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New refinements of Chebyshev–Pólya–Szegő-type inequalities via generalized fractional integral operators

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Abstract

Fractional analysis, as a rapidly developing area, is a tool to bring new derivatives and integrals into the literature with the effort put forward by many researchers in recent years. The theory of inequalities is a subject of many mathematicians' work in the last century and has contributed to other areas with its applications. Especially in recent years, these two fields, fractional analysis and inequality theory, have shown a synchronous development. Inequality studies have been carried out by using new operators revealed in the fractional analysis. In this paper, by combining two important concepts of these two areas we obtain new inequalities of Chebyshev–Pólya–Szegő type by means of generalized fractional integral operators. Our results are concerned with the integral of the product of two functions and the product of two integrals. They improve the results in the paper (J. Math. Inequal. 10(2):491–504, 2016).

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1 Introduction and preliminaries

The Chebyshev inequality, which has a well-known place in inequality theory, generates limit values for synchronous functions and helps to produce new variance inequalities of many different types. The basis for this inequality lies in the following Chebyshev functional (see [3]):

$$T(f, g) = \frac{1}{b-a} \int_a^b f(x)g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right), \quad (1)$$

where f and g are two integrable functions that are synchronous on $[a, b]$, that is,

$$(f(x) - f(y))(g(x) - g(y)) \geq 0$$

for $x, y \in [a, b]$. The Chebyshev inequality states that $T(f, g) \geq 0$.

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Many studies have been done on the Chebyshev inequality and its generalizations, iterations, extensions, and modifications for various classes of functions. They have found a wide usage in numerical analysis, functional analysis, and statistics; for these results, we refer the reader to [3, 4, 17].

Another aesthetic and useful inequality is known the Pólya–Szegő inequality, which constitutes the main motivation point in our study. It is expressed as follows (see [19]):

$$\frac{\int_a^b f^2(x) dx \int_a^b g^2(x) dx}{\left(\int_a^b f(x)g(x) dx\right)^2} \leq \frac{1}{4} \left(\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}} \right)^2.$$

The following inequality for bounded and positive functions was proved by Dragomir and Diamond [10] as a good example of the use of Pólya–Szegő inequality in achieving a Grüss-type inequality.

Theorem 1 *Let $f, g : [a, b] \rightarrow \mathbb{R}_+$ be integrable functions such that*

$$0 < m \leq f(x) \leq M < \infty$$

and

$$0 < n \leq g(x) \leq N < \infty$$

for $x \in [a, b]$. Then we have

$$|T(f, g; a, b)| \leq \frac{1}{4} \frac{(M - m)(N - n)}{\sqrt{mnMN}} \left(\frac{1}{b - a} \int_a^b f(x) dx \right) \left(\frac{1}{b - a} \int_a^b g(x) dx \right). \tag{2}$$

The constant $\frac{1}{4}$ is the best possible in (2) in the sense it cannot be replaced by a smaller one.

Since one of the main motivation points of fractional analysis is obtaining more general and useful integral operators, the generalized fractional integral operator is a good tool to generalize many previous studies and results (see [6–9, 14, 18, 20]). Similarly, in inequality theory, researchers use such general operators to generalize and extend their inequalities (see [1, 11, 13, 16, 22–28, 30]). Now we will remind some important concepts, which guide the researchers working in fractional analysis.

Definition 1 (Diaz and Parigun [5]) The k -gamma function Γ_k , a generalization of the classical gamma function, is defined as

$$\Gamma_k(x) = \lim_{n \rightarrow \infty} \frac{n! k^n (nk)^{\frac{x}{k}} - 1}{(x)_{n,k}}, \quad k > 0.$$

It is shown that Mellin transform of the exponential function $e^{-\frac{t^k}{k}}$ is the k -gamma function given by

$$\Gamma_k(\alpha) = \int_0^\infty e^{-\frac{t^k}{k}} t^{\alpha-1} dt.$$

Obviously, $\Gamma_k(x + k) = x\Gamma_k(x)$, $\Gamma(x) = \lim_{k \rightarrow 1} \Gamma_k(x)$, and $\Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma(\frac{x}{k})$.

Definition 2 (See [20]) We define

$$\mathcal{F}_{\rho,\lambda}^{\sigma,k}(x) = \sum_{m=0}^{\infty} \frac{\sigma(m)}{k\Gamma_k(\rho km + \lambda)} x^m \quad (\rho, \lambda > 0; |x| < \mathcal{R}),$$

where the coefficients $\sigma(m)$ for $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ form a bounded sequence of positive real numbers, and \mathcal{R} is the set of real numbers.

