Che et al. *Journal of Inequalities and Applications* (2020) 2020:188
https://doi.org/10.1186/s13660-020-02454-1

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

New lower bounds for the minimum M-eigenvalue of elasticity M-tensors and applications

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Abstract

M-eigenvalues of elasticity M-tensors play an important role in nonlinear elasticity and materials. In this paper, we present several new lower bounds for the minimum M-eigenvalue of elasticity M-tensors and propose numerical examples to illustrate the efficiency of the obtained results. As applications, we provide several checkable sufficient conditions for the strong ellipticity and positive definiteness of irreducible elasticity M-tensors.

MSC: 15A06; 74B20; 47J25

Keywords: M-eigenvalue; Elasticity M-tensors; Strong ellipticity; M-positive definiteness

1 Introduction

A tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ is called a fourth-order real partially symmetric tensor if

$$a_{ijkl} = a_{jikl} = a_{ijlk}, \quad i, j, l, k \in [n],$$

where $[n] = \{1, 2, \dots, n\}$. The tensor of elastic moduli for a linearly anisotropic elastic solid is a fourth-order real partially symmetric tensor [1], and the components of such a tensor are considered as the coefficients of the following optimization problem:

$$\begin{cases} \min \quad f(\mathbf{x}, \mathbf{y}) = \mathcal{A}\mathbf{xyxy} = \sum_{i,j,k,l \in [n]} a_{ijkl} x_i x_j y_k y_l, \\ \text{s.t.} \quad \mathbf{x}^T\mathbf{x} = 1, \mathbf{y}^T\mathbf{y} = 1, \\ \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \end{cases} \tag{1.1}$$

Problem (1.1) has applications in the ordinary ellipticity and strong ellipticity and nonlinear elastic materials analysis [2–28]. The strong ellipticity condition is stated as $f(\mathbf{x}, \mathbf{y}) > 0$ for all nonzero vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, which guarantees the existence of solutions of basic boundary-value problems of elastostatics and ensures an elastic material to satisfy some mechanical properties [29]. In fact, the KKT condition of (1.1) can be regarded as the following definition of M-eigenvalues.

Definition 1.1 ([1]) Let $\mathcal{A} \in \mathbb{E}_{4,n}$. If there are $\lambda \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ such that

$$\begin{cases} \mathcal{A}\mathbf{x}\mathbf{y}^2 = \lambda\mathbf{x}, \\ \mathcal{A}\mathbf{x}^2\mathbf{y} = \lambda\mathbf{y}, \\ \mathbf{x}^T\mathbf{x} = 1, \\ \mathbf{y}^T\mathbf{y} = 1, \end{cases} \tag{1.2}$$

where $(\mathcal{A}\mathbf{x}\mathbf{y}^2)_i = \sum_{j,k,l\in[n]} a_{ijkl}x_jy_ky_l$, and $(\mathcal{A}\mathbf{x}^2\mathbf{y})_l = \sum_{i,j,k\in[n]} a_{ijkl}x_ix_jy_k$, then the scalar λ is called an M-eigenvalue of $\mathcal{A}$, and $\mathbf{x}$, $\mathbf{y}$ are called the corresponding left and right M-eigenvectors of $\mathcal{A}$, respectively.

Furthermore, Han et al. revealed that the strong ellipticity condition holds if and only if the smallest M-eigenvalue is positive [1]. Recently, Ding et al. [30] investigated a fourth-order structured partially symmetric tensors named elasticity M-tensors, and some sufficient conditions for the strong ellipticity were provided. Since the strong ellipticity condition and M-positive definiteness can be identified by the smallest M-eigenvalue, He et al. [31] proposed some lower bounds for the minimum M-eigenvalue of elasticity M-tensors.

In this paper, we present several new bounds for the minimum M-eigenvalue of elasticity M-tensors. We prove that the bounds are tighter than those proposed in [31]. Numerical examples illustrate the efficiency of the obtained results. As applications, we give some checkable sufficient conditions for the strong ellipticity and positive definiteness of elasticity tensors.

2 Main results

For an elasticity tensor $\mathcal{A} \in \mathbb{E}_{4,n}$, its M-spectral radius is denoted by

$$\rho(\mathcal{A}) = \max\bigl\{|\lambda| : \lambda \text{ is an M-eigenvalue of } \mathcal{A}\bigr\}.$$

The identity tensor $\mathcal{I} = (e_{ijkl}) \in \mathbb{E}_{4,n}$ is defined by

$$e_{ijkl} = \begin{cases} 0 & \text{if } i = j, k = l, \\ 1 & \text{otherwise.} \end{cases}$$

Let $\alpha_i = \max_{l\in[n]}\{a_{iill}\}$, $\beta_l = \max_{i\in[n]}\{a_{iill}\}$, and

$$r_i(\mathcal{A}) = \sum_{j,k,l\in[n],k\neq l} |a_{ijkl}|, \qquad \gamma_i = \sum_{j\in[n],j\neq i} \max_{l\in[n]}\bigl\{|a_{ijll}|\bigr\},$$

$$r_i^i(\mathcal{A}) = \sum_{k,l\in[n],k\neq l} |a_{iikl}|, \qquad R_i(\mathcal{A}) = r_i(\mathcal{A}) + \gamma_i,$$

$$c_l(\mathcal{A}) = \sum_{i,j,k\in[n],i\neq j} |a_{ijkl}|, \qquad \delta_l = \sum_{k\in[n],k\neq l} \max_{i\in[n]}\bigl\{|a_{iikl}|\bigr\},$$

$$c_l^l(\mathcal{A}) = \sum_{i,j\in[n],i\neq j} |a_{ijll}|, \qquad C_l(\mathcal{A}) = c_l(\mathcal{A}) + \delta_l.$$

To continue, we need the following definitions and technical results.

Definition 2.1 ([30]) A tensor $\mathcal{A} \in \mathbb{E}_{4,n}$ is called an elasticity M-tensor if there exist a nonnegative tensor $\mathcal{B} \in \mathbb{E}_{4,n}$ and a real number $s \geq \rho(\mathcal{B})$ such that $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, where $\rho(\mathcal{B})$ is the M-spectral radius of $\mathcal{B}$. Furthermore, if $s > \rho(\mathcal{B})$, then $\mathcal{A}$ is called a nonsingular elasticity M-tensor.

Definition 2.2 ([32]) A tensor $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$ of order m and dimension n is called reducible if there exists a nonempty proper index subset $J \in \{1, 2, \dots, n\} \subset [n]$ such that

$$a_{i_1 i_2 \dots i_m} = 0, \quad \forall i_1 \in J, \forall i_2 \dots i_m \in [n] \backslash J.$$

If $\mathcal{A}$ is not reducible, then we say that $\mathcal{A}$ is irreducible.

