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On statistical convergence and strong Cesàro convergence by moduli

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Abstract

In this paper we will establish a result by Connor, Khan and Orhan (Analysis 8:47–63, 1988; Publ. Math. (Debr.) 76:77–88, 2010) in the framework of the statistical convergence and the strong Cesàro convergence defined by a modulus function f. Namely, for every modulus function f, we will prove that a f-strongly Cesàro convergent sequence is always f-statistically convergent and uniformly integrable. The converse of this result is not true even for bounded sequences. We will characterize analytically the modulus functions f for which the converse is true. We will prove that these modulus functions are those for which the statistically convergent sequences are f-statistically convergent, that is, we show that Connor–Khan–Orhan's result is sharp in this sense.

Keywords: Statistical convergence; Strong Cesàro convergence; Modulus function; Uniformly bounded sequence

1 Introduction

A sequence (x_k) on a normed space $(X, \|\cdot\|)$ is said to be strongly Cesàro convergent to L if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \|x_k - L\| = 0.$$

The strong Cesàro convergence for real numbers was introduced by Hardy–Littlewood [14] and Fekete [12] in connection with the convergence of Fourier series (see [35], for historical notes, and the most recent monograph [5]).

A sequence (x_n) is statistically convergent to L if for any $\varepsilon > 0$ the subset $\{k : |x_k - L| < \varepsilon\}$ has density 1 on the natural numbers. The term statistical convergence was first presented by Fast [11] and Steinhaus [34] independently in the same year 1951. Actually, a root of the notion of statistical convergence can be detected in [36], where he used the term almost convergence which turned out to be equivalent to the concept of statistical convergence.

Both concepts were developed independently and surprisingly enough, both are related thanks to a result by Connor ([6]) which was sharpened by Khan and Orhan ([15]). Among other results, Khan and Orhan show that a sequence is strongly Cesàro convergent if and only if it is statistically convergent and uniformly integrable. In this circle of ideas, a significant number of deep and beautiful results have been obtained by Connor, Fridy, Khan, Mursaleen, Orhan... and many others (see [2, 9, 13, 26, 27, 31–33]). Moreover, the convergence methods are an active research area with important applications (see the recent monograph by Mursaleen [24]). For instance, there are applications in many fields, such as approximation theory [3, 10, 22, 23, 25, 28]. On the other hand, from the point of view of infinite dimensional spaces, there are interesting results that characterize properties of normed spaces by means of some convergence types (see for instance [8, 16–18]).

Let us recall that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a modulus function if it satisfies:

(1) $f(x) = 0$ if and only if $x = 0$.

(2) $f(x+y) \leq f(x) + f(y)$ for every $x, y \in \mathbb{R}^+$.

(3) f is increasing.

(4) f is continuous from the right at 0.

In [19, 29, 30] the authors extended the notion of strong Cesàro convergence with respect to a modulus function, and in [1] it was introduced the concept of f-statistical convergence in which underlies a new concept of f-density of subsets of natural numbers (where f is a modulus function). A modulus function f was used by Maddox ([20]) to obtain a representation of statistical convergence in terms of strong summability. Later, it was used by Connor ([7]) to study the concepts of strong matrix summability with respect to a modulus. In this paper we will consider only unbounded modulus functions, since the bounded case is reduced only to trivial examples.

In [4] the authors studied the relationship between the f-statistical convergence and other Cesàro convergence types defined with respect to a modulus f. It has been observed that there is not enough structure to establish Connor–Khan–Orhan's result in any direction. The aim of this paper is to establish, for the f-statistical convergence and a suitable f-strong Cesàro convergence that will be introduced in Sect. 2, equivalences analogous to those obtained in Connor–Khan–Orhan's result.

The notion of f-strong Cesàro convergence that we introduce is very handy to use, and it fits as a glove to the f-statistical convergence. In fact, we will prove that if a sequence (x_n) is f-strongly Cesàro convergent to L then (x_n) is f-statistically convergent to L and it is uniformly integrable.

However, the converse of the above result, is not always true, even for bounded sequences. Thus, the following questions arises:

> For which modulus functions f it is possible to obtain the converse of Connor–Khan–Orhan's result. That is, for which modulus functions do we find that all uniformly integrable and f-statistically convergent sequences are f-strongly Cesàro convergent?

