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Spacelike submanifolds with parallel mean curvature vector in a de Sitter space $S_q^{n+p}(c)$

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Abstract

Spacelike submanifolds usually appear in the study of questions related to causality in general relativity. In this paper, we study an n-dimensional spacelike submanifold in $(n+p)$-dimensional connected de Sitter space $S_q^{n+p}(c)$ of index q $(1 \le q \le p)$ and of constant curvature c, and we obtain some integral inequalities of Simons type and rigidity theorems.

MSC: 53A40; 53B25

Keywords: De Sitter space; Spacelike submanifold; Totally umbilical; Rigidity theorems

1 Introduction

During the last decades, the study of spacelike submanifolds in semi-Riemannian manifolds has got increasing interest motivated by their importance in problems related to Physics, such as the theory of general relativity. Furthermore, the unique properties of spacelike submanifolds are of great significance for solving the Cauchy initial value problem of hypersurfaces and the propagation of gravity in arbitrary space-time (see, for example, [1–3]). Therefore, many authors have focused on the development of spacelike submanifolds in semi-Riemannian manifolds; see, for example, [4–7] and the reference therein.

Let M be a finite dimensional manifold, we assume that M can be endowed with a Riemannian metric to become a Riemannian manifold. The structure and pinching problem of some special submanifolds such as totally geodesic submanifolds, minimal submanifolds, submanifolds with parallel mean curvature vector and totally umbilical submanifolds are the research focus of submanifolds on Riemannian manifolds. The pinching problem of submanifolds is to restrict norm square of the second fundamental form, sectional curvature, Ricci curvature and scalar curvature of submanifolds, so as to obtain some special properties.

Let $N_q^{n+p}(c)$ be an $(n+p)$-dimensional connected semi-Riemannian manifold with constant curvature c, and of index q, where $1 \le q \le p$. It is called an indefinite space form of index q. More specifically, it may be considered, up to isometries, as de Sitter space $S_q^{n+p}(c)$, semi-Euclidean space R_q^{n+p}, and semi-hyperbolic space $H_q^{n+p}(c)$, if $c > 0$, $c = 0$, and

$c < 0$, respectively. A submanifold immersed in $N_q^{n+p}(c)$ is said to be spacelike, timelike, lightlike, if the metric in M^n induced by that of the ambient space $N_q^{n+p}(c)$ is positive definite, negative definite, vanishing, respectively. As is usual, the spacelike submanifold is said to be complete if the Riemannian induced metric is a complete metric. For further details, see [8].

Let $\varphi : M \to N_q^{n+p}(c)$ be an n-dimensional spacelike submanifold in $N_q^{n+p}(c)$. If $q = p = 1$, this initial step in this context is due to Goddard's conjecture in 1977 (see [9]) that complete spacelike hypersurfaces of S_1^{n+1} with constant mean curvature H must be totally umbilical. In order to prove this conjecture, many researchers began to study spacelike submanifolds in constant curvature space. Although the conjecture turned out to be false in its original statement, it motivated a great deal of work of several authors trying to find a positive answer to the conjecture under appropriate additional hypotheses (see [10–15]). In the case of higher codimension (i.e. $q = p > 1$) in $N_q^{n+p}(c)$, several fruitful results have been achieved in recent years. Among them, Cheng [16] extended Akugatawa's result [10] to complete spacelike submanifolds with parallel mean curvature vector fields in de Sitter space S_p^{n+p}. Li [17] proved that the conclusion of Motiel [12] is still valid in spacelike submanifolds. For relevant conclusions, refer to [18–26].

When $q = p$, we note here that complete maximal spacelike submanifold M in $N_p^{n+p}(c)$ is totally geodesic for $c \geq 0$ (see [27]). Thus the class of all such submanifolds is very small. While if $0 \leq q < p$, and M is a complete minimal submanifold in sphere $S^m(c)$ $(m > n)$, which is embedded in $S_q^{m+q}(c)$ as a totally geodesic spacelike submanifold such that $m - n + q = p$, we know from [28] and [29] that M is a complete maximal spacelike submanifold in $S_q^{n+p}(c)$. This implies that the class of complete maximal spacelike submanifold in $S_q^{n+p}(c)$ is very large. From the above discussion, it is necessary and important to study the classification of spacelike submanifold in $S_q^{n+p}(c)$ $(1 \leq q < p)$. But to the best of our knowledge, the progress of this research topic is slow.

There are several authors have tried relevant topic and obtained some important properties. By calculating the divergence of certain tangent vector fields and using the divergence theorem, Alías and Romero [28] proved an integral formula for the compact spacelike n-dimensional submanifolds in a de Sitter spaces $S_q^{n+p}(c)$ $(1 \leq q < p)$, and obtained a Bernstein type result for the complete maximal submanifolds in $S_q^{n+p}(c)$ $(1 \leq q < p)$. Cheng and Ishikawa [29] studied compact maximal spacelike submanifold in $S_q^{n+p}(c)$ $(1 \leq q < p)$, and obtained some important results in terms of the pinching conditions on scalar curvature, sectional curvature and Ricci curvature, respectively. Under the assumption that the second fundamental form of M is locally timelike, Mariano [30] obtained some results of complete spacelike submanifold with parallel mean curvature vector in $S_q^{n+p}(c)$ $(1 \leq q < p)$. And Yang and Li [31] obtained some classification results for spacelike submanifold in $S_q^{n+p}(c)$ $(1 \leq q < p)$, but they not only assume the mean curvature vector is parallel but also it is spacelike or timelike.

