^2 + 1\bigr). \tag{5.2}$$

Since $\int_{C(i,\eta)} f_n \, d\mu = n$ for any $n \geq i$, we have

$$\sup_{n \geq i} \left| \int_{C(i,\eta)} f_n \, d\mu \right| = \infty. \tag{5.3}$$

Hence the equality in (4.3) never holds for any $\eta > 0$.

Let us now turn to the uniform integrability condition (4.5). Let $h \in \mathcal{L}^1_+$ and $n \in \mathbb{N}$. We have

$$\int_{|f_n| \geq h} |f_n| \, d\mu \geq \int_{n+n^2}^{n+n^2+1} \mathbb{1}\bigl\{n \geq h(\omega)\bigr\} n \, d\omega \tag{5.4}$$

$$= n\left(1 - \int_{n+n^2}^{n+n^2+1} \mathbb{1}\bigl\{h(\omega) > n\bigr\} \, d\omega\right), \tag{5.5}$$

where $\mathbb{1}\{\cdot\}$ is the indicator function. Note that $h(\omega) \geq 1$ if $h(\omega) > n$ (≥ 1), and that $h(\omega) \geq 0$ otherwise; thus $\mathbb{1}\{h(\omega) > n\} \leq h(\omega)$ for all $\omega \in \Omega$. Hence

$$\lim_{n \uparrow \infty} \int_{n+n^2}^{n+n^2+1} \mathbb{1}\bigl\{h(\omega) > n\bigr\} \, d\omega \leq \lim_{n \uparrow \infty} \int_{n+n^2}^{n+n^2+1} h(\omega) \, d\omega = 0, \tag{5.6}$$

where the equality holds by integrability of h. From (5.4)–(5.6), we have

$$\sup_{n \in \mathbb{N}} \int_{|f_n| \geq h} |f_n| \, d\mu = \infty. \tag{5.7}$$

Since this is true for any $h \in \mathcal{L}^1_+$, it follows that $\{f_n\}$ is not uniformly integrable.

To see that our results apply even to this example, let $A_i = [0, i)$ for $i \in \mathbb{N}$. Then $\{A_i\}_{i \in \mathbb{N}}$ is a σ-finite exhausting sequence. Note that for each $i \in \mathbb{N}$, we have $f_n = f = 0$ on A_i for all $n \geq i$. Thus both (2.3) and (3.4) trivially hold. For any $i \in \mathbb{N}$, we have

$$\forall n \geq i, \quad \int_{\Omega \setminus A_i} f_n \, d\mu = \int_{[i,\infty)} f_n \, d\mu = 0. \tag{5.8}$$

Hence (2.4) also holds. It follows that both Conditions 3.2 and 3.3 hold, so that Theorems 2.1 and 3.1 apply here.

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Abbreviations
a.e., almost everywhere.

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