We answer the above question by characterizing analytically such modulus functions, which will be called *compatible modulus functions*. And surprisingly, we can show that such compatible modulus functions are those for which all statistically convergent sequences are f-statistically convergent, that is, in some sense Connor–Khan–Orhan's result is quite sharp. The paper concludes with a brief section devoted to related issues, references and open questions.

2 Preliminary results

Let X be a normed space. A sequence $(x_n) \subset X$ is said to be uniformly integrable if

$$\lim_{c\to\infty} \sup_n \frac{1}{n} \sum_{\substack{k=1 \\ \|x_k\| \geq c}}^{n} \|x_k\| = 0.$$

If a sequence (x_n) is uniformly integrable then $(x_n - L)$ is also uniformly integrable for every $L \in X$. If we consider $L^1_\mu[0,1]$ where μ is the Lebesgue measure, a sequence (x_n) is uniformly integrable if and only if the set of simple functions

$$g_n(s) = \sum_{k=0}^{n} \|x_{k+1}\| \chi_{[\frac{k}{n}, \frac{k+1}{n})}(s)$$

is uniformly integrable in $L^1_\mu[0,1]$ (here $\chi_A(\cdot)$ denotes the characteristic function of A). This measure theoretic approach was used by Khan and Orhan in [15], providing an answer to a problem posed by Connor [7] in the A-statistical-convergence setting and to another open question posed by Miller ([21]).

Next we define f-strong Cesàro convergence:

Definition 2.1 Let f be a modulus function. A sequence (x_n) is said to be f-strongly Cesàro convergent to L if

$$\lim_{n\to\infty} \frac{f(\sum_{k=1}^{n} \|x_k - L\|)}{f(n)} = 0.$$

Let us observe that if f is bounded, then the constant sequence $x_n = L$ is the only sequence which is f-strongly Cesàro convergent to L. Indeed, if for some k, $\|x_k - L\| = c > 0$ then

$$\frac{f(c)}{\|f\|_\infty} \leq \frac{f(\sum_{k=1}^{n} \|x_k - L\|)}{f(n)},$$

which gives the desired result.

In [1], by means of a new concept of density of a subset of $\mathbb{N}$, it is defined the following non-matrix concept of convergence.

Definition 2.2 A sequence (x_n) is said to be f-statistically convergent to L if for every $\varepsilon > 0$.

$$\lim_{n\to\infty} \frac{f(\#\{k \leq n : \|x_k - L\| > \varepsilon\})}{f(n)} = 0.$$

Analogously, if f is bounded, the only sequences (x_n) that converge f-statistically are the constant sequences. Thus, in what follows we will suppose that f is an unbounded modulus function.

Let us observe that if f is a modulus function, for all $x \in \mathbb{R}^+$ and $m \in \mathbb{N}$ we have $f(\frac{x}{m}) \geq \frac{1}{m} f(x)$. Indeed, $f(x) = f(\frac{1}{m} m x) \leq m f(\frac{x}{m})$. As a consequence, it was pointed out in [1] that if x_n is f-statistically convergent to L, then (x_n) is statistically convergent to L. A similar result remains true for the f-strong Cesàro convergence.

Proposition 2.3 *Let f be a modulus function. If (x_n) is f-strongly Cesàro convergent to L, then (x_n) is strongly Cesàro convergent to L.*

Proof Indeed, for all $p \geq 1$, there exists n_0, such that

$$\frac{f(\sum_{k=1}^{n} \|x_k - L\|)}{f(n)} \leq \frac{1}{p}$$

for all $n \geq n_0$. That is,

$$f\left(\sum_{k=1}^{n} \|x_k - L\|\right) \leq \frac{1}{p} f(n) \leq f\left(\frac{n}{p}\right)$$

and since f is increasing, we have

$$\sum_{k=1}^{n} \|x_k - L\| \leq \frac{n}{p}$$

for all $n \geq n_0$, that is, (x_n) is strongly Cesàro convergent to L. □

However, the converse of the above statement is not true, as it is shown in the following example.