Definition 3 (See [29]) Let $k > 0$, and let $g : [a, b] \rightarrow \mathcal{R}$ be an increasing function having a continuous derivative $g'(x)$ on (a, b) . The left- and right-side generalized k -fractional integrals of a function f with respect to g on $[a, b]$ are respectively defined as follows:

$$\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k,g} f(x) = \int_a^x \frac{g'(t)}{(g(x) - g(t))^{1-\frac{\lambda}{k}}} \mathcal{F}_{\rho,\lambda}^{\sigma,k}[\omega(g(x) - g(t))^\rho] f(t) dt, \quad x > a, \tag{3a}$$

and

$$\mathcal{J}_{\rho,\lambda,b-;\omega}^{\sigma,k,g} f(x) = \int_x^b \frac{g'(t)}{(g(t) - g(x))^{1-\frac{\lambda}{k}}} \mathcal{F}_{\rho,\lambda}^{\sigma,k}[\omega(g(t) - g(x))^\rho] f(t) dt, \quad x < b, \tag{3b}$$

where $\lambda, \rho > 0$ and $\omega \in \mathcal{R}$.

Remark 1 (See [29]) The significant particular cases of the integral operators given in (3a) and (3b) are as follows:

1. For $k = 1$, the operator in (3a) leads to the generalized fractional integral of f with respect to g on $[a, b]$ given as

$$\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,g} f(x) = \int_a^x \frac{g'(t)}{(g(x) - g(t))^{1-\lambda}} \mathcal{F}_{\rho,\lambda}^{\sigma}[\omega(g(x) - g(t))^\rho] f(t) dt, \quad x > a.$$

2. For $g(t) = t$, the operator in (3a) leads to the generalized k -fractional integral of f given as

$$\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k} f(x) = \int_a^x (x - t)^{\frac{\lambda}{k}-1} \mathcal{F}_{\rho,\lambda}^{\sigma,k}[\omega(x - t)^\rho] f(t) dt, \quad x > a.$$

3. For $g(t) = \ln(t)$, the operator in (3a) leads to the generalized Hadamard k -fractional integral of f given as

$$\mathcal{H}_{\rho,\lambda,a+;\omega}^{\sigma,k} f(x) = \int_a^x \left(\ln \frac{x}{t}\right)^{\frac{\lambda}{k}-1} \mathcal{F}_{\rho,\lambda}^{\sigma,k} \left[\omega \left(\ln \frac{x}{t}\right)^\rho\right] f(t) \frac{dt}{t}, \quad x > a.$$

4. For $g(t) = \frac{t^{s+1}}{s+1}$, $s \in \mathcal{R} - \{-1\}$, the operator in (3a) leads to the generalized (k, s) -fractional integral of f given as

$$\begin{aligned} & s\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma,k} f(x) \\ &= (1 + s)^{1-\frac{\lambda}{k}} \int_a^x (x^{s+1} - t^{s+1})^{\frac{\lambda}{k}-1} t^s \mathcal{F}_{\rho,\lambda}^{\sigma,k} \left[\omega \left(\frac{x^{s+1} - t^{s+1}}{s+1}\right)^\rho\right] f(t) dt, \quad x > a. \end{aligned}$$

Remark 2 Similarly, all these particular cases can be generalized for the operator given in (3b).

Remark 3 For $k = 1$ and $g(t) = t$ the operators defined in (3a) and (3b) take the generalized fractional integral operators defined by Agarwal (see [3]) and Raina et al. (see [6]) given as

$$\mathcal{J}_{\rho,\lambda,a+;\omega}^\sigma f(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma [\omega(x-t)^\rho] f(t) dt, \quad x > a, \tag{4}$$

$$\mathcal{J}_{\rho,\lambda,b-;\omega}^\sigma f(x) = \int_x^b (t-x)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^\sigma [\omega(t-x)^\rho] f(t) dt, \quad x < b. \tag{5}$$

Remark 4 For different choices of g , we can obtain other new generalized fractional integrals operators.

Remark 5 For $\omega = 0, \lambda = \alpha$, and $\sigma(0) = 1$ in Definition 3, we get the generalized fractional operators defined by Akkurt et al. [2].

