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Optimal bounds for Toader mean in terms of general means

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Abstract

In this paper, we present the best possible parameters $\alpha(r)$, $\beta(r)$ such that the double inequality

$$[\alpha(r)M^r(a,b)+(1-\alpha(r))N^r(a,b)]^{1/r} < \mathrm{TD}\big[M(a,b),N(a,b)\big] < \big[\beta(r)M^r(a,b)+(1-\beta(r))N^r(a,b)\big]^{1/r},$$

holds for all $r \le 1$ and $a,b>0$ with $a \neq b$, where

$$\mathrm{TD}(a,b) := \int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta$$

is the Toader mean, and M, N are means. As applications, we attain the optimal bounds for the Toader mean in terms of arithmetic, contraharmonic, centroidal and quadratic means, and then we provide some new bounds for the complete elliptic integral of the second kind.

MSC: 26E60

Keywords: Toader mean; Double inequality; Optimal bounds; Complete elliptic integral

1 Introduction

Let $I \subset \mathbb{R}$ be an open interval. A two-variable function $M : I^2 \to I$ is called a *mean* on the interval I if

$$\min\{a,b\} \le M(a,b) \le \max\{a,b\}, \quad a,b \in I.$$

If for all $a,b \in I$, $a \neq b$, these inequalities are strict, M is called *strict mean*. M is called *symmetric* if $M(b,a) = M(a,b)$ holds for all $a,b \in I$. If $M(ta,tb) = tM(a,b)$ holds for all $a,b,t \in \mathbb{R}_+$, then M is called *homogeneous*.

For all $a, b \in \mathbb{R}_+$, the *power mean* or the *Hölder mean* M_p for $p \in \mathbb{R}$ is defined by

$$M_p(a,b) = \begin{cases} (\frac{a^p+b^p}{2})^{\frac{1}{p}}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases}$$

As some special cases,

$$M_1 = \frac{a+b}{2} =: A(a,b), \qquad M_0 = \sqrt{ab} =: G(a,b),$$
$$M_{-1} = \frac{2ab}{a+b} =: H(a,b), \qquad M_2 = \sqrt{a^2+b^2} =: Q(a,b),$$

are, respectively, the classical arithmetic mean, geometric mean, harmonic mean and quadratic mean.

For $a, b \in \mathbb{R}_+$, the Gauss-iteration of the arithmetic mean A and the geometric mean G defined by

$$a_1 := a, \qquad b_1 := b; \qquad a_{n+1} := \frac{a_n + b_n}{2}, \qquad b_{n+1} := \sqrt{a_n b_n}, \quad n \in \mathbb{N},$$

satisfies

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n =: A \otimes G(a,b),$$

which is called the *Gauss arithmetic–geometric mean* [1]. As is well known, Gauss found the general formula for $A \otimes G$ as follows:

$$A \otimes G(a,b) = \left(\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{a^2\cos^2 t + b^2 \sin^2 t}}\right)^{-1}, \quad a, b \in \mathbb{R}_+,$$

which can be rewritten as

$$A \otimes G(a,b) = \begin{cases} (\frac{2a}{\pi}\kappa(\sqrt{1-(\frac{b}{a})^2}))^{-1}, & \text{if } a \geq b, \\ (\frac{2b}{\pi}\kappa(\sqrt{1-(\frac{a}{b})^2}))^{-1}, & \text{if } a < b, \end{cases} \quad a, b \in \mathbb{R}_+,$$

where

$$\kappa(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1-r^2\sin^2\theta}}\,d\theta, \quad r \in (0,1), \tag{1.1}$$

is the complete elliptic integral of the first kind.

In 1991, Haruki considered a more general mean [2]:

$$M_{\varphi,n}(a,b) := \varphi^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi} \varphi\big(r_n(\theta)\big)\,d\theta\right),$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}$ is strictly monotonic function and $r_n(\theta)$ is given by

$$r_n(\theta) = \begin{cases} (a^n\cos^2\theta + b^n\sin^2\theta)^{1/n}, & \text{if } n \neq 0, \\ a^{\cos^2\theta} b^{\sin^2\theta}, & \text{if } n = 0, \end{cases} \quad a, b \in \mathbb{R}_+.$$

It is well known that $M_{\varphi,n} = A \otimes G$ for the case $n = 2$ and $\varphi(x) = x^{-1}$.

In 1998, Toader found that, for the case $n = 2$ and $\varphi(x) = x$, the mean $\varphi_{\varphi,n}(a,b)$ becomes another new mean $\mathrm{TD}(a,b)$, called the *Toader mean* later, which has a close relationship with the complete elliptic integral of the second kind [3], that is,

$$\mathrm{TD}(a,b) = \int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\, d\theta, \quad a,b \in \mathbb{R}_+. \tag{1.2}$$

It can be rewritten as

$$\mathrm{TD}(a,b) = \begin{cases} \frac{2a}{\pi}\varepsilon(\sqrt{1-(\frac{b}{a})^2}), & \text{if } a \geq b, \\ \frac{2b}{\pi}\varepsilon(\sqrt{1-(\frac{a}{b})^2}), & \text{if } a < b, \end{cases} \quad a,b \in \mathbb{R}_+,$$

where

$$\varepsilon(r) = \int_0^{\frac{\pi}{2}} \sqrt{1-r^2\sin^2\theta}\, d\theta, \quad r \in (0,1), \tag{1.3}$$

is the complete elliptic integral of the second kind.

