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Study of fractional integral inequalities involving Mittag-Leffler functions via convexity

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Abstract

This paper studies fractional integral inequalities for fractional integral operators containing extended Mittag-Leffler (ML) functions. These inequalities provide upper bounds of left- and right-sided fractional integrals for $(\alpha, h-m)$ convex functions. A generalized fractional Hadamard inequality is established. All the results hold for h-convex, (h,m)-convex, (α,m)-convex, (s,m)-convex, and associated functions.

Keywords: Convex function; $(\alpha, h-m)$-convex function; Mittag-Leffler function; Fractional integral operators

1 Introduction

Convexity was introduced at the beginning of the twentieth century. Due to having many fascinating and important properties, a convex function plays a vital role in almost all areas of mathematical analysis, probability theory, optimization theory, graph theory, etc. It has been defined in different convenient ways, for example, graph of a convex function always lies below the chord joining any two points lying on its graph, the derivative of a differentiable convex function is increasing and vice versa, a convex function has line of support at each point of the interior of its domain, and many others. In the theory of inequalities it is frequently defined in the form of an inequality which can be interpreted very nicely in the plane. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ satisfying the inequality $f(ta + (1-t)b) \leq tf(a) + (1-t)f(b)$, where I is an interval, $t \in [0,1]$, and $a, b \in I$, is called convex.

This analytic form of presentation of a convex function motivated the authors to define other types of convex functions for example m-convex, s-convex, (s,m)-convex, h-convex, (h,m)-convex, (α,m)-convex, exponentially convex, etc. In this age convex functions lead to the theory of convex analysis, theory of inequalities, a lot of research articles and books are dedicated to the literature which has been developed due to convex function, see [1, 3, 4, 20, 22, 25, 31].

The goal of this paper is to study the bounds of fractional integral operators involving Mittag-Leffler (ML) functions in their kernels by utilizing a generalized form of convex functions, namely $(\alpha, h-m)$-convex functions which unify h-convex, (h,m)-convex,

(α, m)-convex, and (s, m)-convex functions. Therefore the results of this paper will simultaneously hold for all these kinds of convex functions.

In 2007, Varošance introduced the h-convex function.

Definition 1 ([30]) A function $f : I \to \mathbb{R}$ is said to be h-convex if the following inequality holds:

$$f\big(ta + (1-t)b\big) \le h(t)f(a) + h(1-t)f(b),$$

where h is a nonnegative function defined on J, $a, b \in I$, $t \in [0,1]$, I and J are real intervals such that $(0,1) \subset J$.

Özdemir introduced a generalization of h-convex function, namely $(h-m)$-convex function.

Definition 2 ([21]) Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$, and let $h : J \to \mathbb{R}$ be a nonnegative function. We say that $f : [0,b] \to \mathbb{R}$ is an $(h-m)$-convex function if f is nonnegative, and for all $u, v \in [0,b]$, $m \in [0,1]$, and $t \in (0,1)$, one has

$$f\big(tu + m(1-t)v\big) \le h(t)f(u) + mh(1-t)f(v). \tag{1.1}$$

Mihesan introduced the notion of (α, m)-convex function as follows.

Definition 3 ([18]) A function $f : [0,b] \subseteq \mathbb{R} \to \mathbb{R}$ is said to be (α, m)-convex function, where $(\alpha, m) \in [0,1]^2$ and $b > 0$, if for every $u, v \in [0,b]$ and $t \in [0,1]$, we have

$$f\big(tu + m(1-t)v\big) \le t^{\alpha} f(u) + m\big(1-t^{\alpha}\big)f(v).$$

Definition 4 ([5]) A function $f : [0,b] \to \mathbb{R}$ is said to be (s, m)-convex function, where $(s, m) \in [0,1]^2$ and $b > 0$, if for every $x, y \in [0,b]$ and $t \in [0,1]$, we have

$$f\big(ta + m(1-t)b\big) \le t^{s} f(a) + m(1-t)^{s} f(b).$$

Farid et al. unified the notions of h-convexity, (α, m)-convexity, (h, m)-convexity, and (s, m)-convexity in a single definition called $(\alpha, h-m)$-convex function given as follows.

Definition 5 ([15]) Let $J \subseteq \mathbb{R}$ be an interval containing $(0,1)$, and let $h : J \to \mathbb{R}$ be a nonnegative function. We say that $f : [0,b] \to \mathbb{R}$ is an $(\alpha, h-m)$-convex function if f is nonnegative, and for all $u, v \in [0,b]$, $(\alpha, m) \in [0,1]^2$, and $t \in (0,1)$, one has

$$f\big(tu + m(1-t)v\big) \le h\big(t^{\alpha}\big)f(u) + mh\big(1-t^{\alpha}\big)f(v). \tag{1.2}$$

Remark 1 By selecting a suitable function h and particular values of m and α, the above definition produces the functions as follows:

(i) By setting $\alpha = 1, h(t) = t^{s}$, an (s, m)-convex function can be obtained.
(ii) By setting $h(t) = t$, an (α, m)-convex function can be obtained.
(iii) By setting $\alpha = 1$, an (h, m)-convex function can be obtained.

(iv) By setting $m = 1, \alpha = 1$, an h-convex function can be obtained.

Next we give the definition of Mittag-Leffler functions and associated definitions of fractional integral operators.

Mittag-Leffler function $E_\xi(\cdot)$ for one parameter is defined as follows [19]:

$$E_\xi(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\xi n + 1)},$$

where $t, \xi \in \mathbb{C}, \Re(\xi) > 0$, and $\Gamma(\cdot)$ is the gamma function. It is a natural extension of exponential, hyperbolic, and trigonometric functions. This function and its extensions are useful in solving fractional integral/differential equations. It is also studied extensively in various fields of sciences; for details, see [2, 7, 10, 16, 17, 24, 26, 27].