Inspired and motivated by the research work above, in this paper, only assuming the mean curvature vector is parallel, we continue to study this topic and prove some integral inequalities of Simons' type and rigidity theorems for n-dimensional spacelike submanifolds in a de Sitter space $S_q^{n+p}(c)$ $(1 \leq q < p)$, which is a further generalization of the results obtained in [29].

It should be noted that Simons' method [32] is widely used in minimal hypersurfaces, or hypersurfaces with constant mean curvature H in S^{n+1}. Many rigidity results are obtained by using Simons' method (see, for example, [32–36]) .

Next, we will make a brief introduction to the main results present in this paper. We denote by ρ^2 the nonnegative function $\rho^2 = S - nH^2$, where S and H are the norm square of the second fundamental form and the mean curvature vector of M, we see that $\rho^2 = 0$ if and only if M is a totally umbilical spacelike submanifold. We also denote by K and Q the functions which assign to each point of M the infimum of the sectional curvature and the Ricci curvature at the point, we will present the following theorems.

Theorem 1.1 *Let $\varphi : M \to S_q^{n+p}(c)$ $(1 \le q < p)$ be an n $(n \ge 2)$-dimensional compact spacelike submanifold in a de Sitter space $S_q^{n+p}(c)$ with parallel mean curvature vector. Then the following integral inequality holds*:

$$\int_M \rho^2 \left\{ a\rho^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} H\rho - n\left(c - H^2\right) \right\} dv \ge 0,$$

where $a = 1$ if $p - q = 1$, and $a = \frac{3}{2}$ if $p - q > 1$.

In particular, if $\rho^2 \le b_+^2(a, H, c)$ and $H^2 \le c$, where $b_+(a, H, c)$ is the nonnegative root of the multinomial

$$P(x, H, a) = ax^2 + \frac{n(n-2)}{\sqrt{n(n-1)}} Hx - n\left(c - H^2\right),$$

then

(i) *if $p - q = 1$, M is totally umbilical, or M lies in the totally geodesic spacelike submanifold $S^{n+1}(c)$ of $S_q^{n+q+1}(c)$ and is isometric to the Clifford torus $S^k(\frac{n}{k}c) \times S^{n-k}(\frac{n}{n-k}c)$;*

(ii) *if $p - q > 1$, M is totally umbilical, or $n = 2$, $p - q = 2$, M lies in the totally geodesic spacelike submanifold $S^4(c)$ of $S_q^{4+q}(c)$ and is isometric to the Veronese surface.*

Theorem 1.2 *Let $\varphi : M \to S_q^{n+p}(c)$ $(1 \le q < p)$ be an n $(n \ge 2)$-dimensional compact spacelike submanifold in a de Sitter space $S_q^{n+p}(c)$ with parallel mean curvature vector. Then the following integral inequality holds*:

$$\int_M \rho^2 \left\{ nK - \left(1 - \frac{1}{p-q}\right)\rho^2 \right\} dv \le 0.$$

In particular, if

$$K \ge \frac{1}{n}\left(1 - \frac{1}{p-q}\right)\rho^2,$$

then M is totally umbilical, or M is a spacelike submanifold with parallel second fundamental form.

Theorem 1.3 *Let $\varphi : M \to S_q^{n+p}(c)$ $(1 \le q < p)$ be an n $(n \ge 2)$-dimensional compact spacelike submanifold in a de Sitter space $S_q^{n+p}(c)$ with parallel mean curvature vector. Then the* *following integral inequality holds*:

$$\int_M \rho^2 \left\{ Q - (n-2)c - nH^2 - \frac{n-2}{\sqrt{n(n-1)}} H\rho - \frac{1}{n}\left(3 - \frac{p+q}{(p-q)q}\right) \right\} dv \leq 0.$$

In particular, if

$$Q \geq (n-2)c + nH^2 + \frac{n-2}{\sqrt{n(n-1)}} H\rho + \frac{1}{n}\left(3 - \frac{p+q}{(p-q)q}\right),$$

then M is totally umbilical, or M is a maximal Einstein submanifold with parallel second fundamental form, and the Ricci curvature

$$\mathrm{Ric}(M) = (n-2)c + \frac{1}{n}\left(3 - \frac{p+q}{(p-q)q}\right).$$

From Theorem 1.3, we also have the following corollary.

Corollary 1.4 *Let $\varphi : M \to S_q^{n+p}(c)$ $(1 \leq q < p)$ be an n $(n \geq 2)$-dimensional compact maximal spacelike submanifold in a de Sitter space $S_q^{n+p}(c)$. Then the following integral inequality holds*:

$$\int_M \rho^2 \left\{ Q - (n-2)c - \frac{1}{n}\left(3 - \frac{p+q}{(p-q)q}\right) \right\} dv \leq 0.$$

In particular, if

$$Q \geq (n-2)c + \frac{1}{n}\left(3 - \frac{p+q}{(p-q)q}\right),$$

then M is totally geodesic, or M is a maximal Einstein submanifold with Ricci curvature

$$\mathrm{Ric}(M) = (n-2)c + \frac{1}{n}\left(3 - \frac{p+q}{(p-q)q}\right).$$

Remark 1 If $H = 0$, i.e. M is maximal, we see that the second part of Theorem 1.1, Theorem 1.2, and Corollary 1.4 are reduced to Theorem 1, Theorem 2 (if $p - q = 1$) and Theorem 3 (if $p = 2$, $q = 1$) of [29], respectively. Thus, we generalize the results of [29] to spacelike submanifold with parallel mean curvature vector for any $1 \leq q < p$.