For $a,b \in \mathbb{R}_+$ with $a \neq b$, the contraharmonic mean $C(a,b)$, the centroidal mean $\overline{C}(a,b)$, the logarithmic mean $L(a,b)$, the identric mean $I(a,b)$ and the first Seiffert mean $P(a,b)$ [4] are, respectively, defined by

$$\begin{aligned} &C(a,b) = \frac{a^2+b^2}{a+b}, \qquad \overline{C}(a,b) = \frac{2(a^2+ab+b^2)}{3(a+b)}, \qquad L(a,b) = \frac{b-a}{\log b - \log a}, \\ &I(a,b) = \frac{1}{e}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}, \qquad P(a,b) = \frac{a-b}{2\arcsin(\frac{a-b}{a+b})}, \end{aligned} \tag{1.4}$$

which satisfy the well-known chain of inequalities

$$\begin{aligned} &H(a,b) < G(a,b) < L(a,b) < P(a,b) < I(a,b) \\ &\quad < A(a,b) < \mathrm{TD}(a,b) < \overline{C}(a,b) < Q(a,b) < C(a,b). \end{aligned}$$

In 1997, Vuorinen [5] conjectured that

$$\mathrm{TD}(a,b) > M_{3/2}(a,b),$$

for all $a,b > 0$ with $a \neq b$. The conjecture was proved by Qiu and Shen [6], and Barnard, Pearce and Richards [7], respectively.

In [8], Alzer and Qiu presented a best possible upper power mean bound for the Toader mean as follows:

$$\mathrm{TD}(a,b) < M_{\log 2/\log(\pi/2)}(a,b),$$

for all $a,b > 0$ with $a \neq b$.

Neuman [9] proved that the inequalities

$$\frac{(a+b)\sqrt{ab}-ab}{A\otimes G(a,b)} < \mathrm{TD}(a,b) < \frac{4(a+b)\sqrt{ab}+(a-b)^2}{8A\otimes G(a,b)}$$

hold for all $a,b>0$ with $a\neq b$.

Kazi and Neuman [10] proved the inequality

$$\mathrm{TD}(a,b) < \frac{1}{4}\left(\sqrt{(2+\sqrt{2})a^2+(2-\sqrt{2})b^2}+\sqrt{(2+\sqrt{2})b^2+(2-\sqrt{2})a^2}\right)$$

holds for all $a,b>0$ with $a\neq b$.

In [11], the authors proved the inequalities

$$\begin{aligned}
&\alpha_1 A(a,b)+(1-\alpha_1)C(a,b) < \mathrm{TD}\big(A(a,b),C(a,b)\big) < \beta_1 A(a,b)+(1-\beta_1)C(a,b),\\
&A^{\alpha_2}(a,b)C^{1-\alpha_2}(a,b) < \mathrm{TD}\big(A(a,b),C(a,b)\big) < A^{\beta_2}(a,b)C^{1-\beta_2}(a,b),\\
&\frac{\alpha_3}{A(a,b)}+\frac{1-\alpha_3}{C(a,b)} < \frac{1}{\mathrm{TD}(A(a,b),C(a,b))} < \frac{\beta_3}{A(a,b)}+\frac{1-\beta_3}{C(a,b)},\\
&C\big(\alpha_4 a+(1-\alpha_4)b,\alpha_4 b+(1-\alpha_4)a\big) < \mathrm{TD}\big(A(a,b),C(a,b)\big)\\
&\qquad < C\big[\beta_4 a+(1-\beta_4)b,\beta_4 b+(1-\beta_4)a\big],
\end{aligned}$$

hold for all $a,b>0$ with $a\neq b$ if and only if

$$\begin{aligned}
&\alpha_1\geq 1/2, \qquad \beta_1\leq 2\left[1-\frac{3}{\pi}\varepsilon(1/3)+\frac{4}{3\pi}\kappa(1/3)\right],\\
&\alpha_2\geq 1/2, \qquad \beta_2\leq \log\big[3\pi/\big(9\varepsilon(1/3)-4\kappa(1/3)\big)\big],\\
&\alpha_3\leq \big[3\pi-9\varepsilon(1/3)+4\kappa(1/3)\big]/\big[9\varepsilon(1/3)-4\kappa(1/3)\big], \qquad \beta_3\geq 1/2,\\
&\alpha_4\leq (2+\sqrt{2})/4, \qquad \beta_4\geq \Big(1+\sqrt{\big[18\varepsilon(1/3)-8\kappa(1/3)\big]/3\pi-1}\Big)/2.
\end{aligned}$$

Recently, there were published numerous articles which focus on the bounds for the Toader mean [12–23]. For example, Zhao, Chu and Zhang [24] presented the best possible parameters $\alpha(r)$ and $\beta(r)$ such that the double inequality

$$\begin{aligned}
&[\alpha(r)A^r(a,b)+(1-\alpha(r)Q^r(a,b))]^{1/r} < \mathrm{TD}\big[A(a,b),Q(a,b)\big]\\
&\qquad < \big[\beta(r)A^r(a,b)+\big(1-\beta(r)Q^r(a,b)\big)\big]^{1/r}
\end{aligned}$$

holds for all $r\leq 1$ and $a,b>0$ with $a\neq b$.

