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Opial integral inequalities for generalized fractional operators with nonsingular kernel

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Abstract

We consider the well-known classes of functions $\mathcal{U}_1(\mathbf{v}, \mathrm{k})$ and $\mathcal{U}_2(\mathbf{v}, \mathrm{k})$, and those of Opial inequalities defined on these classes. In view of these indices, we establish new aspects of the Opial integral inequality and related inequalities, in the context of fractional integrals and derivatives defined using nonsingular kernels, particularly the Caputo–Fabrizio (CF) and Atangana–Baleanu (AB) models of fractional calculus.

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1 Introduction

Fractional calculus, or specifically the subject of fractional differential equations, is usually considered as a generalization of ordinary differential equations. In many fields they have been applied as more appropriate models of real world problems, for example in biology, engineering, finance, and physics [1–10]. Recently, the applications of fractional calculus have been developing, including control system, anomalous diffusion, petroleum engineering, multi-strain tuberculosis model, and viscoelastic mechanics. For this reason, we advice the reader to read the book [11] carefully, which includes a collection of different fractional models.

Many mathematical inequalities are simulated via the fractional calculus that lead to fractional integral inequalities. Fractional integral inequalities form a basic field of study within mathematical analysis [12, 13]. They have been used in the study of fractional ordinary and partial differential equations [14–17]. Specifically, they have been used in finding the uniqueness of solutions for a certain fractional differential equations and in providing bounds to solve certain fractional boundary value problems [18, 19].

There are many inequalities nowadays which has interested many mathematicians and in the past ten years much research has been done around this topic; see [20–35] for further details on various mathematical inequalities. One of the well-known inequalities arising in the theory of mathematical analysis, namely, is the Opial integral inequality. It was firstly found by Opial in 1960 [36], and his result is as follows: for a function $\eth(t) \in C^1[0, h]$

with $\eth(0) = \eth(h) = 0$ and $\eth(t) > 0$ for $t \in (0, h)$

$$\int_0^h \left|\eth(t)\eth'(t)\right| \mathrm{d}t \le \int_0^h \left(\eth'(t)\right)^2 \mathrm{d}t. \tag{1.1}$$

A best possibility here is $\frac{h}{4}$.

The Opial integral inequality (1.1) has received substantial attention in various field of mathematics. Moreover, a large number of articles dealing with extensions, new results, variants, generalizations and discrete analogues of Opial's integral inequality can be found in the literature [33, 37–40].

The idea we have considered in the current paper is Farid et al.'s idea [37]. Their results are as follows.

Definition 1.1 Let $\mathbf{v}$ be a continuous function. Denote by $\mathcal{U}_1(\mathbf{v}, \mathtt{k})$ the class of functions $\mathfrak{u} : [\rtimes_1, \rtimes_2] \to \mathbb{R}$ having the representation

$$\mathfrak{u}(\eta) = \int_{\rtimes_1}^{\eta} \mathtt{k}(\eta, t)\mathbf{v}(t)\, \mathrm{d}t,$$

where $\mathtt{k} = \mathtt{k}(\eta, t)$ is an arbitrary nonnegative kernel such that $\mathtt{k}(\eta, t) = 0$ for $t > \eta$, and $\mathbf{v}(\eta) > 0$ implies $\mathfrak{u}(\eta) > 0$ for every $\eta \in [\rtimes_1, \rtimes_2]$.

Definition 1.2 Let $\mathbf{v}$ be a continuous function. Denote by $\mathcal{U}_2(\mathbf{v}, \mathtt{k})$ the class of functions $\mathfrak{u} : [\rtimes_1, \rtimes_2] \to \mathbb{R}$ having the representation

$$\mathfrak{u}(\eta) = \int_{\eta}^{\rtimes_2} \mathtt{k}(\eta, t)\mathbf{v}(t)\, \mathrm{d}t,$$

where $\mathtt{k} = \mathtt{k}(\eta, t)$ is an arbitrary nonnegative kernel such that $\mathtt{k}(\eta, t) = 0$ for $t < \eta$, and $\mathbf{v}(\eta) > 0$ implies $\mathfrak{u}(\eta) > 0$ for every $\eta \in [\rtimes_1, \rtimes_2]$.

Theorem 1.1 *Suppose $\sigma, \eth : [0, \infty) \to \mathbb{R}$ are two differentiable convex and increasing functions with $\sigma(g(0)) = 0$. Also, suppose $\mathfrak{u} \in \mathcal{U}_1(\eth \circ \mathbf{v}, \mathtt{k})$ such that $|\mathtt{k}(\eta, t)| \le \mathcal{K}$ =constant. Then we have*

$$\begin{aligned}
&\int_{\rtimes_1}^{\rtimes_2} \sigma'\big(\eth\big(|\mathfrak{u}(\eta)|\big)\big)\eth'\big(|\mathfrak{u}(\eta)|\big)\left|\eth \circ \mathbf{v}(\eta)\right| \mathrm{d}\eta \\
&\quad \le \frac{1}{\mathcal{K}}\sigma\left(\eth\left(\mathcal{K}\int_{\rtimes_1}^{\rtimes_2} \left|\eth \circ \mathbf{v}(t)\right| \mathrm{d}t\right)\right) \\
&\quad \le \frac{1}{\mathcal{K}(\rtimes_2 - \rtimes_1)} \int_{\rtimes_1}^{\rtimes_2} \sigma\left(\eth\big(\mathcal{K}(\rtimes_2 - \rtimes_1)\left|\eth \circ \mathbf{v}(t)\right|\big)\right) \mathrm{d}t.
\end{aligned} \tag{1.2}$$