2 Preliminaries

In this section, we will introduce some basic facts and notations that will appear on the paper. Let $N_q^{n+p}(c)$ be an $(n+p)$-dimensional indefinite space form with index q $(1 \leq q \leq p)$. Let M be an n-dimensional connected spacelike submanifold immersed in $N_q^{n+p}(c)$. We choose a local field of semi-Riemannian orthonormal frames $e_1, \dots, e_{n+p}$ in $N_q^{n+p}(c)$, such that at each point of M, $e_1, \dots, e_n$ span the tangent space of M and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leq A, B, C, \ldots \leq n+p, \qquad 1 \leq i, j, k, \ldots \leq n, \qquad n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p.$$

Let $\omega_1, \dots, \omega_{n+p}$ be its dual frame field, so that the semi-Riemannian metric of $N_q^{n+p}(c)$ is given by $d\bar{s}^2 = \sum_A \varepsilon_A \omega_A^2$, where

$$\varepsilon_A = \begin{cases} 1, & 1 \le A \le n+p-q, \\ -1, & n+p-q+1 \le A \le n+p. \end{cases}$$

Then the structure equations of $N_q^{n+p}(c)$ are given by (see [29])

$$d\omega_A = -\sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D K_{ABCD} \omega_C \wedge \omega_D,$$

$$K_{ABCD} = c\varepsilon_A \varepsilon_B (\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}).$$

If we restrict these form to M, then $\omega_\alpha = 0$ $(n+1 \le \alpha \le n+p)$, and

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \qquad h_{ij}^\alpha = h_{ji}^\alpha.$$

The second fundamental form II, the mean curvature vector $\vec{H}$ of M are defined by

$$II = \sum_{\alpha,i,j} \varepsilon_\alpha h_{ij}^\alpha \omega_i \omega_j e_\alpha, \qquad \vec{H} = \sum_\alpha \varepsilon_\alpha H^\alpha e_\alpha, \qquad H^\alpha = \frac{1}{n} \sum_k h_{kk}^\alpha.$$

The norm square of the second fundamental form and the mean curvature of M are defined by

$$S = |II|^2 = \sum_{i,j,\alpha} \left(\varepsilon_\alpha h_{ij}^\alpha\right)^2 = \sum_{i,j,\alpha} \left(h_{ij}^\alpha\right)^2, \qquad H = |\vec{H}| = \sqrt{\sum_\alpha \left(H^\alpha\right)^2}.$$

The Gauss equations are

$$R_{ijkl} = c(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}) + \sum_\alpha \varepsilon_\alpha \left(h_{il}^\alpha h_{jk}^\alpha - h_{ik}^\alpha h_{jl}^\alpha\right), \tag{2.1}$$

$$R_{jk} = (n-1)c\delta_{jk} + \sum_\alpha \varepsilon_\alpha \left(\sum_i h_{ii}^\alpha h_{jk}^\alpha - \sum_i h_{ik}^\alpha h_{ji}^\alpha\right). \tag{2.2}$$

Defining the first and the second covariant derivatives of h_{ij}^α, say h_{ijk}^α and h_{ijkl}^α by

$$\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha - \sum_k h_{ik}^\alpha \omega_{kj} - \sum_k h_{jk}^\alpha \omega_{ki} - \sum_\beta \varepsilon_\beta h_{ij}^\beta \omega_{\beta\alpha}, \tag{2.3}$$

$$\sum_l h_{ijkl}^\alpha \omega_l = dh_{ijk}^\alpha - \sum_m h_{mjk}^\alpha \omega_{mi} - \sum_m h_{imk}^\alpha \omega_{mj} - \sum_m h_{ijm}^\alpha \omega_{mk} - \sum_\beta \varepsilon_\beta h_{ijk}^\beta \omega_{\beta\alpha}, \tag{2.4}$$

we have the Codazzi equations and the Ricci identities

$$h_{ijk}^\alpha = h_{ikj}^\alpha, \tag{2.5}$$

$$h^\alpha_{ijkl} - h^\alpha_{ijlk} = -\sum_m h^\alpha_{im} R_{mjkl} - \sum_m h^\alpha_{jm} R_{mikl} - \sum_\beta \varepsilon_\beta h^\beta_{ij} R_{\beta\alpha kl}. \tag{2.6}$$

The Ricci equations are

$$R_{\alpha\beta ij} = -\sum_m \left(h^\alpha_{im} h^\beta_{mj} - h^\alpha_{jm} h^\beta_{mi} \right). \tag{2.7}$$

The Laplacian of h^α_{ij} is defined by $\Delta h^\alpha_{ij} = \sum_k h^\alpha_{ijkk}$. From (2.6), we obtain for any α $(n+1 \leq \alpha \leq n+p)$,

$$\Delta h^\alpha_{ij} = \sum_k h^\alpha_{kkij} - \sum_{k,m} h^\alpha_{km} R_{mijk} - \sum_{k,m} h^\alpha_{im} R_{mkjk} - \sum_{k,\beta} \varepsilon_\beta h^\beta_{ik} R_{\beta\alpha jk}.$$

We need the following lemma (see [37]).