Theorem 2.1 ([31]) *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an irreducible and nonnegative partially symmetric tensor, and let $\tau(\mathcal{A})$ be the minimal M-eigenvalue of $\mathcal{A}$. Then $\tau(\mathcal{A}) \geq 0$ is an M-eigenvalue of $\mathcal{A}$ with positive eigenvectors. Moreover, there exist a nonnegative tensor $\mathcal{B}$ and a real number $c \geq \rho(\mathcal{B})$ such that $\mathcal{A} = c\mathcal{I} - \mathcal{B}$.*

Theorem 2.2 ([31]) *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an irreducible elasticity M-tensor. Then*

$$\tau(\mathcal{A}) \leq \min_{i,l \in [n]} \{a_{iill}\}.$$

Theorem 2.3 ([31]) *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an irreducible elasticity M-tensor. Then*

$$\tau(\mathcal{A}) \geq \max\left\{\min_{i \in [n]} \alpha_i - R_i(\mathcal{A}), \min_{l \in [n]} \beta_l - C_l(\mathcal{A})\right\}.$$

Now we are in a position to propose some lower bounds for $\tau(\mathcal{A})$.

Theorem 2.4 *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an irreducible elasticity M-tensor. Then the minimum M-eigenvalue satisfies*

$$\tau(\mathcal{A}) \geq \max\left\{\min_{i,j \in [n], i \neq j} \{\eta_1(\mathcal{A})\}, \min_{k,l \in [n], k \neq l} \{\eta_2(\mathcal{A})\}\right\},$$

where $\eta_1(\mathcal{A}) = \frac{\alpha_i - r_i^i(\mathcal{A}) + \alpha_j - \Delta_{i,j}^{\frac{1}{2}}}{2}$, $\eta_2(\mathcal{A}) = \frac{\beta_k - c_k^k(\mathcal{A}) + \beta_l - \Theta_{k,l}^{\frac{1}{2}}}{2}$, *and*

$$\Delta_{i,j} = \left(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j\right)^2 + 4\left(r_i(\mathcal{A}) - r_i^i(\mathcal{A}) + \gamma_i\right)R_j(\mathcal{A}),$$
$$\Theta_{k,l} = \left(\beta_k - c_k^k(\mathcal{A}) - \beta_l\right)^2 + 4\left(c_k(\mathcal{A}) - c_k^k(\mathcal{A}) + \delta_k\right)C_l(\mathcal{A}).$$

Proof By Theorem 2.1 suppose that $\mathbf{x} = \{x_i\}_{i=1}^n > 0 \in \mathbb{R}^n$ and $\mathbf{y} = \{y_l\}_{l=1}^n > 0 \in \mathbb{R}^n$ are the corresponding left and right M-eigenvectors, respectively. Let $x_p \geq x_s \geq \max_{i \in [n], i \neq p,s}\{x_i\}$.

From the pth equation of $\mathcal{A}\mathbf{x}\mathbf{y}^2 = \tau(\mathcal{A})\mathbf{x}$ in (1.2) we obtain

$$\begin{aligned}\tau(\mathcal{A})x_p &= \sum_{j,k,l\in[n]} a_{pjkl}x_jy_ky_l \\ &= \sum_{k,l\in[n],k\neq l} a_{ppkl}x_py_ky_l + \sum_{j,k,l\in[n],j\neq p,k\neq l} a_{pjkl}x_jy_ky_l \\ &\quad + \sum_{j,l\in[n],j\neq p} a_{pjll}x_jy_l^2 + \sum_{l\in[n]} a_{ppll}x_py_l^2,\end{aligned}$$

that is,

$$\begin{aligned}&\sum_{l\in[n]} a_{ppll}x_py_l^2 - \tau(\mathcal{A})x_p \\ &\quad = -\sum_{k,l\in[n],k\neq l} a_{ppkl}x_py_ky_l - \sum_{j,k,l\in[n],j\neq p,k\neq l} a_{pjkl}x_jy_ky_l - \sum_{j,l\in[n],j\neq p} a_{pjll}x_jy_l^2.\end{aligned}$$

Let $\alpha_p = \min_{l\in[n]}\{a_{ppll}\}$. It follows from Theorem 2.2 that

$$\begin{aligned}0 &\leq \big(\alpha_p - \tau(\mathcal{A})\big)x_p \leq \Big(\sum_{l\in[n]} a_{ppll}x_py_l^2 - \tau(\mathcal{A})\Big)x_p \\ &\leq \sum_{k,l\in[n],k\neq l} |a_{ppkl}|x_p + \sum_{j,k,l\in[n],j\neq p,k\neq l} |a_{pjkl}|x_s + \sum_{j,l\in[n],j\neq p} |a_{pjll}|x_s\big|y_l^2\big|.\end{aligned}$$

Note that

$$\begin{aligned}\sum_{j,l\in[n],j\neq p} |a_{pjll}|x_s\big|y_l^2\big| &= \sum_{j\in[n],j\neq p}\Big(\sum_{l\in[n]} |a_{pjll}|\big|y_l^2\big|\Big)x_s \\ &\leq \sum_{j\in[n],j\neq p} \max_{l\in[n]}|a_{pjll}|\Big(\sum_{l\in[n]}\big|y_l^2\big|\Big)x_s \\ &= \sum_{j\in[n],j\neq p} \max_{l\in[n]}\big\{|a_{pjll}|\big\}x_s.\end{aligned}$$

Furthermore,

$$\big(\alpha_p - \tau(\mathcal{A}) - r_p^p(\mathcal{A})\big)x_p \leq \big(r_p(\mathcal{A}) - r_p^p(\mathcal{A}) + \gamma_p\big)x_s. \tag{2.1}$$

From the sth equation of $\mathcal{A}\mathbf{x}\mathbf{y}^2 = \tau(\mathcal{A})\mathbf{x}$ in (1.2) we have

$$\begin{aligned}\tau(\mathcal{A})x_s &= \sum_{j,k,l\in[n]} a_{sjkl}x_jy_ky_l \\ &= \sum_{j,k,l\in[n],k\neq l} a_{sjkl}x_jy_ky_l + \sum_{j,l\in[n],j\neq s} a_{sjll}x_jy_l^2 + \sum_{l\in[n]} a_{ssll}y_l^2x_s.\end{aligned}$$

Let $\alpha_s = \min_{l\in[n]}\{a_{ssll}\}$. It follows from Theorem 2.2 that

$$\big(\alpha_s - \tau(\mathcal{A})\big)x_s \leq R_s(\mathcal{A})x_p. \tag{2.2}$$

Multiplying (2.1) and (2.2), we have

$$\big(\alpha_p - \tau(\mathcal{A}) - r_p^p(\mathcal{A})\big)\big(\alpha_s - \tau(\mathcal{A})\big) \le \big(r_p(\mathcal{A}) - r_p^p(\mathcal{A}) + \gamma_p\big)R_s(\mathcal{A}),$$

which means that

$$\tau(\mathcal{A}) \ge \frac{\alpha_p - r_p^p(\mathcal{A}) + \alpha_s - \Delta_{p,s}^{\frac{1}{2}}}{2}, \tag{2.3}$$

where $\Delta_{p,s} = (\alpha_p - r_p^p(\mathcal{A}) - \alpha_s)^2 + 4(r_p(\mathcal{A}) - r_p^p(\mathcal{A}) + \gamma_p)R_s(\mathcal{A})$.