Theorem 1.2 *Suppose $\sigma, : [0, \infty) \to \mathbb{R}$ is a differentiable convex and increasing function with $\sigma(0) = 0$. Also, suppose $\mathfrak{u} \in \mathcal{U}_1(\eth \circ \mathbf{v}, \mathtt{k})$ such that $|\mathtt{k}(\eta, t)| \le \mathcal{K}$ =constant. Then we have*

$$\begin{aligned}
&\int_{\rtimes_1}^{\rtimes_2} \sigma'\big(|\mathfrak{u}(\eta)|^{\bar{\varrho}}\big)\big(|\mathfrak{u}(\eta)|\big)^{\bar{\varrho}-1}\left|\big(\mathbf{v}(\eta)\big)^{\bar{\varrho}}\right| \mathrm{d}\eta \\
&\quad \le \frac{1}{\bar{\varrho}\mathcal{K}}\sigma\left(\eth\left(\mathcal{K}\int_{\rtimes_1}^{\rtimes_2} \left|\mathbf{v}(t)\right|^{\bar{\varrho}} \mathrm{d}t\right)^{\bar{\varrho}}\right)
\end{aligned}$$

$$\leq \frac{1}{\bar{\varrho}\mathcal{K}(\rtimes_2 - \rtimes_1)} \int_{\rtimes_1}^{\rtimes_2} \sigma\left(\eth\left(\mathcal{K}(\rtimes_2 - \rtimes_1)\left|\mathbf{v}(t)\right|^{\bar{\varrho}}\right)^{\bar{\varrho}}\right) \mathrm{d}t, \tag{1.3}$$

for $\bar{\varrho} \geq 1$.

The most efficient branch of mathematical analysis is fractional calculus, which involves integrals and derivatives taken to fractional orders, orders outside of the integer or natural numbers. Here, we present the Riemann–Liouville (RL) definition to facilitate the discussion of the aforementioned operations, which is most commonly used for fractional derivatives and integrals.

Definition 1.3 ([1, 2]) For any function $\mathbf{f}$ which is L^1 on an interval $[\rtimes_1, \rtimes_2]$, and for any $\eta \in [\rtimes_1, \rtimes_2]$, the $\bar{\nu}$th left-RL fractional integral of $\mathbf{f}(\eta)$ is defined by the following integral transform, for $\operatorname{Re}(\bar{\nu}) > 0$:

$${}^{RL}I_{\rtimes_1+}^{\bar{\nu}}\mathbf{f}(\eta) := \frac{1}{\Gamma(\bar{\nu})} \int_{\rtimes_1}^{\eta} (\eta - \xi)^{\bar{\nu}-1}\mathbf{f}(\xi)\,\mathrm{d}\xi. \tag{1.4}$$

For any function $\mathbf{f}$ which is C^n on an interval $[\rtimes_1, \rtimes_2]$, and for any $\eta \in [\rtimes_1, \rtimes_2]$, the $\bar{\nu}$th left-RL fractional derivative of $\mathbf{f}(\eta)$ is defined, for $n - 1 \leq \operatorname{Re}(\bar{\nu}) < n$, as follows:

$${}^{RL}D_{\rtimes_1+}^{\bar{\nu}}\mathbf{f}(\eta) := \frac{\mathrm{d}^n}{\mathrm{d}\eta^n}\,{}^{RL}I_{\rtimes_1+}^{n-\bar{\nu}}\mathbf{f}(\eta). \tag{1.5}$$

Between them, these two definitions cover orders of differentiation throughout the entire complex plane, where we interpret ${}^{RL}D_{\rtimes_1+}^{-\bar{\nu}}\mathbf{f}(\eta) = {}^{RL}I_{\rtimes_1+}^{\bar{\nu}}\mathbf{f}(\eta)$. Equation (1.5) is the analytic continuation in $\bar{\nu}$ of the formula (1.4); thus, differentiation and integration are now unified in a single operator which we call *differintegration*.

Definition 1.4 ([1, 2]) For any function $\mathbf{f}$ which is L^1 on an interval $[\rtimes_1, \rtimes_2]$, and for any $\eta \in [\rtimes_1, \rtimes_2]$, the $\bar{\nu}$th right-RL fractional integral of $\mathbf{f}(\eta)$ is defined by the following integral transform, for $\operatorname{Re}(\bar{\nu}) > 0$:

$${}^{RL}I_{\rtimes_2-}^{\bar{\nu}}\mathbf{f}(\eta) := \frac{1}{\Gamma(\bar{\nu})} \int_{\eta}^{\rtimes_2} (\xi - x)^{\bar{\nu}-1}\mathbf{f}(\xi)\,\mathrm{d}\xi. \tag{1.6}$$

For any function $\mathbf{f}$ which is C^n on an interval $[\rtimes_1, \rtimes_2]$, and for any $\eta \in [\rtimes_1, \rtimes_2]$, the $\bar{\nu}$th right-RL fractional derivative of $\mathbf{f}(\eta)$ is defined, for $n - 1 \leq \operatorname{Re}(\bar{\nu}) < n$, as follows:

$${}^{RL}D_{\rtimes_2-}^{\bar{\nu}}\mathbf{f}(\eta) := (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}\eta^n}\,{}^{RL}I_{\rtimes_2-}^{n-\bar{\nu}}\mathbf{f}(\eta). \tag{1.7}$$

Recently, many possible definitions have been proposed for fractional integrals and derivatives, starting from the classical RL formula (1.4)–(1.7) and its modifying and generalisations it by replacing the power function kernel with other kernel functions; like the Caputo [41], Hilfer [42, Chapter II], Atangana–Baleanu [43, 44] and Prabhakar [45, 46] definitions.

Many of those definitions have different properties from the classical RL model. Also, the definition discussed in this article, namely the Caputo–Fabrizio (CF) definition, is designed with the convolution of an exponential function with an ordinary derivative but it has the same supplemental properties of configuration and heterogeneous [47–49] with different kernels as occur in the Caputo and RL fractional derivatives. The CF-fractional derivative [50] has formed a new dimension in the study of fractional differential equations with its nonsingular kernel.