Remark 6 For $\omega = 0, \lambda = \alpha$, and $\sigma(0) = 1$ in Definition 3, we have the following particular cases:

1. for $k = 1$, the fractional integrals of a function f with respect to a function g (see [14]);
2. for $g(t) = t$, the k -fractional integrals (see [15]);
3. for $g(t) = \ln(t)$ and $k = 1$, the Hadamard fractional integrals (see [14]);
4. for $g(t) = \frac{t^{s+1}}{s+1}, s \in \mathcal{R} - \{-1\}$, the (k, s) -fractional integral operators (see [21]);
5. for $g(t) = \frac{t^{s+1}}{s+1}, s \in \mathcal{R} - \{-1\}$, and $k = 1$, the Katugampola fractional integral operators (see [12]).

In the main section of this paper, using the generalized fractional integral operator mentioned in the Introduction, we obtain new and motivating Pólya–Szegő- and Chebyshev-type inequalities. We emphasize that the findings obtained by including the particular cases of the results are general.

2 Main results

We start with certain Pólya–Szegő-type integral inequalities for positive integral functions involving generalized fractional integral operator.

Theorem 2 *Assume that*

- f and g are two positive integrable functions on $[0, \infty)$;
- $h : [a, b] \rightarrow \mathcal{R}$ is an increasing positive function with continuous derivative on (a, b) ;
- there exist four positive integrable functions v_1, v_2, w_1 , and w_2 such that

$$0 < v_1(t) \leq f(t) \leq v_2(t), \quad 0 < w_1(t) \leq g(t) \leq w_2(t) \quad (t \in [0, x], x > 0), \tag{6a}$$

$$0 < v_1(t) \leq f(t) \leq v_2(t), \quad 0 < w_1(t) \leq g(t) \leq w_2(t) \quad (t \in [0, x], x > 0). \tag{6b}$$

Then we have the following inequality for generalized fractional integral operators:

$$\frac{\mathcal{J}_{\rho,\lambda,0^+;\omega}^{\sigma,k,h} \{w_1 w_2 f^2\}(x) \mathcal{J}_{\rho,\lambda,0^+;\omega}^{\sigma,k,h} \{v_1 v_2 g^2\}(x)}{(\mathcal{J}_{\rho,\lambda,0^+;\omega}^{\sigma,k,h} \{(v_1 w_1 + v_2 w_2)fg\}(x))^2} \leq \frac{1}{4}. \tag{7}$$

Proof From (6a) and (6b), for $t \in [0, x], x > 0$, we have

$$\left(\frac{v_2(t)}{w_1(t)} - \frac{f(t)}{g(t)}\right) \geq 0 \tag{8}$$

and

$$\left(\frac{f(t)}{g(t)} - \frac{v_1(t)}{w_2(t)}\right) \geq 0. \tag{9}$$

By multiplying (8) and (9) we get

$$\left(\frac{v_2(t)}{w_1(t)} - \frac{f(t)}{g(t)}\right)\left(\frac{f(t)}{g(t)} - \frac{v_1(t)}{w_2(t)}\right) \geq 0.$$

By the last inequality we can write

$$\begin{aligned} & (v_1(t)w_1(t) + v_2(t)w_2(t))f(t)g(t) \\ & \geq w_1(t)w_2(t)f^2(t) + v_1(t)v_2(t)g^2(t). \end{aligned} \tag{10}$$

Multiplying both sides of (10) by

$$\frac{h'(t)}{(h(x) - h(t))^{1-\frac{1}{k}}} \mathcal{F}_{\rho, \lambda}^{\sigma, k} [\omega (h(x) - h(t))^\rho]$$

and integrating the resulting inequality with respect to t over $(0, x)$, we get

$$\mathcal{J}_{\rho, \lambda, 0^+; \omega}^{\sigma, k, h} \{(v_1 w_1 + v_2 w_2)fg\}(x) \geq \mathcal{J}_{\rho, \lambda, 0^+; \omega}^{\sigma, k, h} \{w_1 w_2 f^2\}(x) + \mathcal{J}_{\rho, \lambda, 0^+; \omega}^{\sigma, k, h} \{v_1 v_2 g^2\}(x), \tag{11}$$

and by applying the AM-GM inequality $a + b \geq 2\sqrt{ab}$, $a, b \in \mathbb{R}^+$, we have

$$\mathcal{J}_{\rho, \lambda, 0^+; \omega}^{\sigma, k, h} \{(v_1 w_1 + v_2 w_2)fg\}(x) \geq 2\sqrt{\mathcal{J}_{\rho, \lambda, 0^+; \omega}^{\sigma, k, h} \{w_1 w_2 f^2\}(x) \times \mathcal{J}_{\rho, \lambda, 0^+; \omega}^{\sigma, k, h} \{v_1 v_2 g^2\}(x)},$$

which implies that

$$\mathcal{J}_{\rho, \lambda, 0^+; \omega}^{\sigma, k, h} \{w_1 w_2 f^2\}(x) \times \mathcal{J}_{\rho, \lambda, 0^+; \omega}^{\sigma, k, h} \{v_1 v_2 g^2\}(x) \leq \frac{1}{4} (\mathcal{J}_{\rho, \lambda, 0^+; \omega}^{\sigma, k, h} \{(v_1 w_1 + v_2 w_2)fg\}(x))^2.$$