Lemma 2.1 *Let A, B be symmetric $n \times n$ matrices satisfying $AB = BA$ and $\operatorname{tr} A = \operatorname{tr} B = 0$. Then*

$$\left| \operatorname{tr} A^2 B \right| \leq \frac{n-2}{\sqrt{n(n-1)}} \left(\operatorname{tr} A^2 \right) \left(\operatorname{tr} B^2 \right)^{1/2},$$

and the equality holds if and only if $(n-1)$ of the eigenvalues x_i of B, and the corresponding eigenvalues y_i of A satisfy $|x_i| = (\operatorname{tr} B^2)^{1/2}/\sqrt{n(n-1)}$, $x_i x_j \geq 0$, $y_i = (\operatorname{tr} A^2)^{1/2}/\sqrt{n(n-1)}$.

3 Basic formulas

This section introduces some basic formulas which plays a crucial role in the proof of the theorems in this paper. Define the tensors

$$\tilde{h}^\alpha_{ij} = h^\alpha_{ij} - H^\alpha \delta_{ij}, \tag{3.1}$$

$$\tilde{\sigma}_{\alpha\beta} = \sum_{i,j} \tilde{h}^\alpha_{ij} \tilde{h}^\beta_{ij}, \qquad \sigma_{\alpha\beta} = \sum_{i,j} h^\alpha_{ij} h^\beta_{ij}. \tag{3.2}$$

Then the $(p \times p)$-matrix $(\tilde{\sigma}_{\alpha\beta})$ is symmetric and can be assumed to be diagonalized for a suitable choice of $e_{n+1}, \ldots, e_{n+p}$. We set

$$\tilde{\sigma}_{\alpha\beta} = \tilde{\sigma}_\alpha \delta_{\alpha\beta}. \tag{3.3}$$

By a direct calculation, we have

$$\sum_k \tilde{h}^\alpha_{kk} = 0, \qquad \tilde{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta} - nH^\alpha H^\beta, \qquad \rho^2 = \sum_\alpha \tilde{\sigma}_\alpha = S - nH^2, \tag{3.4}$$

where $\tilde{\sigma}_\alpha = \tilde{\sigma}_{\alpha\alpha}$.

From (2.5) and (2.6), we have

$$\begin{aligned}\frac{1}{2}\Delta S &= \sum_{i,j,k,\alpha}\left(h_{ijk}^{\alpha}\right)^2 + \sum_{i,j,\alpha} h_{ij}^{\alpha}\Delta h_{ij}^{\alpha} \\ &= |\nabla h|^2 + \sum_{i,j,\alpha} h_{ij}^{\alpha}\left(nH^{\alpha}\right)_{,ij} - \sum_{\alpha}\sum_{i,j,k,l} h_{ij}^{\alpha}h_{kl}^{\alpha}R_{lijk} \\ &\quad - \sum_{\alpha}\sum_{i,j,k,l} h_{ij}^{\alpha}h_{li}^{\alpha}R_{lkjk} - \sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_{\beta}h_{ij}^{\alpha}h_{ki}^{\beta}R_{\beta\alpha jk}. \end{aligned} \tag{3.5}$$

In general, for a matrix $A = (a_{ij})$ we denote by $N(A)$ the square of the norm of A, that is, $N(A) = \operatorname{trace}(A\cdot A^t) = \sum_{i,j}(a_{ij})^2$. Clearly, $N(A) = N(T^tAT)$ for any orthogonal matrix T. From (2.7), we have

$$\begin{aligned} -\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_{\beta}h_{ij}^{\alpha}h_{ki}^{\beta}R_{\beta\alpha jk} &= -\sum_{\alpha,\beta}\sum_{i,j,k,l}\varepsilon_{\beta}h_{ij}^{\alpha}h_{ki}^{\beta}\left(h_{kl}^{\beta}h_{lj}^{\alpha} - h_{jl}^{\beta}h_{lk}^{\alpha}\right) \\ &= -\frac{1}{2}\sum_{\alpha,\beta,j,k}\varepsilon_{\beta}\left(\sum_{l} h_{kl}^{\beta}h_{lj}^{\alpha} - \sum_{l} h_{kl}^{\alpha}h_{lj}^{\beta}\right)^2 \\ &= -\frac{1}{2}\sum_{\alpha,\beta,j,k}\varepsilon_{\beta}\left(\sum_{l} \tilde{h}_{kl}^{\beta}\tilde{h}_{lj}^{\alpha} - \sum_{l} \tilde{h}_{kl}^{\alpha}\tilde{h}_{lj}^{\beta}\right)^2 \\ &= -\frac{1}{2}\sum_{\alpha,\beta}\varepsilon_{\beta}N(\tilde{A}_{\alpha}\tilde{A}_{\beta} - \tilde{A}_{\beta}\tilde{A}_{\alpha}), \end{aligned} \tag{3.6}$$

where $\tilde{A}_{\alpha} := (\tilde{h}_{ij}^{\alpha}) = (h_{ij}^{\alpha} - H^{\alpha}\delta_{ij})$.