On the other hand, let $|y_q| \ge |y_t| \ge \max_{l\in[n],l\neq q,t} |y_l|$. From the qth equation of $\mathcal{A}\mathbf{x}^2\mathbf{y} = \tau(\mathcal{A})\mathbf{y}$ in (1.2) it follows that

$$\begin{aligned}\tau(\mathcal{A})y_q &= \sum_{i,j,k\in[n]} a_{ijkq}x_ix_jy_k \\ &= \sum_{i,j\in[n],i\neq j} a_{ijqq}x_ix_jy_q + \sum_{i,j,k\in[n],i\neq j,k\neq q} a_{ijkq}x_ix_jy_k + \sum_{i,k\in[n],k\neq q} a_{iikq}x_i^2y_k + \sum_{i\in[n]} a_{iiqq}y_qx_i^2.\end{aligned}$$

Let $\beta_q = \min_{i\in[n]}\{a_{iiqq}\}$. It follows from Theorem 2.2 that

$$\begin{aligned}0 &\le \big(\beta_q - \tau(\mathcal{A})\big)y_q \le \Big(\sum_{i\in[n]} a_{iiqq}y_qx_i^2 - \tau(\mathcal{A})\Big)y_q \\ &= -\sum_{i,j\in[n],i\neq j} a_{ijqq}x_ix_jy_q - \sum_{i,j,k\in[n],i\neq j,k\neq q} a_{ijkq}x_ix_jy_k - \sum_{i,k\in[n],k\neq q} a_{iikq}x_i^2y_k \\ &\le \sum_{i,j\in[n],i\neq j} |a_{ijqq}|y_q + \sum_{i,j,k\in[n],i\neq j,k\neq q} |a_{ijkq}|y_t + \sum_{i,k\in[n],k\neq q} |a_{iikq}|y_t,\end{aligned}$$

that is,

$$\big(\beta_q - \tau(\mathcal{A}) - c_q^q(\mathcal{A})\big)y_q \le \big(c_q(\mathcal{A}) - c_q^q(\mathcal{A}) + \delta_q\big)y_t. \tag{2.4}$$

From the tth equation of $\mathcal{A}\mathbf{x}^2\mathbf{y} = \tau(\mathcal{A})\mathbf{y}$ in (1.2) we obtain

$$\begin{aligned}\tau(\mathcal{A})y_t &= \sum_{i,j,k\in[n]} a_{ijkt}x_ix_jy_k \\ &= \sum_{i,j,k\in[n],i\neq j} a_{ijkt}x_ix_jy_k + \sum_{i,k\in[n],k\neq t} a_{iikt}x_i^2y_k + \sum_{i\in[n]} a_{iitt}x_i^2y_t.\end{aligned}$$

Let $\beta_t = \min_{i\in[n]}\{a_{iitt}\}$. This yields

$$\big(\beta_t - \tau(\mathcal{A})\big)y_t \le C_t(\mathcal{A})y_q. \tag{2.5}$$

Multiplying (2.4) and (2.5), we have

$$\big(\beta_q - \tau(\mathcal{A}) - c_q^q(\mathcal{A})\big)\big(\beta_t - \tau(\mathcal{A})\big) \le \big(c_q(\mathcal{A}) - c_q^q(\mathcal{A}) + \delta_q\big)C_t(\mathcal{A}), \tag{2.6}$$

which means that

$$\tau(\mathcal{A}) \geq \frac{\beta_q - c_q^q(\mathcal{A}) + \beta_t - \Theta_{q,t}^{\frac{1}{2}}}{2}, \tag{2.7}$$

where $\Theta_{q,t} = (\beta_q - c_p^p(\mathcal{A}) - \beta_t)^2 + 4(c_p(\mathcal{A}) - c_p^p(\mathcal{A}) + \delta_q)C_t(\mathcal{A})$. Then the conclusion follows. □

Next, we compare the bound in Theorem 2.3 with that in Theorem 2.4 and obtain the following conclusion.

Theorem 2.5 *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an irreducible elasticity M-tensor. Then*

$$\begin{aligned}\tau(\mathcal{A}) &\geq \max\Bigl\{\min_{i,j\in[n],i\neq j}\bigl\{\eta_1(\mathcal{A})\bigr\}, \min_{k,l\in[n],k\neq l}\bigl\{\eta_2(\mathcal{A})\bigr\}\Bigr\} \\ &\geq \max\Bigl\{\min_{i\in[n]}\bigl\{\alpha_i - R_i(\mathcal{A})\bigr\}, \min_{l\in[n]}\bigl\{\beta_l - C_l(\mathcal{A})\bigr\}\Bigr\}.\end{aligned}$$

Proof We first show that $\min_{i,j\in[n],i\neq j}\{\eta_1(\mathcal{A})\} \geq \min_{i\in[n]}\{\alpha_i - R_i(\mathcal{A})\}$ and divide the argument into two cases.

Case 1. For any $i, j \in [n]$, $i \neq j$, if $\alpha_i - R_i(\mathcal{A}) \leq \alpha_j - R_j(\mathcal{A})$, then

$$\alpha_j - \alpha_i + R_i(\mathcal{A}) \geq R_j(\mathcal{A}) \geq 0. \tag{2.8}$$