So, we get the desired result. □

Corollary 1 *If $v_1 = m, v_2 = M, w_1 = n$, and $w_2 = N$, then we have*

$$\frac{\mathcal{J}_{\rho, \lambda, 0^+; \omega}^{\sigma, k, h} (nNf^2)(x) \mathcal{J}_{\rho, \lambda, 0^+; \omega}^{\sigma, k, h} (mMg^2)(x)}{(\mathcal{J}_{\rho, \lambda, 0^+; \omega}^{\sigma, k, h} (mn + MNfg)(x))^2} \leq \frac{1}{4}.$$

Theorem 3 *Let*

- f and g be two positive integrable functions on $[0, \infty)$;
- $h : [a, b] \rightarrow \mathcal{R}$ be an increasing positive function with a continuous derivative on (a, b) ;

- there exist four positive integrable functions $v_1, v_2, w_1,$ and w_2 such that condition (6a) and (6b) is satisfied.

Then we have the following inequality for generalized fractional integral operators:

$$\begin{aligned} & \mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h}\{v_1 v_2\}(x) \mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h}\{w_1 w_2\}(x) \times \mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h}\{f^2\}(x) \mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h}\{g^2\}(x) \\ & \leq \frac{1}{4} \left(\mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h}\{v_1 f\}(x) \mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h}\{w_1 g\}(x) \right. \\ & \quad \left. + \mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h}\{v_2 f\}(x) \mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h}\{w_2 g\}(x) \right)^2. \end{aligned} \tag{12}$$

Proof From conditions (6a) and (6b) we get

$$\left(\frac{v_2(t)}{w_1(\xi)} - \frac{f(t)}{g(\xi)} \right) \geq 0$$

and

$$\left(\frac{f(t)}{g(\xi)} - \frac{v_1(t)}{w_2(\xi)} \right) \geq 0,$$

which leads to

$$\left(\frac{v_1(t)}{w_2(\xi)} + \frac{v_2(t)}{w_1(\xi)} \right) \frac{f(t)}{g(\xi)} \geq \frac{f^2(t)}{g^2(\xi)} + \frac{v_1(t)v_2(t)}{w_1(\xi)w_2(\xi)}. \tag{13}$$

Multiplying both sides of (13) by $w_1(\xi)w_2(\xi)g^2(\xi)$, we have

$$\begin{aligned} & v_1(t)f(t)w_1(\xi)g(\xi) + v_2(t)f(t)w_2(\xi)g(\xi) \\ & \geq w_1(\xi)w_2(\xi)f^2(t) + v_1(t)v_2(t)g^2(\xi). \end{aligned} \tag{14}$$

Multiplying both sides of (10) by

$$\frac{h'(\xi)}{(h(x) - h(\xi))^{1-\frac{\lambda_1}{k}}} \mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} [\omega(h(x) - h(\xi))^\rho]$$

and integrating the resulting inequality with respect to t and ξ over $(0, x)^2$, we get

$$\begin{aligned} & \mathcal{J}_{\rho,\lambda_1}^{\sigma,k,h}\{v_1 f\}(x) \mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h}\{w_1 g\}(x) + \mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h}\{v_2 f\}(x) \mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h}\{w_2 g\}(x) \\ & \geq \mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h}\{f^2\}(x) \mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h}\{w_1 w_2\}(x) + \mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h}\{v_1 v_2\}(x) \mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h}\{g^2\}(x). \end{aligned}$$

Applying the AM-GM inequality, we obtain

$$\begin{aligned} & \mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h}\{v_1 f\}(x) \mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h}\{w_1 g\}(x) + \mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h}\{v_2 f\}(x) \mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h}\{w_2 g\}(x) \\ & \geq 2 \sqrt{\mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h}\{f^2\}(x) \mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h}\{w_1 w_2\}(x) \times \mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h}\{v_1 v_2\}(x) \mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h}\{g^2\}(x)}, \end{aligned}$$