Combining (2.1), (2.7), (3.1) (3.2), (3.4) and (3.6), we conclude that

$$\begin{aligned} &-\sum_{\alpha}\sum_{i,j,k,l} h_{ij}^{\alpha}\left(h_{kl}^{\alpha}R_{lijk} + h_{li}^{\alpha}R_{lkjk}\right) \\ &= nc\rho^2 - \sum_{\alpha,\beta}\varepsilon_{\beta}\sigma_{\alpha\beta}^2 + n\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_{\beta}H^{\beta}h_{kj}^{\beta}h_{ij}^{\alpha}h_{ik}^{\alpha} - \sum_{\alpha,\beta,i,j,k}\varepsilon_{\beta}h_{ji}^{\alpha}h_{ik}^{\beta}R_{\beta\alpha jk} \\ &= nc\rho^2 - \sum_{\alpha,\beta}\varepsilon_{\beta}\tilde{\sigma}_{\alpha\beta}^2 - 2n\sum_{\alpha,\beta}\sum_{i,j}\varepsilon_{\beta}H^{\alpha}H^{\beta}\tilde{h}_{ij}^{\alpha}\tilde{h}_{ij}^{\beta} - n^2\sum_{\alpha}\left(H^{\alpha}\right)^2\sum_{\beta}\varepsilon_{\beta}\left(H^{\beta}\right)^2 \\ &\quad + n\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_{\beta}H^{\beta}\tilde{h}_{kj}^{\beta}\tilde{h}_{ij}^{\alpha}\tilde{h}_{ik}^{\alpha} + n\rho^2\sum_{\beta}\varepsilon_{\beta}\left(H^{\beta}\right)^2 + 2n\sum_{\alpha,\beta}\sum_{i,j}\varepsilon_{\beta}H^{\alpha}H^{\beta}\tilde{h}_{ij}^{\alpha}\tilde{h}_{ij}^{\beta} \\ &\quad + n^2\sum_{\alpha}\left(H^{\alpha}\right)^2\sum_{\beta}\varepsilon_{\beta}\left(H^{\beta}\right)^2 - \frac{1}{2}\sum_{\alpha,\beta}\varepsilon_{\beta}N(\tilde{A}_{\alpha}\tilde{A}_{\beta} - \tilde{A}_{\beta}\tilde{A}_{\alpha}) \\ &= nc\rho^2 - \sum_{\alpha,\beta}\varepsilon_{\beta}\tilde{\sigma}_{\alpha\beta}^2 + n\rho^2\sum_{\beta}\varepsilon_{\beta}\left(H^{\beta}\right)^2 + n\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_{\beta}H^{\beta}\tilde{h}_{kj}^{\beta}\tilde{h}_{ij}^{\alpha}\tilde{h}_{ik}^{\alpha} \\ &\quad - \frac{1}{2}\sum_{\alpha,\beta}\varepsilon_{\beta}N(\tilde{A}_{\alpha}\tilde{A}_{\beta} - \tilde{A}_{\beta}\tilde{A}_{\alpha}). \end{aligned} \tag{3.7}$$

Since the mean curvature vector is parallel, that is, $|\nabla^{\perp}\vec{H}|^2 = \sum_{i,\alpha}(H_{,i}^{\alpha})^2 = 0$, we see that $H_{,i}^{\alpha} = 0$ for all i, α and H^{α} are constant for all α, this implies that H is constant. Putting

(3.6) and (3.7) into (3.5), we have

$$\frac{1}{2}\Delta\rho^2 = |\nabla h|^2 + nc\rho^2 + n\rho^2\sum_{\beta}\varepsilon_\beta\left(H^\beta\right)^2 + n\sum_{\alpha,\beta}\sum_{i,j,k}\varepsilon_\beta H^\beta\tilde{h}^\beta_{kj}\tilde{h}^\alpha_{ij}\tilde{h}^\alpha_{ik} - \sum_{\alpha,\beta}\varepsilon_\beta N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) - \sum_{\alpha,\beta}\varepsilon_\beta\tilde{\sigma}^2_{\alpha\beta}. \tag{3.8}$$

4 Proofs of theorems

Proof of Theorem 1.1 We first have the following:

$$\begin{aligned} n\rho^2\sum_{\beta}\varepsilon_\beta\left(H^\beta\right)^2 &= n\rho^2\sum_{\beta=n+1}^{n+p-q}\left(H^\beta\right)^2 - n\rho^2\sum_{\beta=n+p-q+1}^{n+p}\left(H^\beta\right)^2 \\ &= 2n\rho^2\sum_{\beta=n+1}^{n+p-q}\left(H^\beta\right)^2 - n\rho^2\sum_{\beta=n+1}^{n+p}\left(H^\beta\right)^2 \geq -n\rho^2H^2. \end{aligned} \tag{4.1}$$

We use $\sum_i(\tilde{h}^\beta_{ii})^2 = \tilde{\sigma}_\beta$, $\sum_i\tilde{h}^\beta_{ii} = 0$, $\sum_i\mu^\alpha_i = 0$, and $\sum_i(\mu^\alpha_i)^2 = \tilde{\sigma}_\alpha$. It follows from Lemma 2.1 that