Motivated by the above mentioned work, in this paper, for two means M, N, we present the best possible parameters $\alpha(r)$, $\beta(r)$ such that the double inequality

$$\begin{aligned}
&[\alpha(r)M^r(a,b)+(1-\alpha(r)N^r(a,b))]^{1/r} < \mathrm{TD}\big[M(a,b),N(a,b)\big]\\
&\qquad < \big[\beta(r)M^r(a,b)+\big(1-\beta(r)N^r(a,b)\big)\big]^{1/r}
\end{aligned}$$

holds for all $r\leq 1$ and $a,b>0$ with $a\neq b$.

2 Lemmas

In what follows, we will need some useful functional relations about complete elliptic integrals which we collect.

Lemma 1 ([1]) *For* $r \in (0,1)$,

(i)

$$\kappa(0^+) = \varepsilon(0^+) = \frac{\pi}{2},$$

(ii)

$$\frac{d\kappa(r)}{dr} = \frac{\varepsilon(r) - (1-r^2)\kappa(r)}{r(1-r^2)}, \qquad \frac{d\varepsilon(r)}{dr} = \frac{\varepsilon(r) - \kappa(r)}{r},$$

(iii)

$$\varepsilon(r) = (1+r')\varepsilon\left(\frac{1-r'}{1+r'}\right) - \frac{2r'}{1+r'}\kappa\left(\frac{1-r'}{1+r'}\right),$$

where $r' = \sqrt{1-r^2}$.

Lemma 2 *For all* $0 < r < 1$, *the following inequalities hold*:

(i) $\varepsilon(r) > \frac{\pi}{4}(1+\sqrt{1-r^2})$,

(ii) $\sqrt{1-r^2}\kappa(r) + \varepsilon(r) < \frac{\pi}{2}(1+\sqrt{1-r^2})$,

(iii) $(1-r^2)\kappa(r) < \varepsilon(r) < (1-\frac{r^2}{2})\kappa(r) < \kappa(r)$.

Proof (i) In fact, by the definition of $\varepsilon(r)$, we get

$$\begin{aligned}
&\varepsilon(r) - \frac{\pi}{4}\left(1+\sqrt{1-r^2}\right)\\
&\quad= \int_0^{\frac{\pi}{2}} \sqrt{1-r^2\sin^2\theta}\,d\theta - \frac{\pi}{4}\left(1+\sqrt{1-r^2}\right)\\
&\quad= \int_0^{\frac{\pi}{4}} \left(\sqrt{1-r^2\sin^2\theta} + \sqrt{1-r^2\cos^2\theta}\right)d\theta - \int_0^{\frac{\pi}{4}}\left(1+\sqrt{1-r^2}\right)d\theta\\
&\quad= \int_0^{\frac{\pi}{4}} \left(\sqrt{2-r^2+2\sqrt{1-r^2+r^4\sin^2\theta\cos^2\theta}} - \sqrt{2-r^2+2\sqrt{1-r^2}}\right)d\theta > 0.
\end{aligned}$$

(ii) By the definition of $\varepsilon(r)$ and $\kappa(r)$, we get

$$\begin{aligned}
&\sqrt{1-r^2}\kappa(r) + \varepsilon(r) - \frac{\pi}{2}\left(1+\sqrt{1-r^2}\right)\\
&\quad= \sqrt{1-r^2}\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-r^2\sin^2\theta}}\,d\theta + \int_0^{\frac{\pi}{2}} \sqrt{1-r^2\sin^2\theta}\,d\theta - \int_0^{\frac{\pi}{2}}\left(1+\sqrt{1-r^2}\right)d\theta\\
&\quad= -\int_0^{\frac{\pi}{2}} \frac{r^2\cos^2\theta(1-\sqrt{1-r^2\sin^2\theta})}{\sqrt{1-r^2\sin^2\theta}(\sqrt{1-r^2}+\sqrt{1-r^2\sin^2\theta})}\,d\theta < 0.
\end{aligned}$$

(iii) By the definition of $\varepsilon(r)$ and $\kappa(r)$, we get

$$\begin{aligned}
&(1-r^2)\kappa(r)-\varepsilon(r)\\
&=(1-r^2)\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-r^2\sin^2\theta}}\,d\theta-\int_0^{\frac{\pi}{2}}\sqrt{1-r^2\sin^2\theta}\,d\theta\\
&=\int_0^{\frac{\pi}{2}}\frac{r^2(\sin^2\theta-1)}{\sqrt{1-r^2\sin^2\theta}}\,d\theta<0
\end{aligned}$$

and

$$\begin{aligned}
&\left(1-\frac{r^2}{2}\right)\kappa(r)-\varepsilon(r)\\
&=\left(1-\frac{r^2}{2}\right)\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1-r^2\sin^2\theta}}\,d\theta-\int_0^{\frac{\pi}{2}}\sqrt{1-r^2\sin^2\theta}\,d\theta\\
&=\int_0^{\frac{\pi}{2}}\frac{r^2(\sin^2\theta-\frac{1}{2})}{\sqrt{1-r^2\sin^2\theta}}\,d\theta\\
&=r^2\int_0^{\frac{\pi}{4}}\left(\frac{\sin^2\theta-\frac{1}{2}}{\sqrt{1-r^2\sin^2\theta}}+\frac{\cos^2\theta-\frac{1}{2}}{\sqrt{1-r^2\cos^2\theta}}\right)d\theta\\
&=r^2\int_0^{\frac{\pi}{4}}\left(\sin^2\theta-\frac{1}{2}\right)\cdot\left(\frac{1}{\sqrt{1-r^2\sin^2\theta}}-\frac{1}{\sqrt{1-r^2\cos^2\theta}}\right)d\theta>0.
\end{aligned}$$