which leads to the desired inequality (12). The proof is completed. □

Corollary 2 *If $v_1 = m, v_2 = M, w_1 = n,$ and $w_2 = N,$ then we have*

$$C_{\lambda_1}(x)C_{\lambda_2}(x) \times \frac{(\mathcal{J}_{\rho,\lambda_1,0^+;\omega}^{\sigma,k,h} f^2)(x)(\mathcal{J}_{\rho,\lambda_2,0^+;\omega}^{\sigma,k,h} g^2)(x)}{((\mathcal{J}_{\rho,\lambda_1,0^+;\omega}^{\sigma,k,h} f)(x)(\mathcal{J}_{\rho,\lambda_2,0^+;\omega}^{\sigma,k,h} g)(x))^2} \leq \frac{1}{4} \left(\sqrt{\frac{mn}{MN}} + \sqrt{\frac{MN}{mn}} \right)^2,$$

where

$$C_{\lambda_1}(x) = (h(x))^{\frac{\lambda_1}{k}} \mathcal{F}_{\rho,\lambda_1+1}^{\sigma,k}(\omega(h(x))^\rho),$$

$$C_{\lambda_2}(x) = (h(x))^{\frac{\lambda_2}{k}} \mathcal{F}_{\rho,\lambda_2+1}^{\sigma,k}(\omega(h(x))^\rho).$$

Theorem 4 *Let*

- f and g be two positive integrable functions on $[0, \infty)$;
- $h : [a, b] \rightarrow \mathcal{R}$ be an increasing positive function with a continuous derivative on (a, b) ;
- there exist four positive integrable functions $v_1, v_2, w_1,$ and w_2 such that condition (6a) and (6b) is satisfied.

Then we have the following inequality for generalized fractional integrals:

$$\mathcal{J}_{\rho,\lambda_1,0^+;\omega}^{\sigma,k,h} \{f^2\}(x) \mathcal{J}_{\rho,\lambda_2,0^+;\omega}^{\sigma,k,h} \{g^2\}(x) \leq \mathcal{J}_{\rho,\lambda_1,0^+;\omega}^{\sigma,k,h} \left\{ \frac{v_2 f g}{w_1} \right\}(x) \mathcal{J}_{\rho,\lambda_2,0^+;\omega}^{\sigma,k,h} \left\{ \frac{w_2 f g}{v_1} \right\}(x). \tag{15}$$

Proof Using condition (6a) and (6b), we get

$$f^2(t) \leq \frac{v_2(t)}{w_1(t)} f(t)g(t). \tag{16}$$

Multiplying both sides of (10) by

$$\frac{h'(t)}{(h(x) - h(t))^{1 - \frac{\lambda_1}{k}}} \mathcal{J}_{\rho,\lambda_1,0^+;\omega}^{\sigma,k,h} [\omega(h(x) - h(t))^\rho]$$

and integrating the resulting inequality with respect to t over $(0, x)$, we obtain

$$\int_0^x \frac{h'(t)}{(h(x) - h(t))^{1 - \frac{\lambda_1}{k}}} \mathcal{J}_{\rho,\lambda_1,0^+;\omega}^{\sigma,k,h} [\omega(h(x) - h(t))^\rho] f^2(t) dt$$

$$\leq \int_0^x \frac{h'(t)}{(h(x) - h(t))^{1 - \frac{\lambda_1}{k}}} \mathcal{J}_{\rho,\lambda_1,0^+;\omega}^{\sigma,k,h} [\omega(h(x) - h(t))^\rho] \frac{v_2(\tau)}{w_1(\tau)} f(\tau)g(\tau) dt,$$

which leads to

$$\mathcal{J}_{\rho,\lambda_1,0^+;\omega}^{\sigma,k,h} \{f^2\}(x) \leq \mathcal{J}_{\rho,\lambda_1,0^+;\omega}^{\sigma,k,h} \left\{ \frac{v_2 f g}{w_1} \right\}(x). \tag{17}$$

Similarly, we can write

$$g^2(\xi) \leq \frac{w_2(\xi)}{v_1(\xi)} f(\xi)g(\xi).$$

By a similar argument we have

$$\begin{aligned} & \int_0^x \frac{h'(\xi)}{(h(x) - h(\xi))^{1-\frac{\lambda_2}{k}}} \mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} [\omega(h(x) - h(\xi))^\rho] g^2(\xi) d\xi \\ & \leq \int_0^x \frac{h'(\xi)}{(h(x) - h(\xi))^{1-\frac{\lambda_2}{k}}} \mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} [\omega(h(x) - h(\xi))^\rho] \frac{w_2(\xi)}{v_1(\xi)} f(\xi) g(\xi) d\xi, \end{aligned}$$

which implies

$$\mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} \{g^2\}(x) \leq \mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} \left\{ \frac{w_2 f g}{v_1} \right\}(x). \tag{18}$$