Andrić et al. introduced the following extended Mittag-Leffler function.

Definition 6 ([6]) Let $\mu, \xi, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\xi), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Then the extended generalized Mittag-Leffler function $E^{\gamma,\delta,k,c}_{\mu,\xi,l}(t;p)$ is defined by

$$E^{\gamma,\delta,k,c}_{\mu,\xi,l}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}}, \tag{1.3}$$

where β_p is defined by

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}}\, dt$$

and $(c)_{nk} = \frac{\Gamma(c+nk)}{\Gamma(c)}$.

A derivative formula of the extended Mittag-Leffler function is given in the following lemma.

Lemma 1 ([6]) *If* $m \in \mathbb{N}, \omega, \mu, \xi, l, \gamma, c \in \mathbb{C}, \Re(\mu), \Re(\xi), \Re(l) > 0, \Re(c) > \Re(\gamma) > 0$ *with* $p \geq 0, \delta > 0$, *and* $0 < k < \delta + \Re(\mu)$, *then*

$$\left(\frac{d}{dt}\right)^m \left[t^{\xi-1} E^{\gamma,\delta,k,c}_{\mu,\xi,l}\left(\omega t^\mu; p\right)\right] = t^{\xi-m-1} E^{\gamma,\delta,k,c}_{\mu,\xi-m,l}\left(\omega t^\mu; p\right), \quad \Re(\xi) > m. \tag{1.4}$$

Remark 2 The extended Mittag-Leffler function (1.3) produces the related functions defined in [23, 24, 26–28], see [29, Remark 1.3].

Next, we give the definition of fractional integral operators containing the extended Mittag-Leffler function (1.3).

Definition 7 ([6]) Let $\omega, \mu, \xi, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\xi), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$, and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then the generalized fractional integral operators containing Mittag-Leffler function are defined by

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,a^+} f\right)(x;p) = \int_a^x (x-t)^{\xi-1} E^{\gamma,\delta,k,c}_{\mu,\xi,l}\left(\omega(x-t)^\mu; p\right) f(t)\, dt, \tag{1.5}$$

$$\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,b^-}f\right)(x;p) = \int_x^b (t-x)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}\left(\omega(t-x)^{\mu};p\right)f(t)\,dt. \tag{1.6}$$

Remark 3 Operators (1.5) and (1.6) produce in particular several kinds of known fractional integral operators, see [29, Remark 1.4].

The classical Riemann–Liouville fractional integral operator is defined as follows.

Definition 8 ([28]) Let $f \in L_1[a,b]$. Then Riemann–Liouville fractional integral operators of order $\xi > 0$ are defined by

$$I^{\xi}_{a^+}f(x) = \frac{1}{\Gamma(\xi)}\int_x^b (x-t)^{\xi-1}f(t)\,dt, \quad x > a, \tag{1.7}$$

$$I^{\xi}_{b^-}f(x) = \frac{1}{\Gamma(\xi)}\int_a^x (t-x)^{\xi-1}f(t)\,dt, \quad x < b. \tag{1.8}$$

It can be noted that $(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,0,a^+}f)(x;0) = I^{\xi}_{a^+}f(x)$ and $(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,0,b^-}f)(x;0) = I^{\xi}_{b^-}f(x)$. From fractional integral operators (1.5) and (1.6) we can write

$$J_{\xi,a^+}(x;p) := \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,a^+}1\right)(x;p) = (x-a)^{\xi}E^{\gamma,\delta,k,c}_{\mu,\xi+1,l}\left(w(x-a)^{\mu};p\right), \tag{1.9}$$

$$J_{\eta,b^-}(x;p) := \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta,l,\omega,b^-}1\right)(x;p) = (b-x)^{\eta}E^{\gamma,\delta,k,c}_{\mu,\eta+1,l}\left(w(b-x)^{\mu};p\right). \tag{1.10}$$

In the upcoming section the extended Mittag-Leffler (ML) function (1.3) and the corresponding generalized fractional integral operators are used to evaluate the bounds of sum of left- and right-sided operators by using $(\alpha,h-m)$-convexity. Their particular cases are also discussed. Furthermore, the lower and upper bounds of sum of these operators are presented in the form of a Hadamard inequality for $(\alpha,h-m)$-convex functions. Also the presented results are connected with several already known results.

2 Main results

Theorem 1 *Let $\varphi : [x_0,y_0] \longrightarrow \mathbb{R}$ be a real-valued function. If φ is positive and $(\alpha,h-m)$-convex, $(\alpha,m) \in [0,1]^2$, $m \neq 0$, then for $\xi,\eta \geq 1$, the following fractional integral inequality for generalized fractional integral operators (1.5) and (1.6) holds*:

$$\begin{aligned}
&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,x_0^+}\varphi\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta,l,\omega,y_0^-}\varphi\right)(x;p) \\
&\quad \leq (x-x_0)J_{\xi-1,x_0^+}(x;p)\left(\varphi(x_0)\int_0^1 h\left(z^{\alpha}\right)dz + m\varphi\left(\frac{x}{m}\right)\int_0^1 h\left(1-z^{\alpha}\right)dz\right) \\
&\qquad + (y_0-x)J_{\eta-1,y_0^-}(x;p)\left(\varphi(y_0)\int_0^1 h\left(z^{\alpha}\right)dz + m\varphi\left(\frac{x}{m}\right)\int_0^1 h\left(1-z^{\alpha}\right)dz\right).
\end{aligned} \tag{2.1}$$

Proof Let $x \in [x_0,y_0]$. Then, for $t \in [x_0,x)$, $\xi \geq 1$, the following inequality holds:

$$(x-t)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}\left(\omega(x-t)^{\mu};p\right) \leq (x-x_0)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}\left(\omega(x-x_0)^{\mu};p\right). \tag{2.2}$$