Definition 1.5 ([1, 2]) For any function $\mathbf{f}$ which is $\mathcal{H}^1(\rtimes_1, \rtimes_2)$ with $0 < \bar{\nu} < 1$, and for any $\eta \in [\rtimes_1, \rtimes_2]$, the $\bar{\nu}$th left-CF-fractional derivative of $\mathbf{f}(\eta)$ in the sense of Caputo is defined by

$$ {}^{CFC}_{\rtimes_1}D^{\bar{\nu}}\mathbf{f}(\eta) := \frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}} \int_{\rtimes_1}^{\eta} \mathbf{f}'(\xi) \exp\bigl(\bar{\lambda}(\eta-\xi)\bigr)\, d\xi, \tag{1.8} $$

and the left-CF-fractional derivative of $\mathbf{f}(\eta)$ in the sense of Caputo is defined by

$$ {}^{CFC}_{\rtimes_1}D^{\bar{\nu}}_{\rtimes_2-}\mathbf{f}(\eta) := -\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}} \int_{\eta}^{\rtimes_2} \mathbf{f}'(\xi) \exp\bigl(\bar{\lambda}(\xi-x)\bigr)\, d\xi, \tag{1.9} $$

where $\bar{\lambda} = -\frac{\bar{\nu}}{1-\bar{\nu}}$ and $\mathcal{B}(\bar{\nu}) > 0$ is a normalization function that satisfies $\mathcal{B}(0) = \mathcal{B}(1) = 1$.

The aim of the present article is to establish new Opial integral inequalities type involving Caputo–Fabrizio fractional models by an extension of Farid et al.'s idea [37] as shown in Theorems 1.1–1.2. For special cases, our results yield some of the recent integral inequalities of Opial type and offer new estimates on such types of inequalities. Furthermore, we improve those findings to the higher order Caputo–Fabrizio (CF) and Atangana–Baleanu (AB) fractional operators, which we define in Sects. 2.2 and 3, respectively.

2 The CF-fractional inequalities

We here utilize CF-fractional integrals and derivatives for the inequalities (1.2) and (1.3) to obtain new corresponding CF-fractional integral inequalities of Opial type.

2.1 The first order CF-fractional inequalities

Theorem 2.1 *Suppose $\sigma, \eth : [0,\infty) \to \mathbb{R}$ are two differentiable convex and increasing functions with $\sigma(g(0)) = 0$ and suppose that $\mathbf{f}' \in L[\rtimes_1, \rtimes_2]$. Then, for $0 < \bar{\nu} < 1$, we have*

$$ \begin{aligned} &\int_{\rtimes_1}^{\rtimes_2} \sigma'\bigl(\eth\bigl(\bigl|{}^{CFC}_{\rtimes_1}D^{\bar{\nu}}\mathbf{f}(\eta)\bigr|\bigr)\bigr)\eth'\bigl(\bigl|{}^{CFC}_{\rtimes_1}D^{\bar{\nu}}\mathbf{f}(\eta)\bigr|\bigr)\bigl|\eth \circ \mathbf{f}'(\eta)\bigr|\, d\eta \\ &\quad \le \frac{1-\bar{\nu}}{\mathcal{B}(\bar{\nu})}\sigma\left(\eth\left(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}\int_{\rtimes_1}^{\rtimes_2}\bigl|\eth \circ \mathbf{f}'(t)\bigr|\, dt\right)\right) \\ &\quad \le \frac{1-\bar{\nu}}{\mathcal{B}(\bar{\nu})(\rtimes_2-\rtimes_1)}\int_{\rtimes_1}^{\rtimes_2}\sigma\left(\eth\left(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}(\rtimes_2-\rtimes_1)\bigl|\eth \circ \mathbf{f}'(t)\bigr|\right)\right) dt, \end{aligned} \tag{2.1} $$

where ${}^{CFC}_{\rtimes_1}D^{\bar{\nu}}\mathbf{f}(\eta)$ is as in (1.8).

Proof Let $\mathbf{v} := \mathbf{f}'$, so $\mathbf{v}$ is a continuous function. Then, we find

$$ u(\eta) := {}^{CFC}_{\rtimes_1}D^{\bar{\nu}}\mathbf{f}(\eta) = \frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}} \int_{\rtimes_1}^{\eta} \exp\bigl(\bar{\lambda}(\eta-\xi)\bigr)\mathbf{v}(\xi)\, d\xi $$

$$= \int_{\rtimes_1}^{\eta} \mathrm{k}(\eta,\xi)\mathbf{v}(\xi)\,\mathrm{d}\xi, \tag{2.2}$$

where

$$\mathrm{k}(\eta,\xi) = \begin{cases} \frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}} \exp(\bar{\lambda}(\eta-\xi)), & \rtimes_1 \le \xi \le \eta, \\ 0, & \eta < \xi \le \rtimes_2. \end{cases} \tag{2.3}$$

Since $\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}} > 0$ and $\bar{\lambda} = \frac{-\bar{\nu}}{1-\bar{\nu}} < 0$ for $0 < \bar{\nu} < 1$, the function $\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}} \exp(\bar{\lambda}(\eta-\xi))$ is a decreasing function on the interval $[\rtimes_1, \rtimes_2]$ and hence

$$\left|\mathrm{k}(\eta,\xi)\right| \le \frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}} := \mathcal{K}.$$

Then, by applying Theorem 1.1 for the above particular findings, we easily obtain inequalities (2.1). □