$$\begin{aligned} & n\sum_{\alpha,\beta,i,j,k}\varepsilon_\beta H^\beta\tilde{h}^\beta_{kj}\tilde{h}^\alpha_{ij}\tilde{h}^\alpha_{ik} \\ &= n\sum_{\alpha,i,j,k}\sum_{\beta=n+1}^{n+p-q}H^\beta\tilde{h}^\beta_{kj}\tilde{h}^\alpha_{ij}\tilde{h}^\alpha_{ik} - n\sum_{\alpha,i,j,k}\sum_{\beta=n+p-q+1}^{n+p}H^\beta\tilde{h}^\beta_{kj}\tilde{h}^\alpha_{ij}\tilde{h}^\alpha_{ik} \\ &= n\sum_{\alpha,i}\sum_{\beta=n+1}^{n+p-q}H^\beta\tilde{h}^\beta_{ii}\left(\mu^\alpha_i\right)^2 - n\sum_{\alpha,i}\sum_{\beta=n+p-q+1}^{n+p}H^\beta\tilde{h}^\beta_{ii}\left(\mu^\alpha_i\right)^2 \\ &\geq -\frac{n(n-2)}{\sqrt{n(n-1)}}\sum_{\alpha}\sum_{\beta=n+1}^{n+p-q}\left|H^\beta\right|\tilde{\sigma}_\alpha\sqrt{\tilde{\sigma}_\beta} - \frac{n(n-2)}{\sqrt{n(n-1)}}\sum_{\alpha}\sum_{\beta=n+p-q+1}^{n+p}\left|H^\beta\right|\tilde{\sigma}_\alpha\sqrt{\tilde{\sigma}_\beta} \\ &= -\frac{n(n-2)}{\sqrt{n(n-1)}}\sum_{\alpha}\tilde{\sigma}_\alpha\left(\sum_{\beta=n+1}^{n+p-q}\left|H^\beta\right|\sqrt{\tilde{\sigma}_\beta} + \sum_{\beta=n+p-q+1}^{n+p}\left|H^\beta\right|\sqrt{\tilde{\sigma}_\beta}\right) \\ &= -\frac{n(n-2)}{\sqrt{n(n-1)}}\sum_{\alpha}\tilde{\sigma}_\alpha\sum_{\beta=n+1}^{n+p}\left|H^\beta\right|\sqrt{\tilde{\sigma}_\beta} \\ &\geq -\frac{n(n-2)}{\sqrt{n(n-1)}}\rho^2\left(\sqrt{\sum_{\beta}\left(H^\beta\right)^2\sum_{\beta}\tilde{\sigma}_\beta}\right) \\ &= -\frac{n(n-2)}{\sqrt{n(n-1)}}H\rho^3. \end{aligned} \tag{4.2}$$

And

$$\begin{aligned} & -\sum_{\alpha,\beta}\varepsilon_\beta N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) - \sum_{\alpha,\beta}\varepsilon_\beta\tilde{\sigma}^2_{\alpha\beta} \\ &= -\sum_{\alpha}\sum_{\beta=n+1}^{n+p-q}N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) + \sum_{\alpha}\sum_{\beta=n+p-q+1}^{n+p}N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) \end{aligned}$$

$$
\begin{aligned}
&-\sum_{\alpha}\sum_{\beta=n+1}^{n+p-q}\tilde{\sigma}_{\alpha\beta}^2+\sum_{\alpha}\sum_{\beta=n+p-q+1}^{n+p}\tilde{\sigma}_{\alpha\beta}^2\\
&=-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}\tilde{\sigma}_{\alpha\beta}^2\\
&\quad+\sum_{\alpha=n+p-q+1}^{n+p}\sum_{\beta=n+p-q+1}^{n+p}N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)+\sum_{\alpha=n+p-q+1}^{n+p}\sum_{\beta=n+p-q+1}^{n+p}\tilde{\sigma}_{\alpha\beta}^2\\
&\geq-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}\tilde{\sigma}_{\alpha\beta}^2,
\end{aligned}
\tag{4.3}
$$

where the inequality $N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)\geq 0$ for any α, β is used.

If $p-q=1$, we have

$$
-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}\tilde{\sigma}_{\alpha\beta}^2=-\tilde{\sigma}_{n+1n+1}^2\geq-\rho^4. \tag{4.4}
$$

If $p-q>1$, from Anmin and Jimin [38, Lemma 1], we have

$$
-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}N(\tilde{A}_\alpha\tilde{A}_\beta-\tilde{A}_\beta\tilde{A}_\alpha)-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q}\tilde{\sigma}_{\alpha\beta}^2\geq-\frac{3}{2}\left(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha^2\right)\geq-\frac{3}{2}\rho^4. \tag{4.5}
$$

From (3.8), (4.1)–(4.5), we have

$$
\frac{1}{2}\Delta\rho^2\geq|\nabla h|^2+\rho^2\left\{n\left(c-H^2\right)-\frac{n(n-2)}{\sqrt{n(n-1)}}H\rho-a\rho^2\right\}, \tag{4.6}
$$

where $a=1$ if $p-q=1$ and $a=\frac{3}{2}$ if $p-q>1$.

From the Stokes formula, we get

$$
\int_M\rho^2\left\{a\rho^2+\frac{n(n-2)}{\sqrt{n(n-1)}}H\rho-n\left(c-H^2\right)\right\}dv\geq 0. \tag{4.7}
$$

In particular, if $\rho^2\leq b_+^2(a,H,c)$ and $H^2\leq c$, since $b_+(a,H,c)$ is the nonnegative root of the multinomial $P(x,H,a)=ax^2+\frac{n(n-2)}{\sqrt{n(n-1)}}Hx-n(c-H^2)$, we easily see that $a\rho^2+\frac{n(n-2)}{\sqrt{n(n-1)}}H\rho-n(c-H^2)\leq 0$. From (4.7), we have $\rho^2=0$ and M is totally umbilical, or $a\rho^2+\frac{n(n-2)}{\sqrt{n(n-1)}}H\rho-n(c-H^2)=0$. In the latter case, if $\rho^2=0$, then M is totally umbilical, if $\rho^2\neq 0$, we see that the equalities in (4.6), (4.1)–(4.3) hold. Thus, we have

$$
\sum_{\beta=n+1}^{n+p-q}\left(H^\beta\right)^2=0,\qquad\sum_{\beta}\left|H^\beta\right|\sqrt{\tilde{\sigma}_\beta}=H\rho,\qquad\sum_{\alpha=n+p-q+1}^{n+p}\sum_{\beta=n+p-q+1}^{n+p}\tilde{\sigma}_{\alpha\beta}^2=0.
$$

This implies that $H^\beta=0$ for $\beta=n+1,\ldots,n+p-q$, and $\tilde{\sigma}_\beta=0$ for $\beta=n+p-q+1,\ldots,n+p$. Therefore, we get $H\rho=\sum_\beta|H^\beta|\sqrt{\tilde{\sigma}_\beta}=0$. Since $\rho\neq 0$, we have $H=0$, that is, M is a compact maximal spacelike submanifold in $S_q^{n+p}(c)$.