Using the definition of $(\alpha,h-m)$-convex function, we get

$$\varphi(t) \leq h\left(\frac{x-t}{x-x_0}\right)^{\alpha}\varphi(x_0) + mh\left(1-\left(\frac{x-t}{x-x_0}\right)^{\alpha}\right)\varphi\left(\frac{x}{m}\right). \tag{2.3}$$

After multiplying (2.2) and (2.3), we integrate over $[x_0, x]$ to obtain the following inequality:

$$\begin{aligned}&\int_{x_0}^{x}(x-t)^{\xi-1}E_{\mu,\xi,l}^{\gamma,\delta,k,c}\big(\omega(x-t)^{\mu};p\big)\varphi(t)\,dt\\&\quad\le(x-x_0)^{\xi-1}E_{\mu,\xi,l}^{\gamma,\delta,k,c}\big(\omega(x-x_0)^{\mu};p\big)\biggl(\varphi(x_0)\int_{x_0}^{x}h\biggl(\frac{x-t}{x-x_0}\biggr)^{\alpha}\,dt\\&\qquad+m\varphi\biggl(\frac{x}{m}\biggr)\int_{x_0}^{x}h\biggl(1-\biggl(\frac{x-t}{x-x_0}\biggr)^{\alpha}\biggr)\,dt\biggr).\end{aligned}$$

Therefore the left fractional integral operator (1.5) satisfies the following upper bound:

$$\begin{aligned}&\bigl(\epsilon_{\mu,\xi,l,\omega,x_0^+}^{\gamma,\delta,k,c}\varphi\bigr)(x;p)\\&\quad\le(x-x_0)J_{\xi-1,x_0^+}(x;p)\biggl(\varphi(x_0)\int_0^1h\bigl(z^{\alpha}\bigr)\,dz+m\varphi\biggl(\frac{x}{m}\biggr)\int_0^1h\bigl(1-z^{\alpha}\bigr)\,dz\biggr).\end{aligned}\tag{2.4}$$

Similarly, for $t\in(x,y_0]$ and $\eta\ge1$, the following inequality holds:

$$(t-x)^{\eta-1}E_{\mu,\eta,l}^{\gamma,\delta,k,c}\big(\omega(t-x)^{\mu};p\big)\le(y_0-x)^{\eta-1}E_{\mu,\eta,l}^{\gamma,\delta,k,c}\big(\omega(y_0-x)^{\mu};p\big),\tag{2.5}$$

again by using $(\alpha,h-m)$-convexity of φ, we have

$$\varphi(t)\le h\biggl(\frac{t-x}{y_0-x}\biggr)^{\alpha}\varphi(y_0)+mh\biggl(1-\biggl(\frac{t-x}{y_0-x}\biggr)^{\alpha}\biggr)\varphi\biggl(\frac{x}{m}\biggr).\tag{2.6}$$

By multiplying (2.5) and (2.6), and integrating over $[x, y_0]$, we have

$$\begin{aligned}&\int_{x}^{y_0}(t-x)^{\eta-1}E_{\mu,\eta,l}^{\gamma,\delta,k,c}\big(\omega(t-x)^{\mu};p\big)\varphi(t)\,dt\\&\quad\le(y_0-x)^{\eta-1}E_{\mu,\eta,l}^{\gamma,\delta,k,c}\big(\omega(y_0-x)^{\mu};p\big)\biggl(\varphi(y_0)\int_{x}^{y_0}h\biggl(\frac{t-x}{y_0-x}\biggr)^{\alpha}\,dt\\&\qquad+m\varphi\biggl(\frac{x}{m}\biggr)\int_{x}^{y_0}h\biggl(1-\biggl(\frac{t-x}{y_0-x}\biggr)^{\alpha}\biggr)\,dt\biggr).\end{aligned}$$

Therefore the right fractional integral operator (1.6) satisfies the following upper bound:

$$\begin{aligned}&\bigl(\epsilon_{\mu,\eta,l,\omega,x_0^+}^{\gamma,\delta,k,c}\varphi\bigr)(x;p)\\&\quad\le(y_0-x)J_{\eta-1,y_0^-}(x;p)\biggl(\varphi(y_0)\int_0^1h\bigl(z^{\alpha}\bigr)\,dz+m\varphi\biggl(\frac{x}{m}\biggr)\int_0^1h\bigl(1-z^{\alpha}\bigr)\,dz\biggr).\end{aligned}\tag{2.7}$$

By adding (2.4) and (2.7), inequality (2.1) is obtained. □

Some particular results are stated in the following corollaries.

Corollary 1 *If we set $\xi=\eta$ in* (2.1), *then the following inequality is obtained*:

$$\bigl(\epsilon_{\mu,\xi,l,\omega,x_0^+}^{\gamma,\delta,k,c}\varphi\bigr)(x;p)+\bigl(\epsilon_{\mu,\xi,l,\omega,y_0^-}^{\gamma,\delta,k,c}\varphi\bigr)(x;p)\tag{2.8}$$

$$\leq (x-x_0)J_{\xi-1,x_0^+}(x;p)\left(\varphi(x_0)\int_0^1 h\left(z^\alpha\right)dz + m\varphi\left(\frac{x}{m}\right)\int_0^1 h\left(1-z^\alpha\right)dz\right)$$
$$+ (y_0-x)J_{\xi-1,y_0^-}(x;p)\left(\varphi(y_0)\int_0^1 h\left(z^\alpha\right)dz + m\varphi\left(\frac{x}{m}\right)\int_0^1 h\left(1-z^\alpha\right)dz\right).$$

Corollary 2 *Along with assumptions of Theorem 1, if $\varphi \in L_\infty[x_0,y_0]$, then the following inequality is obtained*:

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,x_0^+}\varphi\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta,l,\omega,y_0^-}\varphi\right)(x;p)\\ &\quad\leq \|\varphi\|_\infty\left[(x-x_0)J_{\xi-1,x_0^+}(x;p) + (y_0-x)J_{\eta-1,y_0^-}(x;p)\right]\\ &\qquad\times\left[\int_0^1 h\left(z^\alpha\right)dz + m\int_0^1 h\left(1-z^\alpha\right)dz\right].\end{aligned} \tag{2.9}$$

Corollary 3 *By setting $\xi = \eta$ in (2.9), we get the following inequality*:

$$\begin{aligned}&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,x_0^+}\varphi\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,y_0^-}\varphi\right)(x;p)\\ &\quad\leq \|\varphi\|_\infty\left[(x-x_0)J_{\xi-1,x_0^+}(x;p) + (y_0-x)J_{\xi-1,y_0^-}(x;p)\right]\\ &\qquad\times\left[\int_0^1 h\left(z^\alpha\right)dz + m\int_0^1 h\left(1-z^\alpha\right)dz\right].\end{aligned} \tag{2.10}$$

Remark 4

(i) If we set $h(t) = t$ in (2.1), then we obtain [12, Theorem 2.1].
(ii) If we set $h(t) = t$ in (2.8), then we obtain [12, Corollary 2.1].
(iii) If we set $\alpha = 1$ in (2.1), then we obtain [9, Theorem 1].
(iv) If we set $\alpha = m = 1$ and $h(t) = t$ in (2.1), then we obtain [9, Corollary 1].
(v) If we set $\alpha = 1, \omega = p = 0$ in (2.1), then we obtain [13, Theorem 1].
(vi) If we set $\alpha = m = 1, \omega = p = 0, h(t) = t$ in (2.1), then we obtain [11, Theorem 1].

Theorem 2 *Along with the assumptions of Theorem 1, if $\varphi \in L_\infty[x_0,y_0]$, then the operators defined in (1.5) and (1.6) are bounded and continuous.*

Proof If $\varphi \in L_\infty[x_0,y_0]$, then from (2.4) we have

$$\begin{aligned}&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,x_0^+}\varphi\right)(x;p)\right|\\ &\quad\leq \|\varphi\|_\infty(x-x_0)J_{\xi-1,x_0^+}(x;p)\int_0^1\left(h\left(z^\alpha\right) + mh\left(1-z^\alpha\right)\right)dz\\ &\quad\leq \|\varphi\|_\infty(y_0-x_0)J_{\xi-1,x_0^+}(x;p)\int_0^1\left(h\left(z^\alpha\right) + mh\left(1-z^\alpha\right)\right)dz,\end{aligned} \tag{2.11}$$

that is,

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,x_0^+}\varphi\right)(x;p)\right| \leq M\|\varphi\|_\infty, \tag{2.12}$$

where $M = (y_0 - x_0)J_{\xi-1,x_0^+}(x;p)\int_0^1 (h(z^\alpha) + mh(1-z^\alpha))\,dz$. On the other hand, from (2.7) we can obtain

$$\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta,l,\omega,y_0^-}\varphi\right)(x;p)\right| \le K\|\varphi\|_\infty, \tag{2.13}$$

where $K = (y_0 - x_0)J_{\eta-1,y_0^-}(x;p)\int_0^1 (h(z^\alpha) + mh(1-z^\alpha))\,dz$. Therefore the operators $(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,x_0^+}\varphi)(x;p)$ and $(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi,l,\omega,x_0^+}\varphi)(x;p)$ are bounded. Also they are linear, hence they are continuous. □

Theorem 3 *Let $\varphi : [x_0, y_0] \longrightarrow \mathbb{R}$ be a real-valued function. If φ is differentiable and $|\varphi'|$ is $(\alpha, h-m)$-convex, $(\alpha, m) \in [0,1]^2$, $m \neq 0$, then for $\xi, \eta \ge 1$, the following fractional integral inequality for generalized fractional integral operators (1.5) and (1.6) holds:*

$$\begin{aligned}
&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,x_0^+}\varphi\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,y_0^-}\varphi\right)(x;p)\right.\\
&\qquad \left. - \left(J_{\xi-1,x_0^+}(x;p)\varphi(x_0) + J_{\eta-1,y_0^-}(x;p)\varphi(y_0)\right)\right|\\
&\quad \le (x-x_0)J_{\xi-1,x_0^+}(x;p)\left(\left|\varphi'(x_0)\right|\int_0^1 h\left(z^\alpha\right)dz + m\left|\varphi'\left(\frac{x}{m}\right)\right|\int_0^1 h\left(1-z^\alpha\right)dz\right)\\
&\qquad + (y_0 - x)J_{\eta-1,y_0^-}(x;p)\\
&\qquad \times \left(\left|\varphi'(y_0)\right|\int_0^1 h\left(z^\alpha\right)dz + m\left|\varphi'\left(\frac{x}{m}\right)\right|\int_0^1 h\left(1-z^\alpha\right)dz\right).
\end{aligned} \tag{2.14}$$

Proof Let $x \in [x_0, y_0]$ and $t \in [x_0, x)$. Then, by using the definition of $(\alpha, h-m)$-convexity of $|\varphi'|$, we have

$$\left|\varphi'(t)\right| \le h\left(\frac{x-t}{x-x_0}\right)^\alpha\left|\varphi'(x_0)\right| + mh\left(1-\left(\frac{x-t}{x-x_0}\right)^\alpha\right)\left|\varphi'\left(\frac{x}{m}\right)\right|. \tag{2.15}$$