From (2.8) we deduce

$$\begin{aligned}&\bigl(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j\bigr)^2 + 4\bigl(r_i(\mathcal{A}) - r_i^i(\mathcal{A}) + \gamma_i\bigr)R_j(\mathcal{A}) \\ &\quad\leq \bigl(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j\bigr)^2 + 4\bigl(r_i(\mathcal{A}) - r_i^i(\mathcal{A}) + \gamma_i\bigr)\bigl(\alpha_j - \alpha_i + R_i(\mathcal{A})\bigr) \\ &\quad= \bigl(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j\bigr)^2 + 4\bigl(R_i(\mathcal{A}) - r_i^i(\mathcal{A})\bigr)\bigl(\alpha_j - \alpha_i + R_i(\mathcal{A})\bigr) \\ &\quad= \bigl(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j\bigr)^2 + 4\bigl(R_i(\mathcal{A}) - r_i^i(\mathcal{A})\bigr)\bigl(\alpha_j - \alpha_i + r_i^i(\mathcal{A}) - r_i^i(\mathcal{A}) + R_i(\mathcal{A})\bigr) \\ &\quad= \bigl(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j\bigr)^2 + 4\bigl(R_i(\mathcal{A}) - r_i^i(\mathcal{A})\bigr)\bigl(\alpha_j - \alpha_i + r_i^i(\mathcal{A})\bigr) + 4\bigl(R_i(\mathcal{A}) - r_i^i(\mathcal{A})\bigr)^2 \\ &\quad= \bigl(\alpha_j - \alpha_i - r_i^i(\mathcal{A}) + 2R_i(\mathcal{A})\bigr)^2.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{1}{2}\Bigl(\alpha_i - r_i^i(\mathcal{A}) + \alpha_j - \sqrt{\bigl(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j\bigr)^2 + 4\bigl(r_i(\mathcal{A}) - r_i^i(\mathcal{A}) + \gamma_i\bigr)R_j(\mathcal{A})}\Bigr) \\ &\quad\geq \frac{1}{2}\bigl(\alpha_i - r_i^i(\mathcal{A}) + \alpha_j - \bigl(\alpha_j - \alpha_i - r_i^i(\mathcal{A}) + 2R_i(\mathcal{A})\bigr)\bigr) \\ &\quad= \alpha_i - R_i(\mathcal{A}),\end{aligned}$$

which means that

$$\begin{aligned}&\frac{1}{2}\min_{i,j\in[n],i\neq j}\Bigl\{\alpha_i - r_i^i(\mathcal{A}) + \alpha_j - \sqrt{\bigl(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j\bigr)^2 + 4\bigl(r_i(\mathcal{A}) - r_i^i(\mathcal{A}) + \gamma_i\bigr)R_j(\mathcal{A})}\Bigr\} \\ &\quad\geq \min_{i\in[n]}\bigl\{\alpha_i - R_i(\mathcal{A})\bigr\}.\end{aligned}$$

Case 2. For any $i, j \in [n]$, $i \neq j$, if $\alpha_i - R_i(\mathcal{A}) \geq \alpha_j - R_j(\mathcal{A})$, then

$$\alpha_i - r_i(\mathcal{A}) - \alpha_j + R_j(\mathcal{A}) \geq \gamma_i. \tag{2.9}$$

From (2.9) we have

$$\begin{aligned}
&\bigl(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j\bigr)^2 + 4\bigl(r_i(\mathcal{A}) - r_i^i(\mathcal{A}) + \gamma_i\bigr)R_j(\mathcal{A}) \\
&\quad\leq \bigl(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j\bigr)^2 + 4\bigl(r_i(\mathcal{A}) - r_i^i(\mathcal{A}) + \alpha_i - r_i(\mathcal{A}) - \alpha_j + R_j(\mathcal{A})\bigr)R_j(\mathcal{A}) \\
&\quad= \bigl(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j\bigr)^2 + 4\bigl(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j + R_j(\mathcal{A})\bigr)R_j(\mathcal{A}) \\
&\quad= \bigl(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j + 2R_j(\mathcal{A})\bigr)^2.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{2}\Bigl(\alpha_i - r_i^i(\mathcal{A}) + \alpha_j - \sqrt{\bigl(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j\bigr)^2 + 4\bigl(r_i(\mathcal{A}) - r_i^i(\mathcal{A}) + \gamma_i\bigr)R_j(\mathcal{A})}\Bigr) \\
&\quad\geq \frac{1}{2}\bigl(\alpha_i - r_i^i(\mathcal{A}) + \alpha_j - \bigl(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j + 2R_j(\mathcal{A})\bigr)\bigr) \\
&\quad= \alpha_j - R_j(\mathcal{A}),
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{1}{2}\min_{i,j\in[n],i\neq j}\Bigl\{\alpha_i - r_i^i(\mathcal{A}) + \alpha_j - \sqrt{\bigl(\alpha_i - r_i^i(\mathcal{A}) - \alpha_j\bigr)^2 + 4\bigl(r_i(\mathcal{A}) - r_i^i(\mathcal{A}) + \gamma_i\bigr)R_j(\mathcal{A})}\Bigr\} \\
&\quad\geq \min_{j\in[n]}\bigl\{\alpha_j - R_j(\mathcal{A})\bigr\}.
\end{aligned}$$

Therefore we obtain $\min_{i,j\in[n],i\neq j}\{\eta_1(\mathcal{A})\} \geq \min_{i\in[n]}\{\alpha_i - R_i(\mathcal{A})\}$.

Similarly, we have $\min_{i,j\in[n],i\neq j}\{\eta_2(\mathcal{A})\} \geq \min_{l\in[n]}\{\beta_l - C_l(\mathcal{A})\}$. Thus we deduce

$$\max\Bigl\{\min_{i,j\in[n],i\neq j}\bigl\{\eta_1(\mathcal{A})\bigr\}, \min_{k,l\in[n],k\neq l}\bigl\{\eta_2(\mathcal{A})\bigr\}\Bigr\} \geq \max\Bigl\{\min_{i\in[n]}\bigl\{\alpha_i - R_i(\mathcal{A})\bigr\}, \min_{l\in[n]}\bigl\{\beta_l - C_l(\mathcal{A})\bigr\}\Bigr\},$$

and the desired result follows. □

In what follows, we propose another lower bound for $\tau(\mathcal{A})$.

Theorem 2.6 *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an irreducible elasticity M-tensor. Then*

$$\begin{aligned}
\tau(\mathcal{A}) \geq \max\Bigl\{&\min_{i,j\in[n],i\neq j}\bigl\{\theta_1(\mathcal{A}), \alpha_i - r_i^i(\mathcal{A}), \alpha_j - r_j^j(\mathcal{A})\bigr\}, \\
&\min_{k,l\in[n],k\neq l}\bigl\{\theta_2(\mathcal{A}), \beta_k - c_k^k(\mathcal{A}), \beta_l - c_l^l(\mathcal{A})\bigr\}\Bigr\},
\end{aligned}$$

where

$$\begin{aligned}
\theta_1(\mathcal{A}) &= \frac{(\alpha_i - r_i^i(\mathcal{A})) + (\alpha_j - r_j^j(\mathcal{A})) - \Omega_{i,j}^{\frac{1}{2}}}{2}, \\
\theta_2(\mathcal{A}) &= \frac{(\beta_k - c_k^k(\mathcal{A})) + (\beta_l - c_l^l(\mathcal{A})) - \Phi_{k,l}^{\frac{1}{2}}}{2},
\end{aligned}$$

$$\Omega_{i,j} = \big(\alpha_i - r_i^i(\mathcal{A}) - \big(\alpha_j - r_j^j(\mathcal{A})\big)\big)^2 + 4\big(R_i(\mathcal{A}) - r_i^i(\mathcal{A})\big)\big(R_j(\mathcal{A}) - r_j^j(\mathcal{A})\big),$$

and

$$\Phi_{k,l} = \big(\beta_k - c_k^k(\mathcal{A}) - \big(\beta_l - c_l^l(\mathcal{A})\big)\big)^2 + 4\big(C_k(\mathcal{A}) - c_k^k(\mathcal{A})\big)\big(C_l(\mathcal{A}) - c_l^l(\mathcal{A})\big).$$