□

Lemma 3 *Let* $0<k_0<1$, $k_0'=\sqrt{1-k_0^2}$, $t\in(0,k_0')$, $\lambda=\frac{1-\frac{2}{\pi}\varepsilon(k_0')}{1-k_0}$ *and*

$$f(t):=\frac{\pi p}{2}\sqrt{1-t^2}+\frac{\pi}{2}(1-p)-\varepsilon(t). \tag{2.1}$$

Then

(i) $f(t)<0$ *for all* $t\in(0,k_0')$ *if and only if* $p\geq 1/2$,
(ii) $f(t)>0$ *for all* $t\in(0,k_0')$ *if and only if* $p\leq\lambda$.

Proof Firstly, we can, respectively, give the first-, second- and third-order derivatives of f as follows:

$$f'(t)=\frac{f_1(t)}{t\sqrt{1-t^2}},\qquad f_1(t)=\sqrt{1-t^2}\big[\kappa(t)-\varepsilon(t)\big]-\frac{\pi p}{2}t^2, \tag{2.2}$$

$$f_1'(t)=\frac{t[2\varepsilon(t)-\kappa(t)]}{\sqrt{1-t^2}}-\pi pt, \tag{2.3}$$

$$f_1''(t)=\frac{(3-2t^2)\varepsilon(t)-(2-t^2)\kappa(t)}{(1-t^2)^{\frac{3}{2}}}-\pi p, \tag{2.4}$$

$$f_1'''(t)=-\frac{(1+t^2)[\kappa(t)-\varepsilon(t)]+t^2\kappa(t)}{t(1-t^2)^{\frac{5}{2}}}<0, \tag{2.5}$$

for all $t\in(0,k_0')$.

Letting $t\to 0$, from (2.1)–(2.4), we have

$$f(0^+)=f_1(0^+)=f_1'(0^+)=0,\qquad f_1''(0^+)=\pi\left(\frac{1}{2}-p\right), \tag{2.6}$$

and substituting $t = k_0'$ into (2.1)–(2.4), we get

$$f(k_0') = \frac{\pi}{2}(1-k_0)(\lambda - p), \qquad f_1(k_0') = \frac{\pi k_0'^2}{2}(\lambda^* - p), \tag{2.7}$$

$$f_1'(k_0') = \pi k_0'(\lambda^{**} - p), \qquad f_1''(k_0') = \pi(\lambda^{***} - p), \tag{2.8}$$

where

$$\lambda = \frac{\pi - 2\varepsilon(k_0')}{\pi(1-k_0)}, \qquad \lambda^* = \frac{2k_0(\kappa(k_0') - \varepsilon(k_0'))}{\pi(1-k_0^2)},$$
$$\lambda^{**} = \frac{(2\varepsilon(k_0') - \kappa(k_0'))}{\pi k_0}, \qquad \lambda^{***} = \frac{(3-2k_0'^2)\varepsilon(k_0') - (2-k_0'^2)\kappa(k_0')}{\pi k_0^3}.$$

By Lemma 2, we can easily prove that

$$\lambda^{***} < \lambda^{**} < \lambda^* < \lambda < \frac{1}{2}. \tag{2.9}$$

Since $f'''(t) < 0$, $t \in (0, k_0')$, $f''(t)$ is strictly decreasing on $(0, k_0')$. Then we divide the proof into six cases in the following.

Case 1 $p \geq 1/2$. Then from (2.6) we can clearly see that

$$f_1''(0^+) \leq 0. \tag{2.10}$$

It follows from (2.10) and the monotonicity of $f_1''(t)$ that $f_1'(t)$ is strictly decreasing on $(0, k_0')$. Therefore $f(t) < 0$ for all $t \in (0, k_0')$, as follows easily from (2.2), (2.6) and the monotonicity of $f_1'(t)$.

Case 2 $\lambda < p < 1/2$. From (2.6), (2.7) and (2.8), we have

$$f(k_0') < 0, \qquad f_1(k_0') < 0, \qquad f_1'(k_0') < 0, \qquad f_1''(k_0') < 0, \tag{2.11}$$

and

$$f_1''(0^+) > 0. \tag{2.12}$$

It follows from (2.11), (2.12) and the monotonicity of $f_1''(t)$ that there exists $t_1 \in (0, k_0')$ such that $f_1'(t)$ is strictly increasing on $(0, t_1]$ and strictly decreasing on $[t_1, k_0')$. Then from (2.6), (2.11) and the piecewise monotonicity of $f_1'(t)$ we clearly see that there exists $t_2 \in (0, k_0')$ such that $f_1(t)$ is strictly increasing on $(0, t_2]$ and strictly decreasing on $[t_2, k_0')$. The piecewise monotonicity of $f_1(t)$ and $f_1(0^+) = 0$, $f_1(k_0') < 0$, show as a result that there exists $t_3 \in (0, k_0')$ such that $f(t)$ is strictly increasing on $(0, t_3]$ and strictly decreasing on $[t_3, k_0')$.