Corollary 2.1 *Let the assumptions of previous theorem be given. Then, for* $0 < \bar{\nu} < 1$ *and* $\bar{\varrho} \ge 1$, *we have*

$$\begin{aligned} &\int_{\rtimes_1}^{\rtimes_2} \sigma'\bigl(\bigl|{}^{CFC}_{\rtimes_1}D^{\bar{\nu}}\mathbf{f}(\eta)\bigr|^{\bar{\varrho}}\bigr)\bigl|{}^{CFC}_{\rtimes_1}D^{\bar{\nu}}\mathbf{f}(\eta)\bigr|^{\bar{\varrho}-1}\bigl|\bigl(\mathbf{f}'(\eta)\bigr)^{\bar{\varrho}}\bigr|\,\mathrm{d}\eta \\ &\quad \le \frac{1-\bar{\nu}}{\bar{\varrho}\mathcal{B}(\bar{\nu})}\sigma\left(\left(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}\int_{\rtimes_1}^{\rtimes_2}\bigl|\mathbf{f}'(t)\bigr|^{\bar{\varrho}}\,\mathrm{d}t\right)^{\bar{\varrho}}\right) \\ &\quad \le \frac{1-\bar{\nu}}{\bar{\varrho}\mathcal{B}(\bar{\nu})(\rtimes_2-\rtimes_1)}\int_{\rtimes_1}^{\rtimes_2}\sigma\left(\left(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}(\rtimes_2-\rtimes_1)\bigl|\mathbf{f}'(t)\bigr|^{\bar{\varrho}}\right)^{\bar{\varrho}}\right)\mathrm{d}t. \end{aligned} \tag{2.4}$$

Proof We know that the function $x^{\bar{\varrho}}$ is an increasing and convex function for $\bar{\varrho} \ge 1$. Then, by applying Theorem 2.1 for such a function $\eth(\eta) := x^{\bar{\varrho}}$, we easily obtain inequalities (2.4). □

We can obtain the same results for the right-sided CF-fractional integral on the class of functions $\mathcal{U}_2(\eth \circ \mathbf{v}, \mathrm{k})$ (see Definition 1.2). These are given in the following.

Theorem 2.2 *Suppose* $\sigma, \eth : [0,\infty) \to \mathbb{R}$ *are two differentiable convex and increasing functions with* $\sigma(g(0)) = 0$ *and suppose that* $\mathbf{f}' \in L[\rtimes_1, \rtimes_2]$. *Then, for* $0 < \bar{\nu} < 1$, *we have*

$$\begin{aligned} &\int_{\rtimes_1}^{\rtimes_2} \sigma'\bigl(\eth\bigl(\bigl|{}^{CFC}D^{\bar{\nu}}_{\rtimes_2-}\mathbf{f}(\eta)\bigr|\bigr)\bigr)\eth'\bigl(\bigl|{}^{CFC}D^{\bar{\nu}}_{\rtimes_2-}\mathbf{f}(\eta)\bigr|\bigr)\bigl|\eth\circ\mathbf{f}'(\eta)\bigr|\,\mathrm{d}\eta \\ &\quad \le \frac{1-\bar{\nu}}{\mathcal{B}(\bar{\nu})}\sigma\left(\eth\left(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}\int_{\rtimes_1}^{\rtimes_2}\bigl|\eth\circ\mathbf{f}'(t)\bigr|\,\mathrm{d}t\right)\right) \\ &\quad \le \frac{1-\bar{\nu}}{\mathcal{B}(\bar{\nu})(\rtimes_2-\rtimes_1)}\int_{\rtimes_1}^{\rtimes_2}\sigma\left(\eth\left(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}(\rtimes_2-\rtimes_1)\bigl|\eth\circ\mathbf{f}'(t)\bigr|\right)\right)\mathrm{d}t, \end{aligned} \tag{2.5}$$

where ${}^{CFC}_{\rtimes_1}D^{\bar{\nu}}_{\rtimes_2-}\mathbf{f}(\eta)$ *is as in* (1.9).

Proof We can use the same method used for Theorem 2.1 to produce the results for Theorem 2.2. □

Corollary 2.2 *Let the assumptions of previous theorem be given. Then, for* $0 < \bar{\nu} < 1$ *and* $\bar{\varrho} \geq 1$, *we have*

$$\begin{aligned}
&\int_{\rtimes_1}^{\rtimes_2} \sigma'\bigl(\bigl|{}^{CFC}D^{\bar{\nu}}_{\rtimes_2 -}\mathbf{f}(\eta)\bigr|^{\bar{\varrho}}\bigr)\bigl|{}^{CFC}D^{\bar{\nu}}_{\rtimes_2 -}\mathbf{f}(\eta)\bigr|^{\bar{\varrho}-1}\bigl|\bigl(\mathbf{f}'(\eta)\bigr)^{\bar{\varrho}}\bigr|\,\mathrm{d}\eta \\
&\quad \leq \frac{1-\bar{\nu}}{\bar{\varrho}\mathcal{B}(\bar{\nu})}\sigma\biggl(\biggl(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}\int_{\rtimes_1}^{\rtimes_2}\bigl|\mathbf{f}'(t)\bigr|^{\bar{\varrho}}\,\mathrm{d}t\biggr)^{\bar{\varrho}}\biggr) \\
&\quad \leq \frac{1-\bar{\nu}}{\bar{\varrho}\mathcal{B}(\bar{\nu})(\rtimes_2-\rtimes_1)}\int_{\rtimes_1}^{\rtimes_2}\sigma\biggl(\biggl(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}(\rtimes_2-\rtimes_1)\bigl|\mathbf{f}'(t)\bigr|^{\bar{\varrho}}\biggr)^{\bar{\varrho}}\biggr)\,\mathrm{d}t.
\end{aligned} \tag{2.6}$$

Proof We can use the same method used for Corollary 2.1 to produce the results for Corollary 2.2. □

2.2 The higher order CF-fractional inequalities

Here we generalize the previous findings to the new corresponding higher order CF-fractional inequalities of Opial type. At first, we have to recall the higher order CF-fractional derivatives.