Example 2.4 Let us consider the modulus function $f(x) = \log(x+1)$ and the sequence (x_k) defined as:

$$x_k = \begin{cases} 1, & k = n^2, \\ 0, & k \neq n^2. \end{cases}$$

Then (x_k) is strongly Cesàro convergent to 0, indeed

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} \|x_k\|}{n} \leq \lim_{n\to\infty} \frac{\sqrt{n}}{n} = 0.$$

However, (x_n) is not f-strongly Cesàro convergent to 0:

$$\lim_{n\to\infty} \frac{\log(\sum_{k=1}^{n^2} \|x_k\|)}{\log(n^2)} = \frac{1}{2}.$$

Definition 2.5 A modulus function f is said to be compatible if for any $\varepsilon > 0$ there exist $\varepsilon' > 0$ and $n_0(\varepsilon) \in \mathbb{N}$ such that $\frac{f(n\varepsilon')}{f(n)} < \varepsilon$ for all $n \geq n_0$.

Example 2.6 The functions $f(x) = x^p + x^q$, $0 < p, q \leq 1$, $f(x) = x^p + \log(x+1)$, $f(x) = x + \frac{x}{x+1}$ are modulus functions which are compatible. And $f(x) = \log(x+1)$, $f(x) = W(x)$ where W is the W-Lambert function restricted to $\mathbb{R}^+$ (that is, the inverse of xe^x) are modulus functions which are not compatible. Indeed, let us show that $f(x) = x + \log(x+1)$ is compatible.

$$\lim_{n\to\infty} \frac{f(n\varepsilon')}{f(n)} = \lim_{n\to\infty} \frac{n\varepsilon' + \log(1 + n\varepsilon')}{n + \log(n+1)} = \varepsilon'.$$

On the other hand if $f(x) = \log(x+1)$, since

$$\lim_{n\to\infty} \frac{\log(1+n\varepsilon')}{\log(1+n)} = 1$$

we find that $f(x) = \log(x+1)$ is not compatible.

Proposition 2.7 *Let f be a compatible modulus function. If (x_n) is statistically convergent to L then (x_n) is f-statistically convergent to L.*

Proof Let us fix $\varepsilon > 0$ arbitrarily small. Since f is compatible, there exist $\varepsilon' > 0$ and $n_0(\varepsilon)$ such that if $n \geq n_0$ then

$$\frac{f(n\varepsilon')}{f(n)} < \varepsilon.$$

Let $c > 0$ and let us fix $\varepsilon' > 0$. Since (x_n) is statistically convergent to L then there exists $m_0(\varepsilon)$ (since ε' depends on ε we find that m_0 depends actually on ε) such that if $n \geq m_0$

$$\#\bigl\{k \leq n : \|x_k - L\| > c\bigr\} \leq n\varepsilon'.$$

Since f is increasing we have

$$\frac{f(\#\{k \leq n : \|x_k - L\| > c\})}{f(n)} \leq \frac{f(n\varepsilon')}{f(n)} < \varepsilon,$$

for all $n \geq \max\{m_0, n_0\}$, which gives the desired result. □

For f-strong Cesàro convergence, we obtain a similar result.

Proposition 2.8 *Let f be a compatible modulus function. Then, if (x_n) is strongly Cesàro convergent to L then (x_n) is f-strongly Cesàro convergent to L.*

Proof Let us suppose that (x_n) is strongly Cesàro convergent to L. Then for any $\varepsilon' > 0$ there exists $n \geq n_0$ such that

$$\sum_{k=1}^{n} \|x_k - L\| \leq n\varepsilon'$$

and since f is increasing, then

$$f\left(\sum_{k=1}^{n} \|x_k - L\|\right) \leq f\bigl(n\varepsilon'\bigr);$$

thus

$$\frac{f(\sum_{k=1}^{n} \|x_k - L\|)}{f(n)} \leq \frac{f(n\varepsilon')}{f(n)}$$

then by applying the same argument as above we get the desired result. □

Proposition 2.9 *Let f be a modulus function.*

(1) *If all statistically convergent sequences are f-statistically convergent, then f must be compatible.*
(2) *If all strongly Cesàro convergent sequences are f-strongly Cesàro convergent, then f must be compatible.*

Proof Let ε_n be a decreasing sequence converging to 0. Since f is not compatible, there exists $c > 0$ such that, for each k, there exists m_k such that $f(m_k\varepsilon_k) > cf(m_k)$. Moreover, we can select m_k inductively satisfying

$$1 - \varepsilon_{k+1} - \frac{1}{m_{k+1}} > \frac{(1-\varepsilon_k)m_k}{m_{k+1}}. \tag{1}$$

Now we use an standard argument used to construct subsets with prescribed densities. Let us denote $\lfloor x \rfloor$ the integer part of $x \in \mathbb{R}$. Set $n_k = \lfloor m_k\varepsilon_k \rfloor + 1$. And extracting a subsequence if it is necessary, we can assume that $n_1 < n_2 < \cdots$, $m_1 < m_2 < \cdots$. Thus, set $A_k = [m_{k+1} - (n_{k+1} - n_k)] \cap \mathbb{N}$. Condition (1) guarantee that $A_k \subset [m_k, m_{k+1}]$.