Therefore, we find that there exists $t_4 \in (0, k_0')$ such that $f(t) > 0$ on $(0, t_4]$ and $f(t) < 0$ $[t_4, k_0')$ as follows easily from $f(0^+) = 0$, $f(k_0') < 0$ and the piecewise monotonicity of $f(t)$.

Case 3 $\lambda^* < p \leq \lambda$. Then (2.7) and (2.8) lead to

$$f(k_0') \geq 0, \qquad f_1(k_0') < 0, \qquad f_1'(k_0') < 0, \qquad f_1''(k_0') < 0,$$

and

$$f_1''(0^+) > 0.$$

Similar to Case 2, the piecewise monotonicity of $f(t)$ can be proved, that is, $f(t)$ is firstly strictly increasing and then strictly decreasing on $(0, k_0')$. It follows from $f(0^+) = 0, f(k_0') \geq 0$ that $f(t) > 0$ for all $t \in (0, k_0')$.

Case 4 $\lambda^{**} < p \leq \lambda^*$. Then (2.7) and (2.8) lead to

$$f(k_0') > 0, \qquad f_1(k_0') \geq 0, \qquad f_1'(k_0') < 0, \qquad f_1''(k_0') < 0,$$

and

$$f_1''(0^+) > 0.$$

We can similarly prove that $f_1(t)$ has piecewise monotonicity, that is, $f_1(t)$ is firstly strictly increasing and then strictly decreasing on $(0, k_0')$.

It follows from $f_1(0^+) = 0, f_1(k_0') \geq 0$ and the piecewise monotonicity of $f_1(t)$ that

$$f_1(t) > 0 \tag{2.13}$$

for all $t \in (0, k_0')$. By (2.2) and (2.13), then $f(t)$ is strictly increasing on $(0, k_0')$. Therefore, we get $f(t) > 0$ for all $t \in (0, k_0')$ from $f(0^+) = 0$.

Case 5 $\lambda^{***} < p \leq \lambda^{**}$. Then (2.7) and (2.8) lead to

$$f(k_0') > 0, \qquad f_1(k_0') > 0, \qquad f_1'(k_0') \geq 0, \qquad f_1''(k_0') < 0,$$

and

$$f_1''(0^+) > 0.$$

Since $f_1''(0^+) > 0, f_1''(k_0') < 0$ and $f_1''(t)$ is strictly decreasing, we find that $f_1'(t)$ is firstly strictly increasing and then strictly decreasing on $(0, k_0')$. Then $f_1'(t) > 0$ holds for all $t \in (0, k_0')$ for $f_1'(0^+) = 0, f_1'(k_0') \geq 0$. Therefore, we get $f(t) > 0$ for all $t \in (0, k_0')$ from (2.2) and (2.6).

Case 6 $p \leq \lambda^{***}$. Then (2.7) and (2.8) lead to

$$f_1''(0^+) > 0, \qquad f_1''(k_0') \geq 0.$$

Since $f_1''(t)$ is strictly decreasing, we have $f_1''(t) > 0$ for all $t \in (0, k_0')$. So $f_1'(t), f_1(t), f(t)$ are strictly increasing from $f_1(0^+) = f_1'(0^+) = 0$. Therefore, we get $f(t) > 0$ for all $t \in (0, k_0')$ from $f(0^+) = 0$.

Therefore, we have $f(t) < 0$ for all $t \in (0, k_0')$ if and only if $p \geq 1/2$, and $f(t) > 0$ for all $t \in (0, k_0')$ if and only if $p \leq \lambda$. □

Lemma 4 *Let* $0 < k_0 < 1, r \in \mathbb{R}, a, b > 0$ *with* $k_0 < a/b < 1$, $c_0 = \frac{2}{\pi}\varepsilon(k_0')$, $k_0' = \sqrt{1 - k_0^2}$, $c_1 = k_0$, $\lambda(r)$ *and* $U(r; a, b)$ *be, respectively, defined by*

$$\lambda(r) = \frac{1 - c_0^r}{1 - c_1^r} \quad (r \neq 0), \qquad \lambda(0) = \frac{\log c_0}{\log c_1}, \tag{2.14}$$

and

$$U(r;a,b)=\left[\lambda(r)a^r+\left(1-\lambda(r)\right)b^r\right]^{\frac{1}{r}}\quad(r\neq 0),\qquad U(0;a,b)=a^{\lambda(0)}b^{1-\lambda(0)}.\tag{2.15}$$

Then the function $r\to U(r;a,b)$ is strictly decreasing on $(-\infty,\infty)$.

Proof When $0<t<1$,

$$\varepsilon\left(\sqrt{1-t^2}\right)-\frac{\pi}{2}t=\int_0^{\frac{\pi}{2}}\left(\sqrt{\cos^2\theta+t^2\sin^2\theta}-t\right)d\theta>0.$$

Hence, $0<c_1<c_0<1$.