Definition 2.1 ([18]) For a function $\mathbf{f}$, $\mathbf{f}^{(n)} \in \mathcal{H}^1(\rtimes_1, \rtimes_2)$ and $n < \bar{\nu} < n+1$, we have the higher order left and right CF-fractional derivatives, respectively, defined by

$$\bigl({}^{CFC}_{\rtimes_1}D^{\bar{\nu}}_{H}\mathbf{f}\bigr)(\eta) := \bigl({}^{CFC}_{\rtimes_1}D^{\bar{\nu}-n}\mathbf{f}^{(n)}\bigr)(\eta) = \frac{\mathcal{B}(\bar{\nu}-n)}{n+1-\bar{\nu}}\int_{\rtimes_1}^{\eta}\mathbf{f}^{(n+1)}(\xi)\exp\bigl(\bar{\lambda}(\eta-\xi)\bigr)\,\mathrm{d}\xi \tag{2.7}$$

and

$$\begin{aligned}
&\bigl({}^{CFC}_{H}D^{\bar{\nu}}_{\rtimes_2}\mathbf{f}\bigr)(\eta) \\
&\quad := \bigl({}^{CFC}D^{\bar{\nu}-n}_{\rtimes_2}(-1)^n\mathbf{f}^{(n)}\bigr)(\eta) = \frac{(-1)^n\mathcal{B}(\bar{\nu}-n)}{n+1-\bar{\nu}}\int_{\eta}^{\rtimes_2}\mathbf{f}^{(n+1)}(\xi)\exp\bigl(\bar{\lambda}(\xi-x)\bigr)\,\mathrm{d}\xi,
\end{aligned} \tag{2.8}$$

where $\bar{\lambda} = -\frac{\bar{\nu}-n}{n+1-\bar{\nu}}$.

Theorem 2.3 *Suppose* $\sigma, \eth : [0,\infty) \to \mathbb{R}$ *are two differentiable convex and increasing functions with* $\sigma(g(0)) = 0$ *and suppose that* $\mathbf{f}^{(n+1)} \in L[\rtimes_1, \rtimes_2]$. *Then, for* $n < \bar{\nu} < n+1$, *we have*

$$\begin{aligned}
&\int_{\rtimes_1}^{\rtimes_2}\sigma'\bigl(\eth\bigl(\bigl|{}^{CFC}_{\rtimes_1}D^{\bar{\nu}}_{H}\mathbf{f}(\eta)\bigr|\bigr)\bigr)\eth'\bigl(\bigl|{}^{CFC}_{\rtimes_1}D^{\bar{\nu}}_{H}\mathbf{f}(\eta)\bigr|\bigr)\bigl|\eth\circ\mathbf{f}^{(n+1)}(\eta)\bigr|\,\mathrm{d}\eta \\
&\quad \leq \frac{n+1-\bar{\nu}}{\mathcal{B}(\bar{\nu}-n)}\sigma\biggl(\eth\biggl(\frac{\mathcal{B}(\bar{\nu}-n)}{n+1-\bar{\nu}}\int_{\rtimes_1}^{\rtimes_2}\bigl|\eth\circ\mathbf{f}^{(n+1)}(t)\bigr|\,\mathrm{d}t\biggr)\biggr) \\
&\quad \leq \frac{n+1-\bar{\nu}}{\mathcal{B}(\bar{\nu}-n)(\rtimes_2-\rtimes_1)}\int_{\rtimes_1}^{\rtimes_2}\sigma\biggl(\eth\biggl(\frac{\mathcal{B}(\bar{\nu}-n)}{n+1-\bar{\nu}}(\rtimes_2-\rtimes_1)\bigl|\eth\circ\mathbf{f}^{(n+1)}(t)\bigr|\biggr)\biggr)\,\mathrm{d}t,
\end{aligned} \tag{2.9}$$

where ${}^{CFC}_{\rtimes_1}D^{\bar{\nu}}\mathbf{f}(\eta)$ *is as in* (2.7).

Proof Let $\mathbf{v} := \mathbf{f}^{(n+1)}$, so $\mathbf{v}$ is a continuous function. Then we find

$$\mathfrak{u}(\eta) := {}^{CFC}_{\rtimes_1}D^{\bar{\nu}}\mathbf{f}(\eta) = \frac{\mathcal{B}(\bar{\nu}-n)}{n+1-\bar{\nu}}\int_{\rtimes_1}^{\eta}\exp\bigl(\bar{\lambda}(\eta-\xi)\bigr)\mathbf{v}(\xi)\,\mathrm{d}\xi$$

$$= \int_{\rtimes_1}^{\eta} \mathrm{k}(\eta,\xi)\mathbf{v}(\xi)\,\mathrm{d}\xi, \tag{2.10}$$

where

$$\mathrm{k}(\eta,\xi) = \begin{cases} \frac{\mathcal{B}(\bar{\nu}-n)}{n+1-\bar{\nu}}\exp(\bar{\lambda}(\eta-\xi)), & \rtimes_1 \le \xi \le \eta, \\ 0, & \eta < \xi \le \rtimes_2. \end{cases} \tag{2.11}$$

From this we have

$$\left|\mathrm{k}(\eta,\xi)\right| \le \frac{\mathcal{B}(\bar{\nu}-n)}{n+1-\bar{\nu}} := \mathcal{K}.$$

Hence by applying Theorem 1.1 for the above particular findings, we easily obtain inequalities (2.9). □