From (2.15), we can write

$$\varphi'(t) \le h\left(\frac{x-t}{x-x_0}\right)^\alpha\left|\varphi'(x_0)\right| + mh\left(1-\left(\frac{x-t}{x-x_0}\right)^\alpha\right)\left|\varphi'\left(\frac{x}{m}\right)\right|. \tag{2.16}$$

Multiplication of (2.2) with (2.16) gives the following inequality:

$$\begin{aligned}
&(x-t)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}\left(\omega(x-t)^\mu;p\right)\varphi'(t)\,dt\\
&\quad \le (x-x_0)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}\left(\omega(x-x_0)^\mu;p\right)\\
&\qquad \times \left(h\left(\frac{x-t}{x-x_0}\right)^\alpha\left|\varphi'(x_0)\right| + mh\left(1-\left(\frac{x-t}{x-x_0}\right)^\alpha\right)\left|\varphi'\left(\frac{x}{m}\right)\right|\right).
\end{aligned} \tag{2.17}$$

Now, integrating over $[x_0, x]$, we get

$$\begin{aligned}
&\int_{x_0}^x (x-t)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}\left(\omega(x-t)^\mu;p\right)\varphi'(t)\,dt\\
&\quad \le (x-x_0)^{\xi-1}E^{\gamma,\delta,k,c}_{\mu,\xi,l}\left(\omega(x-x_0)^\mu;p\right)\left(\left|\varphi'(x_0)\right|\int_{x_0}^x h\left(\frac{x-t}{x-x_0}\right)^\alpha dt\right.
\end{aligned}$$

$$
\begin{aligned}
&+ m\left|\varphi'\left(\frac{x}{m}\right)\right| \int_{x_0}^{x} h\left(1-\left(\frac{x-t}{x-x_0}\right)^{\alpha}\right)\Bigg)\\
&= (x-x_0)^{\xi} E_{\mu,\xi,l}^{\gamma,\delta,k,c}\left(\omega(x-x_0)^{\mu};p\right)\\
&\times \left(\left|\varphi'(x_0)\right| \int_0^1 h\left(z^{\alpha}\right) dz + m\left|\varphi'\left(\frac{x}{m}\right)\right| \int_0^1 h\left(1-z^{\alpha}\right) dz\right).
\end{aligned}
\tag{2.18}
$$

The left-hand side of (2.18) is computed as follows:

$$
\int_{x_0}^{x} (x-t)^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}\left(\omega(x-t)^{\mu};p\right)\varphi'(t)\, dt. \tag{2.19}
$$

Substituting $x - t = r$, using the derivative property (1.4) of Mittag-Leffler function, we have

$$
\begin{aligned}
&\int_0^{x-x_0} r^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}\left(\omega r^{\mu};p\right)\varphi'(x-r)\, dr\\
&= (x-x_0)^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}\left(\omega(x-x_0)^{\mu};p\right)\varphi(x_0) - \int_0^{x-x_0} r^{\xi-2} E_{\mu,\xi,l}^{\gamma,\delta,k,c}\left(\omega r^{\mu};p\right)\varphi(x-r)\, dr.
\end{aligned}
$$

Now, for $x - r = t$ in the second term of the right-hand side of the above equation and then using (1.5), we get

$$
\begin{aligned}
&\int_0^{x-x_0} r^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}\left(\omega r^{\mu};p\right)\varphi'(x-r)\, dr\\
&= (x-x_0)^{\xi-1} E_{\mu,\xi,l}^{\gamma,\delta,k,c}\left(\omega(x-x_0)^{\mu};p\right)\varphi(x_0) - \left(\epsilon_{\mu,\xi+1,l,\omega,x_0^+}^{\gamma,\delta,k,c}\varphi\right)(x;p).
\end{aligned}
$$

Therefore (2.18) becomes

$$
\begin{aligned}
&J_{\xi-1,x_0^+}(x;p)\varphi(x_0) - \left(\epsilon_{\mu,\xi+1,l,\omega,x_0^+}^{\gamma,\delta,k,c}\varphi\right)(x;p)\\
&\le (x-x_0)J_{\alpha-1,x_0^+}(x;p)\left(\left|\varphi'(x_0)\right| \int_0^1 h\left(z^{\alpha}\right) dz + m\left|\varphi'\left(\frac{x}{m}\right)\right| \int_0^1 h\left(1-z^{\alpha}\right) dz\right).
\end{aligned}
\tag{2.20}
$$

Again from (2.15) we have

$$
\varphi'(t) \ge -\left(h\left(\frac{x-t}{x-x_0}\right)^{\alpha}\left|\varphi'(x_0)\right| + mh\left(1-\left(\frac{x-t}{x-x_0}\right)^{\alpha}\right)\left|\varphi'\left(\frac{x}{m}\right)\right|\right). \tag{2.21}
$$

Similar as we did for (2.16), we can obtain

$$
\begin{aligned}
&\left(\epsilon_{\mu,\xi+1,l,\omega,x_0^+}^{\gamma,\delta,k,c}\varphi\right)(x;p) - J_{\xi-1,x_0^+}(x;p)\varphi(x_0)\\
&\le (x-x_0)J_{\xi-1,x_0^+}(x;p)\left(\left|\varphi'(x_0)\right| \int_0^1 h\left(z^{\alpha}\right) dz + m\left|\varphi'\left(\frac{x}{m}\right)\right| \int_0^1 h\left(1-z^{\alpha}\right) dz\right).
\end{aligned}
\tag{2.22}
$$

From (2.20) and (2.22), we get

$$
\begin{aligned}
&\left|\left(\epsilon_{\mu,\xi+1,l,\omega,x_0^+}^{\gamma,\delta,k,c}\varphi\right)(x;p) - J_{\xi-1,x_0^+}(x;p)\varphi(x_0)\right|\\
&\le (x-x_0)J_{\xi-1,x_0^+}(x;p)\left(\left|\varphi'(x_0)\right| \int_0^1 h\left(z^{\alpha}\right) dz + m\left|\varphi'\left(\frac{x}{m}\right)\right| \int_0^1 h\left(1-z^{\alpha}\right) dz\right).
\end{aligned}
\tag{2.23}
$$