Multiplying (17) and (18), we get (15). The proof is completed. □

Corollary 3 *If $v_1 = m, v_2 = M, w_1 = n,$ and $w_2 = N,$ then we have*

$$\frac{(\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} f^2)(x) (\mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} g^2)(x)}{((\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} f g)(x) (\mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} f g)(x))^2} \leq \frac{MN}{mn}.$$

Theorem 5 *Let*

- f and g be two positive integrable functions on $[0, \infty)$;
- $h : [a, b] \rightarrow \mathcal{R}$ be an increasing positive function with a continuous derivative on (a, b) ;
- there exist four positive integrable functions $v_1, v_2, w_1,$ and w_2 such that condition (6a) and (6b) is satisfied.

Then we have the following inequality generalized fractional integral operators:

$$\begin{aligned} & |C_{\lambda_2}(x) (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} f g)(x) + C_{\lambda_1}(x) (\mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} f g)(x) \\ & \quad - (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} f)(x) (\mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} g)(x) - (\mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} f)(x) (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} g)(x)| \\ & \leq |A_1(f, v_1, v_2)(x) + A_2(f, v_1, v_2)(x)|^{1/2} \\ & \quad \times |A_1(g, w_1, w_2)(x) + A_2(g, w_1, w_2)(x)|^{1/2}, \end{aligned} \tag{19}$$

where

$$A_1(u, v, w)(x) = C_{\lambda_2}(x) \frac{(\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} \{(v + w)u\}(x))^2}{4 \mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} \{vw\}(x)} - (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} u)(x) (\mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} u)(x)$$

and

$$A_2(u, v, w)(x) = C_{\lambda_1}(x) \frac{(\mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} \{(v + w)u\}(x))^2}{4 \mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} \{vw\}(x)} - (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} u)(x) (\mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} u)(x).$$

Proof Let f and g be two positive integrable functions on $[0, \infty)$. For $t, \xi \in (0, x)$ with $x > 0,$ we define

$$H(t, \xi) = (f(t) - f(\xi))(g(t) - g(\xi)),$$

that is,

$$H(t, \xi) = f(t)g(t) + f(\xi)g(\xi) - f(t)g(\xi) - f(\xi)g(t). \tag{20}$$

Multiplying both sides of (20) by

$$\frac{h'(t)}{(h(x) - h(t))^{1-\frac{\lambda_1}{k}}} \mathcal{F}_{\rho, \lambda_1}^{\sigma, k} [\omega(h(x) - h(t))^\rho] \times \frac{h'(\xi)}{(h(x) - h(\xi))^{1-\frac{\lambda_2}{k}}} \mathcal{F}_{\rho, \lambda_2}^{\sigma, k} [\omega(h(x) - h(\xi))^\rho]$$

and integrating the resulting inequality with respect to t and ξ over $(0, x)^2$, we get

$$\begin{aligned} & \int_0^x \frac{h'(t)}{(h(x) - h(t))^{1-\frac{\lambda_1}{k}}} \mathcal{F}_{\rho, \lambda_1}^{\sigma, k} [\omega(h(x) - h(t))^\rho] \\ & \quad \times \frac{h'(\xi)}{(h(x) - h(\xi))^{1-\frac{\lambda_2}{k}}} \mathcal{F}_{\rho, \lambda_2}^{\sigma, k} [\omega(h(x) - h(\xi))^\rho] H(t, \xi) dt d\xi \\ & = C_{\lambda_2}(x) (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} fg)(x) + C_{\lambda_1}(x) (\mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} fg)(x) \\ & \quad - (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} f)(x) (\mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} g)(x) - (\mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} f)(x) (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} g)(x). \end{aligned}$$