Corollary 2.3 *Let the assumptions of the previous theorem be given. Then, for $n < \bar{\nu} < n+1$ and $\bar{\varrho} \ge 1$, we have*

$$\begin{aligned} &\int_{\rtimes_1}^{\rtimes_2} \sigma'\big(\big|{}^{CFC}_{\rtimes_1}D^{\bar{\nu}}_{H}\mathbf{f}(\eta)\big|^{\bar{\varrho}}\big)\big|{}^{CFC}_{\rtimes_1}D^{\bar{\nu}}_{H}\mathbf{f}(\eta)\big|^{\bar{\varrho}-1}\big|\big(\mathbf{f}^{(n+1)}(\eta)\big)^{\bar{\varrho}}\big|\,\mathrm{d}\eta \\ &\quad\le \frac{n+1-\bar{\nu}}{\bar{\varrho}\mathcal{B}(\bar{\nu}-n)}\sigma\left(\left(\frac{\mathcal{B}(\bar{\nu}-n)}{n+1-\bar{\nu}}\int_{\rtimes_1}^{\rtimes_2}\big|\mathbf{f}^{(n+1)}(t)\big|^{\bar{\varrho}}\,\mathrm{d}t\right)^{\bar{\varrho}}\right) \\ &\quad\le \frac{n+1-\bar{\nu}}{\bar{\varrho}\mathcal{B}(\bar{\nu}-n)(\rtimes_2-\rtimes_1)}\int_{\rtimes_1}^{\rtimes_2}\sigma\left(\left(\frac{\mathcal{B}(\bar{\nu}-n)}{n+1-\bar{\nu}}(\rtimes_2-\rtimes_1)\big|\mathbf{f}^{(n+1)}(t)\big|^{\bar{\varrho}}\right)^{\bar{\varrho}}\right)\mathrm{d}t. \end{aligned} \tag{2.12}$$

Proof We know that the function $x^{\bar{\varrho}}$ is an increasing and convex function for $\bar{\varrho} \ge 1$. Then, by applying Theorem 2.3 for such a function $\eth(\eta) := x^{\bar{\varrho}}$, we easily obtain inequalities (2.12). □

We can obtain the same results for the right-sided higher order CF-fractional integral on the class of functions $\mathcal{U}_2(\eth\circ\mathbf{v},\mathrm{k})$ (see Definition 1.2). These are given in the following.

Theorem 2.4 *Suppose $\sigma,\eth : [0,\infty) \to \mathbb{R}$ are two differentiable convex and increasing functions with $\sigma(g(0)) = 0$ and suppose that $\mathbf{f}^{(n+1)} \in L[\rtimes_1,\rtimes_2]$. Then, for $n < \bar{\nu} < n+1$, we have*

$$\begin{aligned} &\int_{\rtimes_1}^{\rtimes_2} \sigma'\big(\eth\big(\big|{}^{CFC}_{H}D^{\bar{\nu}}_{\rtimes_2-}\mathbf{f}(\eta)\big|\big)\big)\eth'\big(\big|{}^{CFC}_{H}D^{\bar{\nu}}_{\rtimes_2-}\mathbf{f}(\eta)\big|\big)\big|\eth\circ\mathbf{f}^{(n+1)}(\eta)\big|\,\mathrm{d}\eta \\ &\quad\le \frac{n+1-\bar{\nu}}{\mathcal{B}(\bar{\nu}-n)}\sigma\left(\eth\left(\frac{\mathcal{B}(\bar{\nu}-n)}{n+1-\bar{\nu}}\int_{\rtimes_1}^{\rtimes_2}\big|\eth\circ\mathbf{f}^{(n+1)}(t)\big|\,\mathrm{d}t\right)\right) \\ &\quad\le \frac{n+1-\bar{\nu}}{\mathcal{B}(\bar{\nu}-n)(\rtimes_2-\rtimes_1)}\int_{\rtimes_1}^{\rtimes_2}\sigma\left(\eth\left(\frac{\mathcal{B}(\bar{\nu}-n)}{n+1-\bar{\nu}}(\rtimes_2-\rtimes_1)\big|\eth\circ\mathbf{f}^{(n+1)}(t)\big|\right)\right)\mathrm{d}t, \end{aligned} \tag{2.13}$$

where ${}^{CFC}_{H}D^{\bar{\nu}}_{\rtimes_2-}\mathbf{f}(\eta)$ is as in (2.8).

Proof We can use the same method used for Theorem 2.3 to produce the results for Theorem 2.4. □

Corollary 2.4 *Let the assumptions of previous theorem be given. Then, for $n < \bar{\nu} < n+1$ and $\bar{\varrho} \geq 1$, we have*

$$\begin{aligned}
&\int_{\rtimes_1}^{\rtimes_2} \sigma'\bigl(\bigl|{}_{H}^{CFC}D_{\rtimes_2-}^{\bar{\nu}}\mathbf{f}(\eta)\bigr|^{\bar{\varrho}}\bigr)\bigl|{}_{H}^{CFC}D_{\rtimes_2-}^{\bar{\nu}}\mathbf{f}(\eta)\bigr|^{\bar{\varrho}-1}\bigl|\bigl(\mathbf{f}^{(n+1)}(\eta)\bigr)^{\bar{\varrho}}\bigr|\,\mathrm{d}\eta \\
&\quad\leq \frac{n+1-\bar{\nu}}{\bar{\varrho}\mathcal{B}(\bar{\nu}-n)}\sigma\biggl(\biggl(\frac{\mathcal{B}(\bar{\nu}-n)}{n+1-\bar{\nu}}\int_{\rtimes_1}^{\rtimes_2}\bigl|\mathbf{f}^{(n+1)}(t)\bigr|^{\bar{\varrho}}\,\mathrm{d}t\biggr)^{\bar{\varrho}}\biggr) \\
&\quad\leq \frac{n+1-\bar{\nu}}{\bar{\varrho}\mathcal{B}(\bar{\nu}-n)(\rtimes_2-\rtimes_1)} \\
&\qquad\times\int_{\rtimes_1}^{\rtimes_2}\sigma\biggl(\biggl(\frac{\mathcal{B}(\bar{\nu}-n)}{n+1-\bar{\nu}}(\rtimes_2-\rtimes_1)\bigl|\mathbf{f}^{(n+1)}(t)\bigr|^{\bar{\varrho}}\biggr)^{\bar{\varrho}}\biggr)\,\mathrm{d}t. \qquad (2.14)
\end{aligned}$$

Proof We can use the same method used for Corollary 2.3 to produce the results for Corollary 2.4. □

3 The ABC-fractional inequalities

In the final section, we improve our study from the CF-fractional operators with a nonsingular kernel to the Atangana–Baleanu (AB) fractional operators with a Mittag-Leffler (ML) kernel. For this, we get new inequalities of Opial type. Let us first recall the AB fractional derivatives.