Let $x=a/b\in(k_0,1)$, $r\neq 0$. Then

$$\log U(r;a,b)=\log b+\frac{1}{r}\log\left[\lambda(r)\left(x^r-1\right)+1\right],\tag{2.16}$$

$$\frac{\partial\log U(r;a,b)}{\partial r}=\frac{\lambda'(r)(x^r-1)+\lambda(r)x^r\log x}{r(\lambda(r)(x^r-1)+1)}-\frac{\log[\lambda(r)(x^r-1)+1]}{r^2},\tag{2.17}$$

where

$$\lambda'(r)=\frac{(c_1^r-1)c_0^r\log c_0-(c_0^r-1)c_1^r\log c_1}{(c_1^r-1)^2}.\tag{2.18}$$

Substituting $x=k_0$ and $x=1$ into (2.17), we get

$$\left.\frac{\partial\log U(r;a,b)}{\partial r}\right|_{x=k_0}=\left.\frac{\partial\log U(r;a,b)}{\partial r}\right|_{x=1}=0.\tag{2.19}$$

Now we give the derivative of (2.17) with respect to x as follows:

$$\frac{\partial^2\log U(r;a,b)}{\partial r\,\partial x}=\frac{\lambda(r)x^{r-1}}{(\lambda(r)(x^r-1)+1)^2}V(r,x),\tag{2.20}$$

where

$$V(r,x):=\left(1-\lambda(r)\right)\log x+\frac{\lambda'(r)}{\lambda(r)}.\tag{2.21}$$

Substituting $x=k_0$ and $x=1$ into (2.21), we get

$$V(r,k_0)=c_0^r\left(\frac{\log c_0}{c_0^r-1}-\frac{\log c_1}{c_1^r-1}\right),\qquad V(r,1)=\frac{\log\frac{1}{c_0}}{(\frac{1}{c_0})^r-1}-\frac{\log\frac{1}{c_1}}{(\frac{1}{c_1})^r-1}.\tag{2.22}$$

Because the function $t\mapsto\log t/(t^r-1)$ is strictly decreasing on $(0,+\infty)$ and $c_0>c_1$, we find that $V(r,k_0)<0$ and $V(r,1)>0$. Note that the function $x\mapsto V(r,x)$ is strictly increasing on $(1,k_0)$ for $\lambda(r)\in(0,1)$. There exists $x_0\in(0,1)$ such that the function $x\mapsto\partial\log U(r;a,b)/\partial r$ is strictly decreasing on $(1,x_0)$ and strictly increasing on (x_0,k_0). Therefore, we have, for all $a,b>0$ with $k_0<a/b<1$, $r\neq 0$,

$$\frac{\partial\log U(r;a,b)}{\partial r}<0.\tag{2.23}$$

Since

$$\lim_{r\to 0}\lambda(r)=\lambda(0),\qquad \lim_{r\to 0}U(r;a,b)=U(0;a,b),$$

the function $r\mapsto U(r;a,b)$ is strictly decreasing on $(-\infty,+\infty)$ from (2.23). □

3 Main result

Theorem 1 *Let* $0<k_0<1$, $k_0'=\sqrt{1-k_0^2}$ *and the functions* $M,N:((0,+\infty),(0,\infty))\mapsto(0,+\infty)$ *be two means which satisfy*

$$k_0<\frac{M(a,b)}{N(a,b)}<1,$$

for all $a,b>0$ *with* $a\neq b$. *Suppose* $c_0=2\varepsilon(k_0')/\pi$, $c_1=k_0$ *and* $\lambda(r)$ *be defined by* (2.14). *Then the double inequality*

$$\begin{aligned}\bigl[\alpha(r)M^r(a,b)+\bigl(1-\alpha(r)N^r(a,b)\bigr)\bigr]^{1/r}&<\mathrm{TD}\bigl[M(a,b),N(a,b)\bigr]\\&<\bigl[\beta(r)M^r(a,b)+\bigl(1-\beta(r)N^r(a,b)\bigr)\bigr]^{1/r}\end{aligned}\tag{3.1}$$

holds for all $r\le 1$ *and* $a,b>0$ *with* $a\neq b$ *if and only if*

$$\alpha(r)\geq 1/2,\qquad \beta(r)\leq\lambda(r),\tag{3.2}$$

where $r=0$ *is the limit value of* $r\to 0$.

Proof We first prove the case $r=1$. Let

$$t=\sqrt{1-\left(\frac{M(a,b)}{N(a,b)}\right)^2}\in\bigl(0,k_0'\bigr),$$

then we get

$$M(a,b)=N(a,b)\sqrt{1-t^2},\qquad \mathrm{TD}\bigl[M(a,b),N(a,b)\bigr]=\frac{2}{\pi}N(a,b)\varepsilon(t),$$
$$pM(a,b)+(1-p)N(a,b)-\mathrm{TD}\bigl[M(a,b),N(a,b)\bigr]=\frac{2}{\pi}N(a,b)f(t),$$

where $f(t)$ is defined as in Lemma 3.

Therefore, by Lemma 3, we get the result for the case $r=1$. So we have

$$\frac{M(a,b)+N(a,b)}{2}<\mathrm{TD}\bigl[M(a,b),N(a,b)\bigr]<\lambda(1)M(a,b)+\bigl(1-\lambda(1)\bigr)N(a,b),\tag{3.3}$$

where $\lambda(1)=\frac{1-\frac{2}{\pi}\varepsilon(k_0')}{1-k_0}:=\lambda$.