Now, for $x \in [x_0, y_0]$ and $t \in (x, y_0]$, again by using the $(\alpha, h-m)$-convexity of $|\varphi'|$, we have

$$\left|\varphi'(t)\right| \leq h\left(\frac{t-x}{y_0-x}\right)^{\alpha}\left|\varphi'(y_0)\right| + mh\left(1-\left(\frac{t-x}{y_0-x}\right)^{\alpha}\right)\left|\varphi'\left(\frac{x}{m}\right)\right|. \tag{2.24}$$

Proceeding along similar lines as we did to get (2.23), we can obtain the following inequality:

$$\begin{aligned}&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,y_0^-}\varphi\right)(x;p) - J_{\eta-1,y_0^-}(x;p)\varphi(y_0)\right| \\ &\quad\leq (y_0-x)J_{\eta-1,y_0^-}(x;p)\left(\left|\varphi'(y_0)\right|\left|\int_0^1 h(z^{\alpha}) + m\left|\varphi'\left(\frac{x}{m}\right)\right|\right|\int_0^1 h\left(1-z^{\alpha}\right)\right).\end{aligned} \tag{2.25}$$

From inequalities (2.23) and (2.25), triangular inequality (2.14) can be obtained. □

Corollary 4 *If we put $\xi = \eta$ in (2.14), then the following inequality is obtained*:

$$\begin{aligned}&\left|\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,x_0^+}\varphi\right)(x;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,y_0^-}\varphi\right)(x;p)\right. \\ &\qquad\left. - \left(J_{\xi-1,x_0^+}(x;p)\varphi(x_0) + J_{\xi-1,y_0^-}(x;p)\varphi(y_0)\right)\right| \\ &\quad\leq (x-x_0)J_{\xi-1,x_0^+}(x;p)\left(\left|\varphi'(x_0)\right|\left|\int_0^1 h(z^{\alpha})\,dz + m\left|\varphi'\left(\frac{x}{m}\right)\right|\right|\int_0^1 h\left(1-z^{\alpha}\right)dz\right) \\ &\qquad + (y_0-x)J_{\xi-1,y_0^-}(x;p) \\ &\qquad \times\left(\left|\varphi'(y_0)\right|\left|\int_0^1 h(z^{\alpha})\,dz + m\left|\varphi'\left(\frac{x}{m}\right)\right|\right|\int_0^1 h\left(1-z^{\alpha}\right)dz\right).\end{aligned} \tag{2.26}$$

Remark 5

(i) If we take $h(t) = t$ in (2.14), then we obtain [12, Theorem 2.2].
(ii) If we take $h(t) = t$ in (2.26), then we obtain [12, Corollary 2.2].
(iii) If we take $\alpha = 1$ in (2.14), then we obtain [9, Theorem 2].
(iv) If we take $\alpha = m = 1$ and $h(t) = t$ in (2.14), then we obtain [9, Corollary 2].
(v) If we take $\alpha = 1, \omega = p = 0$ in (2.14), then we obtain [13, Theorem 2].
(vi) If we take $\alpha = m = 1, \omega = p = 0$, and $h(t) = t$ in (2.14), then we obtain [11, Theorem 2].

It is easy to prove the next lemma which will be helpful to establish estimations in the form of a Hadamard-type inequality.

Lemma 2 *Let $\varphi : [x_0, my_0] \to \mathbb{R}$ be an $(\alpha, h-m)$-convex function. If $\varphi(\frac{x_0+my_0-x}{m}) = \varphi(x)$ and $(\alpha, m) \in [0,1]^2$, $m \neq 0$, then the following inequality holds*:

$$\varphi\left(\frac{x_0+my_0}{2}\right) \leq \varphi(x)\left(h\left(\frac{1}{2^{\alpha}}\right) + mh\left(1-\frac{1}{2^{\alpha}}\right)\right). \tag{2.27}$$

Proof Since φ is an $(\alpha, h-m)$-convex function, for $t \in [0,1]$, we have

$$\varphi\left(\frac{x_0+my_0}{2}\right) \leq h\left(\frac{1}{2^{\alpha}}\right)\varphi\left((1-t)x_0+mty_0\right)+mh\left(1-\frac{1}{2^{\alpha}}\right)\varphi\left(\frac{ta+m(1-t)y_0}{m}\right). \tag{2.28}$$

Let $x = x_0(1-t) + mty_0$. Then we have $x_0 + my_0 - x = ta + m(1-t)y_0$.

$$\varphi\left(\frac{x_0+my_0}{2}\right) \leq h\left(\frac{1}{2^\alpha}\right)\varphi(x) + mh\left(1-\frac{1}{2^\alpha}\right)\varphi\left(\frac{x_0+my_0-x}{m}\right). \tag{2.29}$$

Hence, by using $\varphi(\frac{x_0+my_0-x}{m}) = \varphi(x)$, inequality (2.27) can be obtained. □