Definition 3.1 ([19, 51]) For any function $\mathbf{f}$ which is $\mathcal{H}^1(a,b)$ with $0 < \rtimes_1 < \rtimes_2$, and for any $x \in [\rtimes_1, \rtimes_2]$, the $\bar{\nu}$th left-AB fractional derivative of $\mathbf{f}(\eta)$ in the sense of Caputo is defined by

$${}_{\rtimes_1+}^{ABC}D^{\bar{\nu}}\mathbf{f}(\eta) := \frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}\int_{\rtimes_1}^{x}\mathbf{f}'(\xi)\mathrm{E}_{\bar{\nu}}\bigl(\bar{\lambda}(x-\xi)^{\bar{\nu}}\bigr)\,\mathrm{d}\xi, \qquad (3.1)$$

and the left-CF-fractional derivative of $\mathbf{f}(\eta)$ in the sense of Caputo is defined by

$${}^{ABC}D_{\rtimes_2-}^{\bar{\nu}}\mathbf{f}(\eta) := -\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}\int_{x}^{\rtimes_2}\mathbf{f}'(\xi)\mathrm{E}_{\bar{\nu}}\bigl(\bar{\lambda}(\xi-x)^{\bar{\nu}}\bigr)\,\mathrm{d}\xi, \qquad (3.2)$$

where $\bar{\lambda}$ and $\mathcal{B}(\bar{\nu})$ are both as before, and $\mathrm{E}_{\bar{\nu}}(\cdot)$ is a Mittag-Leffler (ML) function defined by [51]

$$\mathrm{E}_{\bar{\nu}}\bigl(\bar{\lambda}z^{\bar{\nu}}\bigr) = \mathrm{E}_{\bar{\nu}}(\bar{\lambda}, z) := \sum_{\ell=0}^{\infty}\bar{\lambda}^{\ell}\frac{z^{\bar{\nu}\ell}}{\Gamma(\bar{\nu}\ell+1)},$$

for $0 \neq \bar{\lambda} \in \mathbb{R}, z \in \mathbb{C}$, and $\Re(\bar{\nu}) > 0$.

Theorem 3.1 *Suppose $\phi, \mathbf{g} : [0,\infty) \to \mathbb{R}$ are two differentiable convex and increasing functions with $\phi(g(0)) = 0$ and suppose that $\mathbf{f}' \in L[\rtimes_1, \rtimes_2]$. Then, for $0 < \bar{\nu} < 1$, we have*

$$\begin{aligned}
&\int_{\rtimes_1}^{\rtimes_2}\phi'\bigl(\mathbf{g}\bigl(\bigl|{}_{\rtimes_1+}^{ABC}D^{\bar{\nu}}\mathbf{f}(\eta)\bigr|\bigr)\bigr)\mathbf{g}'\bigl(\bigl|{}_{\rtimes_1+}^{ABC}D^{\bar{\nu}}\mathbf{f}(\eta)\bigr|\bigr)\bigl|\mathbf{g}\circ\mathbf{f}'(\eta)\bigr|\,\mathrm{d}\eta \\
&\quad\leq \frac{1-\bar{\nu}}{\mathcal{B}(\bar{\nu})}\phi\biggl(\mathbf{g}\biggl(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}\int_{\rtimes_1}^{\rtimes_2}\bigl|\mathbf{g}\circ\mathbf{f}'(t)\bigr|\,\mathrm{d}t\biggr)\biggr)
\end{aligned}$$

$$\leq \frac{1-\bar{\nu}}{\mathcal{B}(\bar{\nu})(\rtimes_2-\rtimes_1)}\int_{\rtimes_1}^{\rtimes_2}\phi\left(\mathbf{g}\left(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}(\rtimes_2-\rtimes_1)\left|\mathbf{g}\circ\mathbf{f}'(t)\right|\right)\right)\mathrm{d}t, \tag{3.3}$$

where ${}^{ABC}_{\rtimes_1+}D^{\bar{\nu}}\mathbf{f}(\eta)$ *is as in* (3.1).

Proof Let $\mathbf{v} := \mathbf{f}'$, so $\mathbf{v}$ is a continuous function. Then we find

$$\begin{aligned}\mathbf{u}(\eta) := {}^{ABC}_{\rtimes_1+}D^{\bar{\nu}}\mathbf{f}(\eta) &= \frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}\int_{\rtimes_1}^{x}E_{\bar{\nu}}\left(\bar{\lambda}(x-\xi)^{\bar{\nu}}\right)\mathbf{v}(\xi)\,\mathrm{d}\xi\\ &= \int_{\rtimes_1}^{x}\mathrm{k}(x,\xi)\mathbf{v}(\xi)\,\mathrm{d}\xi,\end{aligned} \tag{3.4}$$

where

$$\mathrm{k}(x,\xi) = \begin{cases}\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}E_{\bar{\nu}}(\bar{\lambda}(x-\xi)^{\bar{\nu}}), & a\leq\xi\leq x,\\ 0, & x<\xi\leq b.\end{cases} \tag{3.5}$$

From [52], we can see that the function $\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}E_{\bar{\nu}}(\bar{\lambda}(x-\xi)^{\bar{\nu}})$ is a monotonically decreasing function on the interval $[\rtimes_1,\rtimes_2]$ and hence

$$\left|\mathrm{k}(x,\xi)\right| \leq \frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}} := \mathcal{K}.$$