Applying the Cauchy–Schwarz inequality, we can write

$$\begin{aligned} & \left| \int_0^x \int_0^x \frac{h'(t)}{(h(x) - h(t))^{1-\frac{\lambda_1}{k}}} \mathcal{F}_{\rho, \lambda_1}^{\sigma, k} [\omega(h(x) - h(t))^\rho] \right. \\ & \quad \times \left. \frac{h'(\xi)}{(h(x) - h(\xi))^{1-\frac{\lambda_2}{k}}} \mathcal{F}_{\rho, \lambda_2}^{\sigma, k} [\omega(h(x) - h(\xi))^\rho] H(t, \xi) dt d\xi \right| \\ & \leq \left[\int_0^x \int_0^x \frac{h'(t)}{(h(x) - h(t))^{1-\frac{\lambda_1}{k}}} \mathcal{F}_{\rho, \lambda_1}^{\sigma, k} [\omega(h(x) - h(t))^\rho] \right. \\ & \quad \times \left. \frac{h'(\xi)}{(h(x) - h(\xi))^{1-\frac{\lambda_2}{k}}} \mathcal{F}_{\rho, \lambda_2}^{\sigma, k} [\omega(h(x) - h(\xi))^\rho] f^2(t) dt d\xi \right. \\ & \quad + \int_0^x \int_0^x \frac{h'(t)}{(h(x) - h(t))^{1-\frac{\lambda_1}{k}}} \mathcal{F}_{\rho, \lambda_1}^{\sigma, k} [\omega(h(x) - h(t))^\rho] \\ & \quad \times \left. \frac{h'(\xi)}{(h(x) - h(\xi))^{1-\frac{\lambda_2}{k}}} \mathcal{F}_{\rho, \lambda_2}^{\sigma, k} [\omega(h(x) - h(\xi))^\rho] f^2(\xi) dt d\xi \right. \\ & \quad - 2 \int_0^x \int_0^x \frac{h'(t)}{(h(x) - h(t))^{1-\frac{\lambda_1}{k}}} \mathcal{F}_{\rho, \lambda_1}^{\sigma, k} [\omega(h(x) - h(t))^\rho] \\ & \quad \times \left. \frac{h'(\xi)}{(h(x) - h(\xi))^{1-\frac{\lambda_2}{k}}} \mathcal{F}_{\rho, \lambda_2}^{\sigma, k} [\omega(h(x) - h(\xi))^\rho] f(t)f(\xi) dt d\xi \right]^{1/2} \\ & \quad \times \left[\int_0^x \int_0^x \frac{h'(t)}{(h(x) - h(t))^{1-\frac{\lambda_1}{k}}} \mathcal{F}_{\rho, \lambda_1}^{\sigma, k} [\omega(h(x) - h(t))^\rho] \right. \\ & \quad \times \left. \frac{h'(\xi)}{(h(x) - h(\xi))^{1-\frac{\lambda_2}{k}}} \mathcal{F}_{\rho, \lambda_2}^{\sigma, k} [\omega(h(x) - h(\xi))^\rho] g^2(t) dt d\xi \right. \\ & \quad + \int_0^x \int_0^x \frac{h'(t)}{(h(x) - h(t))^{1-\frac{\lambda_1}{k}}} \mathcal{F}_{\rho, \lambda_1}^{\sigma, k} [\omega(h(x) - h(t))^\rho] \end{aligned}$$

$$\begin{aligned} & \times \frac{h'(\xi)}{(h(x) - h(\xi))^{1-\frac{\lambda_2}{k}}} \mathcal{F}_{\rho,\lambda_2}^{\sigma,k} [\omega(h(x) - h(\xi))^\rho] g^2(\xi) dt d\xi \\ & - 2 \int_0^x \int_0^x \frac{h'(t)}{(h(x) - h(t))^{1-\frac{\lambda_1}{k}}} \mathcal{F}_{\rho,\lambda_1}^{\sigma,k} [\omega(h(x) - h(t))^\rho] \\ & \times \frac{h'(\xi)}{(h(x) - h(\xi))^{1-\frac{\lambda_2}{k}}} \mathcal{F}_{\rho,\lambda_2}^{\sigma,k} [\omega(h(x) - h(\xi))^\rho] g(t)g(\xi) dt d\xi \Big]^{1/2}. \end{aligned}$$

As a consequence,

$$\begin{aligned} & \left| \int_0^x \int_0^x \frac{h'(t)}{(h(x) - h(t))^{1-\frac{\lambda_1}{k}}} \mathcal{F}_{\rho,\lambda_1}^{\sigma,k} [\omega(h(x) - h(t))^\rho] \right. \\ & \quad \times \left. \frac{h'(\xi)}{(h(x) - h(\xi))^{1-\frac{\lambda_2}{k}}} \mathcal{F}_{\rho,\lambda_2}^{\sigma,k} [\omega(h(x) - h(\xi))^\rho] H(t, \xi) dt d\xi \right| \\ & \leq [C_{\lambda_2}(x) (\mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} f^2)(x) + C_{\lambda_1}(x) (\mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h} f^2)(x)](x) \\ & \quad - 2 (\mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} f)(x) (\mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h} f)(x)]^{1/2} \\ & \quad \times [C_{\lambda_2}(x) (\mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} g^2)(x) + C_{\lambda_1}(x) (\mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h} g^2)(x)](x) \\ & \quad - 2 (\mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} g)(x) (\mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h} g)(x)]^{1/2}. \end{aligned}$$