Theorem 4 *Let $\varphi : [x_0, y_0] \longrightarrow \mathbb{R}$, $0 \leq x_0 < my_0$, be a real-valued function. If φ is positive $(\alpha, h-m)$-convex, $(\alpha, m) \in [0,1]^2$, $m \neq 0$, and $\varphi(\frac{x_0+my_0-x}{m}) = \varphi(x)$, then for $\xi, \eta > 0$, the following fractional integral inequality holds*:

$$\begin{aligned}
&\frac{1}{h(\frac{1}{2^\alpha}) + mh(1-\frac{1}{2^\alpha})}\varphi\left(\frac{x_0+my_0}{2}\right)\left[J_{\eta+1,y_0^-}(x_0;p) + J_{\xi+1,x_0^+}(y_0;p)\right] \\
&\quad\leq \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,y_0^-}\varphi\right)(x_0;p) + \left(\epsilon^{\gamma,\delta,k,c}_{\mu,\xi+1,l,\omega,x_0^+}\varphi\right)(y_0;p) \\
&\quad\leq \left[J_{\eta-1,y_0^-}(x_0;p) + J_{\xi-1,x_0^+}(y_0;p)\right](y_0-x_0)^2 \\
&\qquad\times\left(\varphi(y_0)\int_0^1 h\left(z^\alpha\right)dz + mf\left(\frac{x_0}{m}\right)\int_0^1 h\left(1-z^\alpha\right)dz\right).
\end{aligned} \tag{2.30}$$

Proof For $x \in [x_0, y_0]$, $\eta > 0$, we have

$$(x-x_0)^\eta E^{\gamma,\delta,k,c}_{\mu,\eta,l}\left(\omega(x-x_0)^\mu;p\right) \leq (y_0-x_0)^\eta E^{\gamma,\delta,k,c}_{\mu,\eta,l}\left(\omega(y_0-x_0)^\mu;p\right). \tag{2.31}$$

Since the function φ is $(\alpha, h-m)$-convex, for $x \in [x_0, y_0]$, we have

$$\varphi(x) \leq h\left(\frac{x-x_0}{y_0-x_0}\right)^\alpha\varphi(y_0) + mf\left(\frac{x_0}{m}\right)h\left(1-\left(\frac{x-x_0}{y_0-x_0}\right)^\alpha\right). \tag{2.32}$$

Multiplying (2.31) and (2.32) and then integrating over $[x_0, y_0]$, we get

$$\begin{aligned}
&\int_{x_0}^{y_0}(x-x_0)^\eta E^{\gamma,\delta,k,c}_{\mu,\eta,l}\left(\omega(x-x_0)^\mu;p\right)\varphi(x)\,dx \\
&\quad\leq (y_0-x_0)^\eta E^{\gamma,\delta,k,c}_{\mu,\eta,l}\left(\omega(y_0-x_0)^\mu;p\right)\left(\varphi(y_0)\int_{x_0}^{y_0}h\left(\frac{x-x_0}{y_0-x_0}\right)^\alpha dx\right. \\
&\qquad\left.+ mf\left(\frac{x_0}{m}\right)\int_{x_0}^{y_0}h\left(1-\left(\frac{x-x_0}{y_0-x_0}\right)^\alpha\right)dx\right),
\end{aligned}$$

from which we can get the following inequality:

$$\begin{aligned}
&\left(\epsilon^{\gamma,\delta,k,c}_{\mu,\eta+1,l,\omega,y_0^-}\varphi\right)(x_0;p) \\
&\quad\leq (y_0-x_0)^2 J_{\eta-1,y_0^-}(x_0;p)\left(\varphi(y_0)\int_0^1 h\left(z^\alpha\right)dz + mf\left(\frac{x_0}{m}\right)\int_0^1 h\left(1-z^\alpha\right)dz\right).
\end{aligned} \tag{2.33}$$

On the other hand, for $x \in [x_0, y_0]$, $\xi > 0$, we have

$$(y_0-x)^\xi E^{\gamma,\delta,k,c}_{\mu,\xi,l}\left(\omega(y_0-x)^\mu;p\right) \leq (y_0-x_0)^\xi E^{\gamma,\delta,k,c}_{\mu,\xi,l}\left(\omega(y_0-x_0)^\mu;p\right). \tag{2.34}$$

Multiplying (2.32) and (2.34), and then integrating over $[x_0, y_0]$, we get

$$\begin{aligned}&\int_{x_0}^{y_0}(y_0-x)^{\xi}E_{\mu,\xi,l}^{\gamma,\delta,k,c}\big(\omega(y_0-x)^{\mu};p\big)\varphi(x)\,dx\\&\quad\le(y_0-x_0)^{\xi}E_{\mu,\xi,l}^{\gamma,\delta,k,c}\big(\omega(y_0-x_0)^{\mu};p\big)\bigg(\varphi(y_0)\int_{x_0}^{y_0}h\bigg(\frac{x-x_0}{y_0-x_0}\bigg)^{\alpha}dx\\&\qquad+mf\bigg(\frac{x_0}{m}\bigg)\int_{x_0}^{y_0}h\bigg(1-\bigg(\frac{x-x_0}{y_0-x_0}\bigg)^{\alpha}\bigg)dx\bigg),\end{aligned}$$

from which we can get the following inequality:

$$\begin{aligned}&\big(\epsilon_{\mu,\xi+1,l,\omega,x_0^+}^{\gamma,\delta,k,c}\varphi\big)(y_0;p)\\&\quad\le(y_0-x_0)^2J_{\xi-1,x_0^+}(y_0;p)\bigg(\varphi(y_0)\int_0^1h\big(z^{\alpha}\big)\,dz+mf\bigg(\frac{x_0}{m}\bigg)\int_0^1h\big(1-z^{\alpha}\big)\,dz\bigg).\end{aligned}\tag{2.35}$$

Adding (2.33) and (2.35), we get

$$\begin{aligned}&\big(\epsilon_{\mu,\eta+1,l,\omega,y_0^-}^{\gamma,\delta,k,c}\varphi\big)(x_0;p)+\big(\epsilon_{\mu,\xi+1,l,\omega,x_0^+}^{\gamma,\delta,k,c}\varphi\big)(y_0;p)\\&\quad\le\big[J_{\eta-1,y_0^-}(x_0;p)+J_{\xi-1,x_0^+}(y_0;p)\big](y_0-x_0)^2\\&\qquad\times\bigg(\varphi(y_0)\int_0^1h\big(z^{\alpha}\big)\,dz+mf\bigg(\frac{x_0}{m}\bigg)\int_0^1h\big(1-z^{\alpha}\big)\,dz\bigg).\end{aligned}\tag{2.36}$$