By Lemma 4 and for the function $r\mapsto[(a^r+b^r)/2]^{1/r}$ being strictly increasing, we get

$$\lambda(1)M(a,b)+\bigl(1-\lambda(1)\bigr)N(a,b)<\bigl[\lambda(r)M^r(a,b)+\bigl(1-\lambda(r)\bigr)N^r(a,b)\bigr]^{1/r}\tag{3.4}$$

and

$$\left[\frac{M^r(a,b)+N^r(a,b)}{2}\right]^{1/r} < \frac{M(a,b)+N(a,b)}{2}, \tag{3.5}$$

hold for all $r < 1$ and $a, b > 0$ with $a \neq b$.

If $\alpha(r) \geq \frac{1}{2}$ and $\beta(r) \leq \lambda(r)$, since $M(a,b) < N(a,b)$,

$$\left[\alpha(r)M^r(a,b) + \left(1-\alpha(r)N^r(a,b)\right)\right]^{1/r} \leq \left[\frac{M^r(a,b)+N^r(a,b)}{2}\right]^{1/r}$$

and

$$\left[\lambda(r)M^r(a,b) + \left(1-\lambda(r)\right)N^r(a,b)\right]^{1/r} \leq \left[\beta(r)M^r(a,b) + \left(1-\beta(r)N^r(a,b)\right)\right]^{1/r}.$$

Then we find that the double inequalities (3.1) hold from (3.3)–(3.5). Thus we prove the "if" part of our theorem.

To prove the converse implication, note that

$$\frac{\mathrm{TD}[M(a,b),N(a,b)]}{[\frac{M^r(a,b)+N^r(a,b)}{2}]^{\frac{1}{r}}} = \frac{2^{1+1/r}}{\pi}\frac{\varepsilon(t)}{[1+(1-t^2)^{r/2}]^{1/r}},$$
$$\frac{\mathrm{TD}[M(a,b),N(a,b)]}{[\lambda(r)M^r(a,b)+(1-\lambda(r))N^r(a,b)]^{\frac{1}{r}}} = \frac{2}{\pi}\frac{\varepsilon(t)}{[\lambda(r)(1-t^2)^{r/2}+1-\lambda(r)]^{1/r}}$$

and

$$\lim_{t\to 0^+}\frac{2^{1+1/r}}{\pi}\frac{\varepsilon(t)}{[1+(1-t^2)^{r/2}]^{1/r}} = \lim_{t\to k_0'}\frac{2}{\pi}\frac{\varepsilon(t)}{[\lambda(r)(1-t^2)^{r/2}+1-\lambda(r)]^{1/r}} = 1,$$

which imply that the bonds for $\alpha(r)$ and $\beta(r)$ given by (3.2) are optimal. This completes the proof. □

Remark 1 Using the symmetry of the Toader mean, we get the result for the case $M(a,b) > N(a,b)$.

4 Some examples

Example 1 Let $c_0 = 2\varepsilon(\sqrt{3}/2)/\pi = 0.770\cdots$, $c_1 = 1/2$ and $\lambda(r)$ be defined by (2.14). Then the double inequality

$$\begin{aligned}[\alpha(r)A^r(a,b) + (1-\alpha(r)C^r(a,b))]^{1/r} &< \mathrm{TD}\left[A(a,b),C(a,b)\right] \\ &< \left[\beta(r)A^r(a,b) + \left(1-\beta(r)C^r(a,b)\right)\right]^{1/r}\end{aligned}$$

holds for all $r \leq 1$ and $a, b > 0$ with $a \neq b$ if and only if $\alpha(r) \geq 1/2$ and $\beta(r) \leq \lambda(r)$, where $r = 0$ is the limit value of $r \to 0$.

Proof Since

$$\frac{1}{2} < \frac{A(a,b)}{C(a,b)} < 1,$$

letting $k_0 = 1/2$, we have $k_0' = \sqrt{3}/2$ and $c_0 = 2\varepsilon(\sqrt{3}/2)/\pi = 0.770\cdots$, $c_1 = 1/2$. By Theorem 1, we get the result. □

Remark 2 (i) Let $r = 1$, Theorem 1 leads that the double inequality

$$\alpha(1)A(x,y) + \bigl(1-\alpha(1)\bigr)C(x,y) < \mathrm{TD}\bigl(A(x,y),C(x,y)\bigr) < \beta(1)A(x,y) + \bigl(1-\beta(1)\bigr)C(x,y)$$

holds if and only if

$$\alpha(1) \geq 1/2, \beta(1) \leq \lambda(1) = 2 - \frac{4}{\pi}\varepsilon(\sqrt{3}/2).$$

It follows from Lemma 1 (iii) that

$$\varepsilon(\sqrt{3}/2) = \frac{3}{2}\varepsilon(1/3) - \frac{2}{3}\kappa(1/3),$$

then

$$\lambda(1) = 2 - \frac{4}{\pi}\varepsilon(\sqrt{3}/2) = 2\left[1 - \frac{3}{\pi}\varepsilon(1/3) + \frac{4}{3\pi}\kappa(1/3)\right].$$

Therefore, the result agrees well with Theorem 3.1 in [11].