Applying Lemma 2 with $w_1(t) = w_2(t) = g(t) = 1$, we get

$$\mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} (f^2)(x) \leq \frac{(\mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} (v_1 + v_2)f)(x)^2}{4 \mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} (v_1 v_2)(x)}.$$

This implies that

$$C_{\lambda_2}(x) \mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} (f^2)(x) \leq C_{\lambda_2}(x) \frac{(\mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} \{(v_1 + v_2)f\}(x))^2}{4 \mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} (v_1 v_2)(x)}.$$

Further calculations produce the following inequalities:

$$\begin{aligned} & C_{\lambda_2}(x) \mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} \{f^2\}(x) - (\mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} f)(x) (\mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h} f)(x) \\ & \leq C_{\lambda_2}(x) \frac{(\mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} \{(v_1 + v_2)f\}(x))^2}{4 \mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} \{v_1 v_2\}(x)} - (\mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} f)(x) (\mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h} f)(x) \\ & = A_1(f, v_1, v_2) \end{aligned} \tag{21}$$

and

$$\begin{aligned} & C_{\lambda_1}(x) \mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h} \{f^2\}(x) - (\mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} f)(x) (\mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h} f)(x) \\ & \leq C_{\lambda_1}(x) \frac{(\mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h} \{(v_1 + v_2)f\}(x))^2}{4 \mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h} \{v_1 v_2\}(x)} - (\mathcal{J}_{\rho,\lambda_1,0^+}^{\sigma,k,h} f)(x) (\mathcal{J}_{\rho,\lambda_2,0^+}^{\sigma,k,h} f)(x) \\ & = A_2(f, v_1, v_2). \end{aligned} \tag{22}$$

Similarly, applying Lemma 2 with $v_1(\tau) = v_2(\tau) = f(\tau) = 1$, we have

$$\begin{aligned}
 & C_{\lambda_2}(x) \mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} \{g^2\}(x) - (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} g)(x) (\mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} g)(x) \\
 & \leq A_1(g, w_1, w_2)
 \end{aligned} \tag{23}$$

and

$$\begin{aligned}
 & C_{\lambda_1}(x) \mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} \{g^2\}(x) - (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} g)(x) (\mathcal{J}_{\rho, \lambda_2, 0^+; \omega}^{\sigma, k, h} g)(x) \\
 & \leq A_2(g, w_1, w_2).
 \end{aligned} \tag{24}$$

Using (21)–(24), we obtain the result. □

Theorem 6 *Let*

- f and g be two positive integrable functions on $[0, \infty)$;
- $h : [a, b] \rightarrow \mathcal{R}$ be an increasing positive function with a continuous derivative on (a, b) ;
- there exist four positive integrable functions v_1, v_2, w_1 , and w_2 such that condition (6a) and (6b) is satisfied.

Then we have the following inequality for generalized fractional integral operators:

$$\begin{aligned}
 & |C_{\lambda_1}(x) \mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} \{fg\}(x) - (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} f)(x) (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} g)(x)| \\
 & \leq |A(f, v_1, v_2)(x) A(g, w_1, w_2)(x)|^{1/2},
 \end{aligned} \tag{25}$$

where

$$A(u, v, w)(x) = C_{\lambda_1}(x) \frac{(\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} \{(v + w)u\}(x))^2}{4 \mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} \{vw\}(x)} - ((\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} u)(x))^2.$$

Proof Setting $\lambda_1 = \lambda_2$ in (19), we obtain (25). □

Corollary 4 *If $v_1 = m, v_2 = M, w_1 = n$, and $w_2 = N$, then we have*

$$\begin{aligned}
 & |C_{\lambda_1}(x) \mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} \{fg\}(x) - (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} f)(x) (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} g)(x)| \\
 & \leq \frac{(M - m)(N - n)}{4\sqrt{MmNn}} \times (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} f)(x) (\mathcal{J}_{\rho, \lambda_1, 0^+; \omega}^{\sigma, k, h} g)(x).
 \end{aligned}$$

3 Conclusion

In this paper, we obtained several new inequalities of Pólya–Szegő- and Chebyshev-type by using generalized fractional integral operators. The main findings offer new estimations for various integral inequalities. Many particular cases can be revealed by using the operators. The interested researchers can investigate similar cases by using different types of integral operators. Also, the problem can be discussed in different spaces by expanding the motivation of this study.

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Authors' contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

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