By Cheng and Ishikawa [29, Theorem 1], if $p-q=1$, we know that M lies in the totally geodesic spacelike submanifold $S^{n+1}(c)$ of $S_q^{n+q+1}(c)$, and is isometric to the Clifford torus

$S^k(\frac{n}{k}c) \times S^{n-k}(\frac{n}{n-k}c)$. If $p-q>1$, we know that $n=p-q=2$, and M lies in the totally geodesic spacelike submanifold $S^4(c)$ of $S_q^{4+q}(c)$, and is isometric to the Veronese surface. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2 For a fixed α, $n+1 \le \alpha \le n+p$, we can take a local orthonormal frame field $\{e_1, \dots, e_n\}$ such that $h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$, then $\tilde{h}_{ij}^\alpha = \mu_i^\alpha \delta_{ij}$ with $\mu_i^\alpha = \lambda_i^\alpha - H^\alpha$, $\sum_i \mu_i^\alpha = 0$. Thus

$$\begin{aligned} -\sum_{\alpha,i,j,k,l} h_{ij}^\alpha \bigl(h_{kl}^\alpha R_{lijk} - h_{li}^\alpha R_{lkjk}\bigr) &= \frac{1}{2}\sum_{\alpha,i,k}\bigl(\lambda_i^\alpha - \lambda_k^\alpha\bigr)^2 R_{kiik} \\ &= \frac{1}{2}\sum_{\alpha,i,k}\bigl(\mu_i^\alpha - \mu_k^\alpha\bigr)^2 R_{kiik} \ge nK\rho^2, \end{aligned} \tag{4.8}$$

where K denotes the infimum of the sectional curvature of M and the equality in (4.8) holds if and only if $R_{kiik} = K$ for any $i \ne k$.

From Chern et al. [39, Lemma 1], we see that

$$\begin{aligned} &-\frac{1}{2}\sum_{\alpha,\beta} \varepsilon_\beta N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) \\ &= -\frac{1}{2}\sum_{\alpha}\sum_{\beta=n+1}^{n+p-q} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) + \frac{1}{2}\sum_{\alpha}\sum_{\beta=n+p-q+1}^{n+p} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) \\ &= -\frac{1}{2}\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) \\ &\quad + \frac{1}{2}\sum_{\alpha=n+p-q+1}^{n+p}\sum_{\beta=n+p-q+1}^{n+p} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) \\ &\ge -\frac{1}{2}\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) \ge -\sum_{\alpha\ne\beta}\tilde{\sigma}_\alpha\tilde{\sigma}_\beta \\ &= -\left(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\right)^2 + \sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha^2 \ge -\left(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\right)^2 + \frac{1}{p-q}\left(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\right)^2 \\ &= -\left(1-\frac{1}{p-q}\right)\left(\sum_{\alpha=n+1}^{n+p-q}\tilde{\sigma}_\alpha\right)^2 \ge -\left(1-\frac{1}{p-q}\right)\rho^4. \end{aligned} \tag{4.9}$$

Thus, from (3.5), (3.6), (4.8), (4.9), we have

$$\frac{1}{2}\Delta\rho^2 \ge |\nabla h|^2 + nK\rho^2 - \left(1-\frac{1}{p-q}\right)\rho^4.$$

From the Stokes formula, we get

$$0 \ge \int_M \rho^2\left\{nK - \left(1-\frac{1}{p-q}\right)\rho^2\right\} dv. \tag{4.10}$$

In particular, if $K \geq \frac{1}{n}(1-\frac{1}{p-q})\rho^2$, from (4.10), we see that $\rho^2 = 0$ and M is totally umbilical, or $K = \frac{1}{n}(1-\frac{1}{p-q})\rho^2$. In the latter case, we see that $|\nabla h| = 0$ and M is a spacelike submanifold with parallel second fundamental form. This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3 From (2.2) and (3.1), we have

$$\begin{aligned} R_{kk} &= (n-1)c + (n-2)\sum_{\alpha} \varepsilon_\alpha H^\alpha \tilde{h}^\alpha_{kk} + (n-1)\sum_{\alpha=n+1}^{n+p-q} (H^\alpha)^2 \\ &\quad - (n-1)\sum_{\alpha=n+p-q+1}^{n+p} (H^\alpha)^2 - \sum_{i,\alpha=n+1}^{n+p-q} (\tilde{h}^\alpha_{ik})^2 + \sum_{i,\alpha=n+p-q+1}^{n+p} (\tilde{h}^\alpha_{ik})^2 \\ &\leq (n-1)c + (n-2)\sum_{\alpha} \varepsilon_\alpha H^\alpha \tilde{h}^\alpha_{kk} + (n-1)H^2 \\ &\quad - \sum_{i,\alpha=n+1}^{n+p-q} (\tilde{h}^\alpha_{ik})^2 + \sum_{i,\alpha=n+p-q+1}^{n+p} (\tilde{h}^\alpha_{ik})^2. \end{aligned} \tag{4.11}$$