Proof Let $\tau(\mathcal{A})$ be the minimal M-eigenvalue of tensor $\mathcal{A}$. From Theorem 2.1 we suppose that $\mathbf{x} = \{x_i\}_{i=1}^n > 0 \in \mathbb{R}^n$ and $\mathbf{y} = \{y_l\}_{l=1}^n > 0 \in \mathbb{R}^n$ are the corresponding left and right M-eigenvectors, respectively. Let $x_p \geq x_s \geq \max_{i\in[n],i\neq p,s}\{x_i\}$. From the sth equation of $\mathcal{A}\mathbf{x}\mathbf{y}^2 = \tau(\mathcal{A})\mathbf{x}$ in (1.2) we have

$$\begin{aligned}\tau(\mathcal{A})x_s &= \sum_{j,k,l\in[n]} a_{sjkl}x_jy_ky_l \\ &= \sum_{k,l\in[n],k\neq l} a_{sskl}x_sy_ky_l + \sum_{j,k,l\in[n],j\neq s,k\neq l} a_{sjkl}x_jy_ky_l + \sum_{j,l\in[n],j\neq s} a_{sjll}x_jy_l^2 + \sum_{l\in[n]} a_{ssll}y_l^2x_s.\end{aligned}$$

Let $\alpha_s = \min_{l\in[n]}\{a_{ssll}\}$. It follows from Theorem 2.2 that

$$\begin{aligned}0 &\leq \big(\alpha_s - \tau(\mathcal{A})\big)x_s \leq \Big(\sum_{l\in[n]} a_{ssll}x_sy_l^2 - \tau(\mathcal{A})\Big)x_p \\ &\leq \sum_{k,l\in[n],k\neq l} |a_{sskl}|x_s + \sum_{j,k,l\in[n],j\neq s,k\neq l} |a_{pjkl}|x_p + \sum_{j,l\in[n],j\neq s} |a_{sjll}|x_p\big|y_l^2\big|.\end{aligned}$$

Moreover,

$$\big(\alpha_s - \tau(\mathcal{A}) - r_s^s(\mathcal{A})\big)x_s \leq \big(r_s(\mathcal{A}) - r_s^s(\mathcal{A}) + \gamma_s\big)x_p. \tag{2.10}$$

When $\alpha_s - r_s^s(\mathcal{A}) > \tau(\mathcal{A})$ or $\alpha_p - r_p^p(\mathcal{A}) > \tau(\mathcal{A})$, multiplying (2.1) and (2.10), we have

$$\big(\alpha_p - \tau(\mathcal{A}) - r_p^p(\mathcal{A})\big)\big(\alpha_s - \tau(\mathcal{A}) - r_s^s(\mathcal{A})\big) \leq \big(r_p(\mathcal{A}) - r_p^p(\mathcal{A}) + \gamma_p\big)\big(r_s(\mathcal{A}) - r_s^s(\mathcal{A}) + \gamma_s\big), \tag{2.11}$$

that is,

$$\tau(\mathcal{A}) \geq \frac{(\alpha_p - r_p^p(\mathcal{A})) + (\alpha_s - r_s^s(\mathcal{A})) - \Omega_{p,s}^{\frac{1}{2}}}{2}, \tag{2.12}$$

where $\Omega_{p,s} = (\alpha_p - r_p^p(\mathcal{A}) - (\alpha_s - r_s^s(\mathcal{A})))^2 + 4(R_p(\mathcal{A}) - r_p^p(\mathcal{A}))(R_s(\mathcal{A}) - r_s^s(\mathcal{A}))$.

On the other hand, let $|y_q| \geq |y_t| \geq \max_{l\in[n],l\neq q,t}|y_l|$. From the tth equation of $\mathcal{A}\mathbf{x}^2\mathbf{y} = \tau(\mathcal{A})\mathbf{y}$ in (1.2) we obtain

$$\begin{aligned}\tau(\mathcal{A})y_t &= \sum_{j,k,l\in[n]} a_{ijkt}x_ix_jy_k \\ &= \sum_{i,j\in[n],i\neq j} a_{ijtt}x_ix_jy_t + \sum_{i,j,k\in[n],i\neq j,k\neq t} a_{ijkt}x_ix_jy_k + \sum_{i,k\in[n],k\neq t} a_{iikt}x_i^2y_k + \sum_{i\in[n]} a_{iitt}x_i^2y_t.\end{aligned}$$

Let $\beta_t = \min_{i\in[n]}\{a_{iitt}\}$. It follows from Theorem 2.2 that

$$\begin{aligned}
0 &\le \bigl(\beta_t - \tau(\mathcal{A})\bigr)y_t \le \Bigl(\sum_{i\in[n]} a_{iitt}y_t x_i^2 - \tau(\mathcal{A})\Bigr)y_t \\
&= -\sum_{i,j\in[n],i\neq j} a_{ijtt}x_i x_j y_t - \sum_{i,j,k\in[n],i\neq j,k\neq t} a_{ijkt}x_i x_j y_k - \sum_{i,k\in[n],k\neq t} a_{iikt}x_i^2 y_k \\
&\le \sum_{i,j\in[n],i\neq j} |a_{ijtt}|y_t + \sum_{i,j,k\in[n],i\neq j,k\neq t} |a_{ijkt}|y_q + \sum_{i,k\in[n],k\neq t} |a_{iikt}|x_i^2 y_q,
\end{aligned}$$

that is,

$$\bigl(\beta_t - \tau(\mathcal{A}) - c_t^t(\mathcal{A})\bigr)y_t \le \bigl(c_t(\mathcal{A}) - c_t^t(\mathcal{A}) + \delta_t\bigr)y_q. \tag{2.13}$$

When $\beta_t - c_t^t(\mathcal{A}) > \tau(\mathcal{A})$ or $\beta_q - c_q^q(\mathcal{A}) > \tau(\mathcal{A})$, multiplying (2.6) and (2.13), we have

$$\bigl(\beta_q - \tau(\mathcal{A}) - c_q^q(\mathcal{A})\bigr)\bigl(\beta_t - \tau(\mathcal{A}) - c_t^t(\mathcal{A})\bigr) \le \bigl(c_q(\mathcal{A}) - c_q^q(\mathcal{A}) + \gamma_q\bigr)\bigl(c_t(\mathcal{A}) - c_t^t(\mathcal{A}) + \gamma_t\bigr), \tag{2.14}$$

which means that

$$\tau(\mathcal{A}) \ge \frac{(\beta_q - c_q^q(\mathcal{A})) + (\beta_t - c_t^t(\mathcal{A})) - \Phi_{q,t}^{\frac{1}{2}}}{2}, \tag{2.15}$$

where $\Phi_{q,t} = (\beta_q - c_q^q(\mathcal{A}) - (\beta_t - c_t^t(\mathcal{A})))^2 + 4(C_q(\mathcal{A}) - c_q^q(\mathcal{A}))(C_t(\mathcal{A}) - c_t^t(\mathcal{A}))$. □

Next, we compare the bound in Theorem 2.3 with that in Theorem 2.6 and obtain the following result.