Then, by applying Theorem 1.1 for the above particular findings, we easily obtain inequalities (3.3). □

Corollary 3.1 *Let the assumptions of the previous theorem be given. Then, for* $0<\bar{\nu}<1$ *and* $\bar{\varrho}\geq 1$, *we have*

$$\begin{aligned}&\int_{\rtimes_1}^{\rtimes_2}\phi'\left(\left|{}^{ABC}_{\rtimes_1+}D^{\bar{\nu}}\mathbf{f}(\eta)\right|^{\bar{\varrho}}\right)\left|{}^{ABC}_{\rtimes_1+}D^{\bar{\nu}}\mathbf{f}(\eta)\right|^{q-1}\left|\left(\mathbf{f}'(\eta)\right)^{\bar{\varrho}}\right|\mathrm{d}\eta\\ &\quad\leq \frac{1-\bar{\nu}}{q\mathcal{B}(\bar{\nu})}\phi\left(\left(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}\int_{\rtimes_1}^{\rtimes_2}\left|\mathbf{f}'(t)\right|^{\bar{\varrho}}\mathrm{d}t\right)^{\bar{\varrho}}\right)\\ &\quad\leq \frac{1-\bar{\nu}}{q\mathcal{B}(\bar{\nu})(\rtimes_2-\rtimes_1)}\int_{\rtimes_1}^{\rtimes_2}\phi\left(\left(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}(\rtimes_2-\rtimes_1)\left|\mathbf{f}'(t)\right|^{\bar{\varrho}}\right)^{\bar{\varrho}}\right)\mathrm{d}t.\end{aligned} \tag{3.6}$$

Proof We can use the same technique used for Corollary 2.1 to produce the results for Corollary 3.2. □

We can obtain the same results for the right-sided AB fractional integral on the class of functions $\mathcal{U}_2(\mathbf{g}\circ\mathbf{v},\mathrm{k})$ (see Definition 1.2). These are given in the following.

Theorem 3.2 *Suppose* $\phi,\mathbf{g}:[0,\infty)\to\mathbb{R}$ *are two differentiable convex and increasing functions with* $\phi(g(0))=0$ *and suppose that* $\mathbf{f}'\in L[\rtimes_1,\rtimes_2]$. *Then, for* $0<\bar{\nu}<1$, *we have*

$$\begin{aligned}&\int_{\rtimes_1}^{\rtimes_2}\phi'\left(\mathbf{g}\left(\left|{}^{ABC}D^{\bar{\nu}}_{\rtimes_2-}\mathbf{f}(\eta)\right|\right)\right)\mathbf{g}'\left(\left|{}^{ABC}D^{\bar{\nu}}_{\rtimes_2-}\mathbf{f}(\eta)\right|\right)\left|\mathbf{g}\circ\mathbf{f}'(\eta)\right|\mathrm{d}\eta\\ &\quad\leq \frac{1-\bar{\nu}}{\mathcal{B}(\bar{\nu})}\phi\left(\mathbf{g}\left(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}\int_{\rtimes_1}^{\rtimes_2}\left|\mathbf{g}\circ\mathbf{f}'(t)\right|\mathrm{d}t\right)\right)\end{aligned}$$

$$\leq \frac{1-\bar{\nu}}{\mathcal{B}(\bar{\nu})(\rtimes_2-\rtimes_1)} \int_{\rtimes_1}^{\rtimes_2} \phi\left(\mathbf{g}\left(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}(\rtimes_2-\rtimes_1)\left|\mathbf{g}\circ\mathbf{f}'(t)\right|\right)\right) dt, \tag{3.7}$$

where ${}^{ABC}D^{\bar{\nu}}_{\rtimes_2-}\mathbf{f}(\eta)$ *is as in* (3.2).

Proof We can use the same technique used for Theorem 3.1 to produce the results for Theorem 3.2. □

Corollary 3.2 *Let the assumptions of the previous theorem be given. Then, for* $0<\bar{\nu}<1$ *and* $\bar{\varrho}\geq 1$, *we have*

$$\begin{aligned}
&\int_{\rtimes_1}^{\rtimes_2} \phi'\left(\left|{}^{ABC}D^{\bar{\nu}}_{\rtimes_2-}\mathbf{f}(\eta)\right|^{\bar{\varrho}}\right)\left|{}^{ABC}D^{\bar{\nu}}_{\rtimes_2-}\mathbf{f}(\eta)\right|^{q-1}\left|\left(\mathbf{f}'(\eta)\right)^{\bar{\varrho}}\right| d\eta \\
&\quad\leq \frac{1-\bar{\nu}}{q\mathcal{B}(\bar{\nu})}\phi\left(\left(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}\int_{\rtimes_1}^{\rtimes_2}\left|\mathbf{f}'(t)\right|^{\bar{\varrho}} dt\right)^{\bar{\varrho}}\right) \\
&\quad\leq \frac{1-\bar{\nu}}{q\mathcal{B}(\bar{\nu})(\rtimes_2-\rtimes_1)}\int_{\rtimes_1}^{\rtimes_2}\phi\left(\left(\frac{\mathcal{B}(\bar{\nu})}{1-\bar{\nu}}(\rtimes_2-\rtimes_1)\left|\mathbf{f}'(t)\right|^{\bar{\varrho}}\right)^{\bar{\varrho}}\right) dt.
\end{aligned} \tag{3.8}$$

Proof We can use the same technique used for Corollary 2.1 to produce the results for Corollary 3.2. □

4 Conclusion

In the current study, we have considered the Opial integral inequalities in the context of generalized fractional operators with nonsingular kernel. Also, we have studied some related integral inequalities for the CFC and ABC-fractional integrals. It can be seen that our obtained formulas will be very helpful in the theoretical study of other models of fractional calculus.

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