Thus

$$nQ \leq \sum_k R_{kk} = n(n-1)c + n(n-1)H^2 - \sum_{i,k,\alpha=n+1}^{n+p-q} (\tilde{h}^\alpha_{ik})^2 + \sum_{i,k,\alpha=n+p-q+1}^{n+p} (\tilde{h}^\alpha_{ik})^2.$$

From (3.2) and (3.3), we have

$$-\sum_{\alpha=n+1}^{n+p-q} \tilde{\sigma}_\alpha + \sum_{\alpha=n+p-q+1}^{n+p} \tilde{\sigma}_\alpha \geq nQ - n(n-1)c - n(n-1)H^2. \tag{4.12}$$

From (4.12), we see that

$$\begin{aligned} -\sum_{\alpha,\beta} \varepsilon_\beta \tilde{\sigma}^2_{\alpha\beta} &= -\sum_{\alpha} \varepsilon_\alpha \tilde{\sigma}^2_\alpha = -\sum_{\alpha=n+1}^{n+p-q} \tilde{\sigma}^2_\alpha + \sum_{\alpha=n+p-q+1}^{n+p} \tilde{\sigma}^2_\alpha \\ &\geq -\left(\sum_{\alpha=n+1}^{n+p-q} \tilde{\sigma}_\alpha\right)^2 + \frac{1}{q}\left(\sum_{\alpha=n+p-q+1}^{n+p} \tilde{\sigma}_\alpha\right)^2 \\ &= -\left(\sum_{\alpha=n+1}^{n+p-q} \tilde{\sigma}_\alpha\right)^2 + \left(\sum_{\alpha=n+p-q+1}^{n+p} \tilde{\sigma}_\alpha\right)^2 + \left(\frac{1}{q}-1\right)\left(\sum_{\alpha=n+p-q+1}^{n+p} \tilde{\sigma}_\alpha\right)^2 \\ &\geq \left(-\sum_{\alpha=n+1}^{n+p-q} \tilde{\sigma}_\alpha + \sum_{\alpha=n+p-q+1}^{n+p} \tilde{\sigma}_\alpha\right)\left(\sum_{\alpha=n+1}^{n+p-q} \tilde{\sigma}_\alpha + \sum_{\alpha=n+p-q+1}^{n+p} \tilde{\sigma}_\alpha\right) - \left(1-\frac{1}{q}\right)\rho^4 \\ &\geq \left(nQ - n(n-1)c - n(n-1)H^2\right)\rho^2 - \left(1-\frac{1}{q}\right)\rho^4. \end{aligned} \tag{4.13}$$

From (4.9), we have

$$-\sum_{\alpha=n+1}^{n+p-q}\sum_{\beta=n+1}^{n+p-q} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) \geq -2\left(1-\frac{1}{p-q}\right)\rho^4. \tag{4.14}$$

Thus, from (3.8), (4.1), (4.2), (4.13) and (4.14), we have

$$\begin{aligned}\frac{1}{2}\Delta\rho^2 &\geq |\nabla h|^2 + nc\rho^2 - n\rho^2H^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\rho^3 - 2\left(1-\frac{1}{p-q}\right)\rho^4 \\ &\quad + \left(nQ - n(n-1)c - n(n-1)H^2\right)\rho^2 - \left(1-\frac{1}{q}\right)\rho^4 \\ &= |\nabla h|^2 + n\rho^2\left\{Q-(n-2)c-nH^2-\frac{n-2}{\sqrt{n(n-1)}}H\rho \right. \\ &\quad \left. -\frac{1}{n}\left(3-\frac{p+q}{(p-q)q}\right)\rho^2\right\}. \end{aligned} \tag{4.15}$$

From the Stokes formula, we get

$$0 \geq \int_M \rho^2\left\{Q-(n-2)c-nH^2-\frac{n-2}{\sqrt{n(n-1)}}H\rho-\frac{1}{n}\left(3-\frac{p+q}{(p-q)q}\right)\rho^2\right\}dv. \tag{4.16}$$

In particular, if $Q \geq (n-2)c + nH^2 + \frac{n-2}{\sqrt{n(n-1)}}H\rho + \frac{1}{n}(3-\frac{p+q}{(p-q)q})$, from (4.16), we see that $\rho^2 = 0$ and M is totally umbilical, or

$$Q = (n-2)c + nH^2 + \frac{n-2}{\sqrt{n(n-1)}}H\rho + \frac{1}{n}\left(3-\frac{p+q}{(p-q)q}\right).$$

In the latter case, we see that the equalities in (4.15), (4.1) and (4.11) hold. Thus, we have

$$|\nabla h| = 0, \qquad \sum_{\alpha=n+1}^{n+p-q}\left(H^\alpha\right)^2 = 0, \qquad \sum_{\alpha=n+p-q+1}^{n+p}\left(H^\alpha\right)^2 = 0.$$

This implies that $H^\alpha = 0$ for $\alpha = n+1,\dots,n+p$, and $H = 0$, that is, M is a compact maximal spacelike submanifold with parallel second fundamental form, and the Ricci curvature $\mathrm{Ric}(M) = (n-2)c + \frac{1}{n}(3-\frac{p+q}{(p-q)q})$. This completes the proof of Theorem 1.3. □

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