Theorem 2.7 *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an irreducible elasticity M-tensor. Then*

$$\begin{aligned}
\tau(\mathcal{A}) &\ge \max\Bigl\{\min_{i,j\in[n],i\neq j}\bigl\{\theta_1(\mathcal{A}), \alpha_i - r_i^i(\mathcal{A}), \alpha_j - r_j^j(\mathcal{A})\bigr\}, \\
&\qquad \min_{k,l\in[n],k\neq l}\bigl\{\theta_2(\mathcal{A}), \beta_k - c_k^k(\mathcal{A}), \beta_l - c_l^l(\mathcal{A})\bigr\}\Bigr\} \\
&\ge \max\Bigl\{\min_{i\in[n]}\bigl\{\alpha_i - R_i(\mathcal{A})\bigr\}, \min_{l\in[n]}\bigl\{\beta_l - C_l(\mathcal{A})\bigr\}\Bigr\}.
\end{aligned}$$

Proof We will show $\min_{i,j\in[n],i\neq j}\{\theta_1(\mathcal{A}), \alpha_i - r_i^i(\mathcal{A}), \alpha_j - r_j^j(\mathcal{A})\} \ge \min_{i\in[n]}\{\alpha_i - R_i(\mathcal{A})\}$ and divide the argument into two cases.

Case 1. For any $i,j \in [n]$, $i \neq j$, if $\alpha_i - R_i(\mathcal{A}) \le \alpha_j - R_j(\mathcal{A})$, then from (2.8) we have

$$\begin{aligned}
&\bigl(\alpha_i - r_i^i(\mathcal{A}) - (\alpha_j - r_j^j(\mathcal{A}))\bigr)^2 + 4\bigl(R_i(\mathcal{A}) - r_i^i(\mathcal{A})\bigr)\bigl(R_j(\mathcal{A}) - r_j^j(\mathcal{A})\bigr) \\
&\quad \le \bigl(\alpha_i - r_i^i(\mathcal{A}) - (\alpha_j - r_j^j(\mathcal{A}))\bigr)^2 \\
&\qquad + 4\bigl(R_i(\mathcal{A}) - r_i^i(\mathcal{A})\bigr)\bigl(\alpha_j - \alpha_i + r_i^i(\mathcal{A}) - r_j^j(\mathcal{A}) + R_i(\mathcal{A}) - r_i^i(\mathcal{A})\bigr) \\
&\quad = \bigl(\alpha_j - \alpha_i + r_i^i(\mathcal{A}) - r_j^j(\mathcal{A}) + 2R_i(\mathcal{A}) - 2r_i^i(\mathcal{A})\bigr)^2.
\end{aligned}$$

Since

$$\begin{aligned} &\alpha_j - \alpha_i + r_i^i(\mathcal{A}) - r_j^j(\mathcal{A}) + 2R_i(\mathcal{A}) - 2r_i^i(\mathcal{A}) \\ &\quad = \alpha_j - \alpha_i + R_i(\mathcal{A}) - R_j(\mathcal{A}) + R_j(\mathcal{A}) - r_j^j(\mathcal{A}) + R_i(\mathcal{A}) - r_i^i(\mathcal{A}) \geq 0, \end{aligned}$$

we have

$$\begin{aligned} \theta_1(\mathcal{A}) &= \frac{(\alpha_i - r_i^i(\mathcal{A})) + (\alpha_j - r_j^j(\mathcal{A})) - \Omega_{i,j}^{\frac{1}{2}}}{2} \\ &\geq \frac{1}{2}\big(\alpha_i - r_i^i(\mathcal{A})\big) + \big(\alpha_j - r_j^j(\mathcal{A}) - \big(\alpha_j - \alpha_i + r_i^i(\mathcal{A}) - r_j^j(\mathcal{A}) + 2R_i(\mathcal{A}) - 2r_i^i(\mathcal{A})\big)\big) \\ &= \alpha_i - R_i(\mathcal{A}), \end{aligned}$$

which means that

$$\min_{i,j\in[n],i\neq j}\big\{\theta_1(\mathcal{A}), \alpha_i - r_i^i(\mathcal{A}), \alpha_j - r_j^j(\mathcal{A})\big\} \geq \min_{i\in[n]}\big\{\alpha_i - R_i(\mathcal{A})\big\}.$$

Case 2. For any $i,j \in [n]$, $i \neq j$, if $\alpha_i - R_i(\mathcal{A}) \geq \alpha_j - R_j(\mathcal{A})$, then

$$\alpha_i - \alpha_j + R_j(\mathcal{A}) \geq R_i(\mathcal{A}). \tag{2.16}$$

From (2.16) we have

$$\begin{aligned} &\big(\alpha_i - r_i^i(\mathcal{A}) - \big(\alpha_j - r_j^j(\mathcal{A})\big)\big)^2 + 4\big(R_i(\mathcal{A}) - r_i^i(\mathcal{A})\big)\big(R_j(\mathcal{A}) - r_j^j(\mathcal{A})\big) \\ &\quad \leq \big(\alpha_i - r_i^i(\mathcal{A}) - \big(\alpha_j - r_j^j(\mathcal{A})\big)\big)^2 + 4\big(\alpha_i - \alpha_j + R_j(\mathcal{A}) - r_i^i(\mathcal{A})\big)\big(R_j(\mathcal{A}) - r_j^j(\mathcal{A})\big) \\ &\quad = \big(\alpha_i - r_i^i(\mathcal{A}) - \big(\alpha_j - r_j^j(\mathcal{A})\big)\big)^2 \\ &\qquad + 4\big(\alpha_i - \alpha_j + r_j^j(\mathcal{A}) - r_i^i(\mathcal{A}) + R_j(\mathcal{A}) - r_j^j(\mathcal{A})\big)\big(R_j(\mathcal{A}) - r_j^j(\mathcal{A})\big) \\ &\quad = \big(\alpha_i - r_i^i(\mathcal{A}) - \big(\alpha_j - r_j^j(\mathcal{A})\big) + 2R_j(\mathcal{A}) - 2r_j^j(\mathcal{A})\big)^2. \end{aligned}$$