(ii) Letting $r = -1$, Theorem 1 shows that the double inequality

$$\frac{\alpha(-1)}{A(x,y)} + \frac{1-\alpha(-1)}{C(x,y)} < \frac{1}{\mathrm{TD}(A(x,y),C(x,y))} < \frac{\beta(-1)}{A(x,y)} + \frac{1-\beta(-1)}{C(x,y)}$$

holds if and only if

$$\alpha(-1) \geq 1/2, \beta(-1) \leq \lambda(-1) = \frac{\pi - 2\varepsilon(\sqrt{3}/2)}{2\varepsilon(\sqrt{3}/2)}.$$

Since

$$\varepsilon(\sqrt{3}/2) = \frac{3}{2}\varepsilon(1/3) - \frac{2}{3}\kappa(1/3),$$

we have

$$\lambda(-1) = \frac{\pi - 2\varepsilon(\sqrt{3}/2)}{2\varepsilon(\sqrt{3}/2)} = \frac{3\pi - 9\varepsilon(1/3) + 4\kappa(1/3)}{9\varepsilon(1/3) - 4\kappa(1/3)}.$$

Therefore, the result agrees well with Theorem 3.3 in [11].

Example 2 Let $c_0 = 2\varepsilon(\sqrt{7}/4)/\pi = 0.879\cdots$, $c_1 = 3/4$, $\lambda(r)$ is defined by (2.14). Then the double inequality

$$\begin{aligned}[\alpha(r)A^r(a,b) + (1-\alpha(r)\overline{C}^r(a,b))]^{1/r} &< \mathrm{TD}\bigl[A(a,b),\overline{C}(a,b)\bigr] \\ &< \bigl[\beta(r)A^r(a,b) + \bigl(1-\beta(r)\overline{C}^r(a,b)\bigr)\bigr]^{1/r}\end{aligned}$$

holds for all $r \leq 1$ and $a,b > 0$, $a \neq b$ if and only if $\alpha_2(r) \geq 1/2$ and $\beta_2(r) \leq \lambda(r)$, where $r = 0$ is the limit value of $r \to 0$.

Proof Since

$$\frac{3}{4} < \frac{A(a,b)}{\overline{C}(a,b)} < 1,$$

letting $k_0 = 3/4$, we have $k_0' = \sqrt{7}/4$ and $c_0 = 2\varepsilon(\sqrt{7}/4)/\pi = 0.879\cdots$, $c_1 = 3/4$. Using Theorem 1, we prove the result. □

Example 3 Let $c_0 = 2\varepsilon(\sqrt{2}/2)/\pi = 0.859\cdots$, $c_1 = \sqrt{2}/2$, $\lambda(r)$ is defined by (2.14). Then the double inequality

$$\begin{aligned}[\alpha(r)A^r(a,b) + (1-\alpha(r)Q^r(a,b))]^{1/r} &< \mathrm{TD}\bigl[A(a,b),Q(a,b)\bigr] \\ &< \bigl[\beta_2(r)A^r(a,b) + \bigl(1-\beta_2(r)Q^r(a,b)\bigr)\bigr]^{1/r}\end{aligned}$$

holds for all $r \leq 1$ and $a,b > 0$, $a \neq b$ if and only if $\alpha(r) \geq 1/2$ and $\beta(r) \leq \lambda(r)$, where $r = 0$ is the limit value of $r \to 0$.

Proof Since

$$\frac{\sqrt{2}}{2} < \frac{A(a,b)}{Q(a,b)} < 1,$$

letting $k_0 = \sqrt{2}/2$, we have $k_0' = \sqrt{2}/2$ and $c_0 = 2\varepsilon(\sqrt{2}/2)/\pi = 0.859\cdots$, $c_1 = \sqrt{2}/2$. Using Theorem 1, we prove the result. □

Remark 3 The same result can be found in [24].

From the case $r = 1$ of Examples 1–3, we get the following results.

Corollary 1

(1) *Let* $\lambda_1 = 2 - 4\varepsilon(\sqrt{3}/2)/\pi = 0.458\cdots$. *Then, for all* $t \in (0, \sqrt{3}/2)$, *the double inequality*

$$\frac{\pi}{4}\sqrt{1-t^2} + \frac{\pi}{4} < \varepsilon(t) < \frac{\pi}{2}\lambda_1\sqrt{1-t^2} + \frac{\pi}{2}(1-\lambda_1) \tag{4.1}$$

holds.

(2) *Let* $\lambda_2 = 4 - 8\varepsilon(\sqrt{7}/4)/\pi = 0.482\cdots$. *Then, for all* $t \in (0, \sqrt{7}/4)$, *the double inequality*

$$\frac{\pi}{4}\sqrt{1-t^2} + \frac{\pi}{4} < \varepsilon(t) < \frac{\pi}{2}\lambda_2\sqrt{1-t^2} + \frac{\pi}{2}(1-\lambda_2) \tag{4.2}$$

holds.

(3) *Let* $\lambda_3 = (2+\sqrt{2})[1 - 2\varepsilon(\sqrt{2}/2)/\pi] = 0.478\cdots$. *Then, for all* $t \in (0, \sqrt{2}/2)$, *the double inequality*

$$\frac{\pi}{4}\sqrt{1-t^2} + \frac{\pi}{4} < \varepsilon(t) < \frac{\pi}{2}\lambda_3\sqrt{1-t^2} + \frac{\pi}{2}(1-\lambda_3) \tag{4.3}$$

holds.

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