Multiplying (2.27) with $(x-x_0)^{\eta}E_{\mu,\eta,l}^{\gamma,\delta,k,c}(\omega(x-x_0)^{\mu};p)$ and integrating over $[x_0, y_0]$, we get

$$\begin{aligned}&\varphi\bigg(\frac{x_0+my_0}{2}\bigg)\int_{x_0}^{y_0}(x-x_0)^{\eta}E_{\mu,\eta,l}^{\gamma,\delta,k,c}\big(\omega(x-x_0)^{\mu};p\big)\,dx\\&\quad\le\bigg(h\bigg(\frac{1}{2^{\alpha}}\bigg)+mh\bigg(1-\frac{1}{2^{\alpha}}\bigg)\bigg)\int_{x_0}^{y_0}(x-x_0)^{\eta}E_{\mu,\eta,l}^{\gamma,\delta,k,c}\big(\omega(x-x_0)^{\mu};p\big)\varphi(x)\,dx.\end{aligned}\tag{2.37}$$

By using (1.6) and (1.9), we get

$$\begin{aligned}&\varphi\bigg(\frac{x_0+my_0}{2}\bigg)J_{\eta+1,y_0^-}(x_0;p)\\&\quad\le\bigg(h\bigg(\frac{1}{2^{\alpha}}\bigg)+mh\bigg(1-\frac{1}{2^{\alpha}}\bigg)\bigg)\big(\epsilon_{\mu,\eta+1,l,\omega,y_0^-}^{\gamma,\delta,k,c}\varphi\big)(x_0;p).\end{aligned}\tag{2.38}$$

Multiplying (2.27) with $(y_0-x)^{\xi}E_{\mu,\xi,l}^{\gamma,\delta,k,c}(\omega(y_0-x)^{\mu};p)$ and integrating over $[x_0, y_0]$, using (1.5) and (1.9), we get

$$\begin{aligned}&\varphi\bigg(\frac{x_0+my_0}{2}\bigg)J_{\xi+1,x_0^+}(y_0;p)\\&\quad\le\bigg(h\bigg(\frac{1}{2^{\alpha}}\bigg)+mh\bigg(1-\frac{1}{2^{\alpha}}\bigg)\bigg)\big(\epsilon_{\mu,\xi+1,l,\omega,x_0^+}^{\gamma,\delta,k,c}\varphi\big)(y_0;p).\end{aligned}\tag{2.39}$$

Adding (2.38) and (2.39), we get

$$\frac{1}{h(\frac{1}{2^{\alpha}})+mh(1-\frac{1}{2^{\alpha}})}\varphi\bigg(\frac{x_0+my_0}{2}\bigg)\big[J_{\eta+1,y_0^-}(x_0;p)+J_{\xi+1,x_0^+}(y_0;p)\big]$$

$$\leq \left(\epsilon_{\mu,\eta+1,l,\omega,y_0^-}^{\gamma,\delta,k,c}\varphi\right)(x_0;p) + \left(\epsilon_{\mu,\xi+1,l,\omega,x_0^+}^{\gamma,\delta,k,c}\varphi\right)(y_0;p). \tag{2.40}$$

Now, by combining (2.36) and (2.40), inequality (2.30) is established. □

Corollary 5 *If we put $\xi = \eta$ in (2.30), then the following inequality is obtained:*

$$\begin{aligned} &\frac{1}{h(\frac{1}{2^\alpha}) + mh(1-\frac{1}{2^\alpha})}\varphi\left(\frac{x_0+my_0}{2}\right)\left[J_{\xi+1,y_0^-}(x_0;p) + J_{\xi+1,x_0^+}(y_0;p)\right] \\ &\quad\leq \left(\epsilon_{\mu,\xi+1,l,\omega,y_0^-}^{\gamma,\delta,k,c}\varphi\right)(x_0;p) + \left(\epsilon_{\mu,\xi+1,l,\omega,x_0^+}^{\gamma,\delta,k,c}\varphi\right)(y_0;p) \\ &\quad\leq \left[J_{\xi-1,y_0^-}(x_0;p) + J_{\xi-1,x_0^+}(y_0;p)\right](y_0-x_0)^2 \\ &\qquad\times \left(\varphi(y_0)\int_0^1 h\left(z^\alpha\right)dz + mf\left(\frac{x_0}{m}\right)\int_0^1 h\left(1-z^\alpha\right)dz\right). \end{aligned} \tag{2.41}$$

Remark 6

(i) If we take $h(t) = t$ in (2.30), then we get [12, Theorem 2.3].
(ii) If we take $h(t) = t$ in (2.41), then we get [12, Corollary 2.3].
(iii) If we take $\alpha = 1$ in (2.30), then we get [9, Theorem 3].
(iv) If we take $\alpha = m = 1$ and $h(t) = t$ in (2.30), then we get [9, Corollary 3].
(v) If we take $\alpha = 1, \omega = p = 0$ in (2.30), then we get [13, Theorem 3].
(vi) If we take $\alpha = m = 1, \omega = p = 0$, and $h(t) = t$ in (2.30), then we obtain [11, Theorem 3].

3 Concluding remarks

In this work, we have studied fractional integral inequalities for a generalized convexity called $(\alpha, h-m)$-convexity. The presented results provide bounds of fractional integral operators involving an extended Mittag-Leffler function and a new fractional Hadamard inequality for $(\alpha, h-m)$-convex functions. The results proved in [8, 9, 11–14] are direct consequences of the theorems of this paper.

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