Since

$$\begin{aligned} &\alpha_i - r_i^i(\mathcal{A}) - \big(\alpha_j - r_j^j(\mathcal{A})\big) + 2R_j(\mathcal{A}) - 2r_j^j(\mathcal{A}) \\ &\quad = \alpha_i - \alpha_j + R_j(\mathcal{A}) - R_i(\mathcal{A}) + R_i(\mathcal{A}) - r_i^i(\mathcal{A}) + R_j(\mathcal{A}) - r_j^j(\mathcal{A}) \geq 0, \end{aligned}$$

we have

$$\begin{aligned} \theta_1(\mathcal{A}) &= \frac{(\alpha_i - r_i^i(\mathcal{A})) + (\alpha_j - r_j^j(\mathcal{A})) - \Omega_{i,j}^{\frac{1}{2}}}{2} \\ &\geq \frac{1}{2}\big(\alpha_i - r_i^i(\mathcal{A})\big) + \big(\alpha_j - r_j^j(\mathcal{A}) - \big(\alpha_i - r_i^i(\mathcal{A}) - \big(\alpha_j - r_j^j(\mathcal{A})\big) + 2R_j(\mathcal{A}) - 2r_j^j(\mathcal{A})\big)\big) \\ &= \alpha_j - R_j(\mathcal{A}), \end{aligned}$$

which means that

$$\min_{i,j\in[n],i\neq j}\big\{\theta_1(\mathcal{A}), \alpha_i - r_i^i(\mathcal{A}), \alpha_j - r_j^j(\mathcal{A})\big\} \geq \min_{i\in[n]}\big\{\alpha_i - R_i(\mathcal{A})\big\}.$$

Similarly, we have $\min_{k,l\in[n],k\neq l}\{\theta_2(\mathcal{A}),\beta_k - c_k^k(\mathcal{A}),\beta_l - c_l^l(\mathcal{A})\} \geq \min_{l\in[n]}\{\beta_l - C_l(\mathcal{A})\}$. Thus we deduce

$$\begin{aligned}\tau(\mathcal{A}) \geq \max\Big\{&\min_{i,j\in[n],i\neq j}\big\{\theta_1(\mathcal{A}),\alpha_i - r_i^i(\mathcal{A}),\alpha_j - r_j^j(\mathcal{A})\big\},\\ &\min_{k,l\in[n],k\neq l}\big\{\theta_2(\mathcal{A}),\beta_k - c_k^k(\mathcal{A}),\beta_l - c_l^l(\mathcal{A})\big\}\Big\}\\ \geq{}& \max\Big\{\min_{i\in[n]}\big\{\alpha_i - R_i(\mathcal{A})\big\},\min_{l\in[n]}\big\{\beta_l - C_l(\mathcal{A})\big\}\Big\},\end{aligned}$$

and the desired result follows. □

The following example shows the superiority of the conclusions obtained in Theorems 2.4 and 2.6.

Example 2.1 ([30]) Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,2}$ be an elasticity M-tensor defined by

$$a_{ijkl} = \begin{cases} a_{1111} = 13, \quad a_{2222} = 12, \quad a_{1122} = 2, \quad a_{2211} = 2,\\ a_{1112} = a_{1121} = -2, \quad a_{2212} = a_{2221} = -1,\\ a_{2111} = a_{1211} = -2, \quad a_{1222} = a_{2122} = -1,\\ a_{1212} = a_{2112} = a_{1221} = a_{2121} = -4. \end{cases}$$

From the matrices

$$\mathcal{A}(\cdot,f_1) = \begin{bmatrix} a_{1111} & a_{1211}\\ a_{2111} & a_{2211}\end{bmatrix} = \begin{bmatrix} 13 & -2\\ -2 & 2\end{bmatrix}, \qquad \mathcal{A}(\cdot,f_2) = \begin{bmatrix} a_{1122} & a_{1222}\\ a_{2122} & a_{2222}\end{bmatrix} = \begin{bmatrix} 2 & -1\\ -4 & 12\end{bmatrix},$$

$$\mathcal{A}(g_1,\cdot) = \begin{bmatrix} a_{1111} & a_{1112}\\ a_{1121} & a_{1122}\end{bmatrix} = \begin{bmatrix} 13 & -2\\ -2 & 2\end{bmatrix}, \qquad \mathcal{A}(g_2,\cdot,) = \begin{bmatrix} a_{2211} & a_{2212}\\ a_{2221} & a_{2222}\end{bmatrix} = \begin{bmatrix} 2 & -1\\ -1 & 12\end{bmatrix},$$

we know that $\mathcal{A}$ is irreducible. By simple computation, $\mathcal{A}$ has six M-eigenvalues: 13.4163, 12.1118, 11.2036, 6.1778, 0.2442, and 0.1964. The minimal M-eigenvalue of $\mathcal{A}$ is 0.1964. Furthermore, we obtain

$$\begin{aligned}&\alpha_1 = 13, \quad \alpha_2 = 12, \quad \beta_1 = 13, \quad \beta_2 = 12,\\ &r_1(\mathcal{A}) = 12, \quad r_2(\mathcal{A}) = 10, \quad c_1(\mathcal{A}) = 12, \quad c_2(\mathcal{A}) = 10,\\ &\gamma_1 = 2, \quad \gamma_2 = 2, \quad \delta_1 = 2, \quad \delta_2 = 2,\\ &r_1^1(\mathcal{A}) = 4, \quad r_2^2(\mathcal{A}) = 2, \quad c_1^1(\mathcal{A}) = 4, \quad c_2^2(\mathcal{A}) = 2,\\ &R_1(\mathcal{A}) = 14, \quad R_2(\mathcal{A}) = 12, \quad C_1(\mathcal{A}) = 14, \quad C_2(\mathcal{A}) = 12.\end{aligned}$$

From Theorems 3.1 and 3.2 in [31] we have $\tau(\mathcal{A}) \geq -1$ and $\tau(\mathcal{A}) \geq -0.8655$, respectively. By Theorem 2.4 we have $\tau(\mathcal{A}) \geq -0.5567$, and by Theorem 2.6 we have $\tau(\mathcal{A}) \geq -0.5125$. Their comparison is drawn in Fig. 1, which reveals that our bounds are tighter than those of [31].

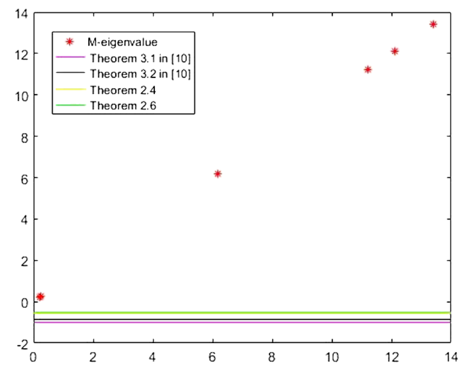

Figure 1 Comparison of the bound estimations of Theorems 3.1 and 3.2 in [31], Theorem 2.4, and Theorem 2.6

3 Strong ellipticity and positive definiteness

In this section, based on the results in Theorems 2.4 and 2.6, we present some sufficient conditions for the strong ellipticity and positive definiteness.

Theorem 3.1 *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an irreducible elasticity M-tensor. If*

$$\max\Big\{\min_{i,j\in[n],i\neq j}\{\eta_1(\mathcal{A})\}, \min_{k,l\in[n],k\neq l}\{\eta_2(\mathcal{A})\}\Big\} > 0,$$

then $\mathcal{A}$ is positive definite, and the strong ellipticity condition holds.

Proof From Theorem 2.4 we have

$$\tau(\mathcal{A}) \geq \max\Big\{\min_{i,j\in[n],i\neq j}\{\eta_1(\mathcal{A})\}, \min_{k,l\in[n],k\neq l}\{\eta_2(\mathcal{A})\}\Big\} > 0.$$

Hence $\mathcal{A}$ is positive definite, and the strong ellipticity condition holds. □

Theorem 3.2 *Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,n}$ be an irreducible elasticity M-tensor. If*

$$\max\Big\{\min_{i,j\in[n],i\neq j}\{\theta_1(\mathcal{A}), \alpha_i - r_i^i(\mathcal{A}), \alpha_j - r_j^j(\mathcal{A})\}, \min_{k,l\in[n],k\neq l}\{\theta_2(\mathcal{A}), \beta_k - c_k^k(\mathcal{A}), \beta_l - c_l^l(\mathcal{A})\}\Big\} > 0,$$

then $\mathcal{A}$ is positive definite, and the strong ellipticity condition holds.

Proof From Theorem 2.6 we have

$$\begin{aligned}\tau(\mathcal{A}) \geq \max\Big\{&\min_{i,j\in[n],i\neq j}\{\theta_1(\mathcal{A}), \alpha_i - r_i^i(\mathcal{A}), \alpha_j - r_j^j(\mathcal{A})\},\\ &\min_{k,l\in[n],k\neq l}\{\theta_2(\mathcal{A}), \beta_k - c_k^k(\mathcal{A}), \beta_l - c_l^l(\mathcal{A})\}\Big\}\\ &> 0.\end{aligned}$$

Hence $\mathcal{A}$ is positive definite, and the strong ellipticity condition holds. □

The following example reveals that Theorems 3.1 and 3.2 can identify the positive definiteness of elasticity M-tensors.

Example 3.1 Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{E}_{4,2}$ be an elasticity M-tensor such that

$$a_{ijkl} = \begin{cases} a_{1111} = 7, \quad a_{2222} = 8, \quad a_{1122} = 7, \quad a_{2211} = 8, \\ a_{1112} = a_{1121} = -1, \quad a_{2212} = a_{2221} = -1, \\ a_{2111} = a_{1211} = -1, \quad a_{1222} = a_{2122} = -1, \\ a_{1212} = a_{2112} = a_{1221} = a_{2121} = -0.5. \end{cases}$$

By a direct computation we have

$$\mathcal{A}(\cdot, f_1) = \begin{bmatrix} 7 & -1 \\ -1 & 8 \end{bmatrix}, \qquad \mathcal{A}(\cdot, f_2) = \begin{bmatrix} 7 & -1 \\ -1 & 8 \end{bmatrix},$$

$$\mathcal{A}(g_1, \cdot) = \begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix}, \qquad \mathcal{A}(g_2, \cdot,) = \begin{bmatrix} 8 & -1 \\ -1 & 8 \end{bmatrix}.$$

Then $\mathcal{A}$ is irreducible. Furthermore, by simple computation we obtain

$$\alpha_1 = 7, \quad \alpha_2 = 8, \quad \beta_1 = 7, \quad \beta_2 = 7,$$

$$r_1(\mathcal{A}) = 3, \quad r_2(\mathcal{A}) = 3, \quad c_1(\mathcal{A}) = 3, \quad c_2(\mathcal{A}) = 3,$$

$$\gamma_1 = 1, \quad \gamma_2 = 1, \quad \delta_1 = 1, \quad \delta_2 = 1,$$

$$r_1^1(\mathcal{A}) = 2, \quad r_2^2(\mathcal{A}) = 2, \quad c_1^1(\mathcal{A}) = 2, \quad c_2^2(\mathcal{A}) = 2,$$

$$R_1(\mathcal{A}) = 4, \quad R_2(\mathcal{A}) = 4, \quad C_1(\mathcal{A}) = 4, \quad C_2(\mathcal{A}) = 4.$$

From Theorem 3.1 we have

$$\tau(\mathcal{A}) \geq \max\Big\{ \min_{i,j\in[n], i\neq j} \{\eta_1(\mathcal{A})\}, \min_{k,l\in[n], k\neq l} \{\eta_2(\mathcal{A})\} \Big\} = 3.2984 > 0.$$

From Theorem 3.2 we have

$$\begin{aligned} \tau(\mathcal{A}) \geq \max\Big\{ & \min_{i,j\in[n], i\neq j} \{\theta_1(\mathcal{A}), \alpha_i - r_i^i(\mathcal{A}), \alpha_j - r_j^j(\mathcal{A})\}, \\ & \min_{k,l\in[n], k\neq l} \{\theta_2(\mathcal{A}), \beta_k - c_k^k(\mathcal{A}), \beta_l - c_l^l(\mathcal{A})\} \Big\} \\ = {} & 3.4384 > 0. \end{aligned}$$

Thus from Theorems 3.1 and 3.2 we obtain that $\mathcal{A}$ is positive definite.

4 Conclusion

In this paper, we present some bounds for the minimum M-eigenvalue of elasticity M-tensors, which are tighter than some existing results. We propose numerical examples that illustrate the efficiency of the obtained results. As applications, we provide some checkable sufficient conditions for the strong ellipticity and positive definiteness.

Acknowledgements

We thank the editor and the anonymous referee for their constructive comments and suggestions, which greatly improved this paper.

Funding
This work was supported by the Natural Science Foundation of China (11401438, 11671228, 11601261, 11571120), Shandong Provincial Natural Science Foundation (ZR2019MA022), Project of Shandong Province Higher Educational Science and Technology Program (Grant No. J14LI52), and China Postdoctoral Science Foundation (Grant No. 2017M622163).

Availability of data and materials
The data used to support the findings of this study are available from the corresponding author upon request.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 16 March 2020 Accepted: 6 July 2020 Published online: 16 July 2020

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