Let us denote $A = \bigcup_k A_k$, and $x_n = \chi_A(n)$. Let us prove that x_n is statistical convergent to 0, but not f-statistical convergent, a contradiction. Indeed, for any m, there exists k such that $m_k < m \leq m_{k+1}$. Moreover, we can suppose without loss that $m \in A$, that is, $m_{k+1} - n_{k+1} + n_k \leq m$. Thus for any $\varepsilon > 0$:

$$\begin{aligned}\frac{\#\{l \leq m : |x_l| > \varepsilon\}}{m} &\leq \frac{\#\{l \leq m_k : |x_k| > \varepsilon\}}{m_k} + \frac{n_{k+1} - n_k}{m_{k+1} - n_{k+1} + n_k} \\ &\leq \frac{n_k}{m_k} + \frac{1}{\frac{m_{k+1}}{n_{k+1}-n_k} - 1} \to 0\end{aligned}$$

as $k \to \infty$. On the other hand, since $\varepsilon_{k+1} < \frac{n_{k+1}}{m_{k+1}}$

$$\frac{f(\#\{n < m_{k+1} : |x_n| > 1/2\})}{f(m_{k+1})} = \frac{f(n_{k+1})}{f(m_{k+1})} \geq \frac{f(m_{k+1}\varepsilon_{k+1})}{f(m_{k+1})} > c,$$

which yields (a) as promised. The part (b) is same proof. Indeed, for the sequence (x_n) defined in part (a), we have that $\frac{f(\sum_{k=1}^{n} |x_n|)}{f(n)} = \frac{f(\{k \leq n | x_k | > \varepsilon\})}{f(n)}$. □

3 Main results

Let us recall Connor–Khan–Orhan's result.

Theorem 3.1 (Connor–Khan–Orhan [6, 15]) *A sequence is strongly Cesàro convergent to L if and only if it is statistically convergent to L and uniformly integrable.*

This result connects two concepts that were introduced historically in different times and by different authors. Sometimes, it is easier to verify that a sequence is strongly convergent than to verify that it is statistically convergent. Conversely, if our sequence is uniformly integrable, we do not know if it has limit and we want to check that the sequence is strongly Cesàro convergent, it is usually simpler to check that the sequence is statistically Cauchy. This great advantage so useful pushes us to know what happens in the f-statistical-convergence setting.

Theorem 3.2 *Let (x_n) be a sequence, then if (x_n) is f-strongly Cesàro convergent to L then (x_n) is f-statistically convergent to L and (x_n) is uniformly integrable.*

Proof In order to prove that (x_n) is f-statistically convergent to L, it is sufficient to show that, for all $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{f(\#\{k \le n : \|x_k - L\| > \frac{1}{m}\})}{f(n)} = 0. \tag{2}$$

Indeed, let $\varepsilon > 0$ and let us consider m such that $\frac{1}{m+1} \le \varepsilon \le \frac{1}{m}$. Then we get

$$\#\{k \le n : \|x_k - L\| > \varepsilon\} \le \#\left\{k \le n : \|x_k - L\| > \frac{1}{m+1}\right\},$$

therefore, since f is increasing

$$\lim_{n\to\infty} \frac{f(\#\{k \le n : \|x_k - L\| > \varepsilon\})}{f(n)} \le \lim_{n\to\infty} \frac{f(\#\{k \le n : \|x_k - L\| > \frac{1}{m+1}\})}{f(n)}$$

thus taking limits on n we obtain what we desired.

Therefore, let $m \in \mathbb{N}$, and let us show Eq. (2). We have

$$f\left(\sum_{k=1}^{n} \|x_k - L\|\right) \ge f\left(\sum_{\substack{k=1\\ \|x_k-L\|\ge\frac{1}{m}}}^{n} \|x_k - L\|\right) \tag{3}$$

$$\ge f\left(\sum_{\substack{k=1\\ \|x_k-L\|\ge\frac{1}{m}}}^{n} \frac{1}{m}\right) \ge \frac{1}{m} f\left(\sum_{\substack{k=1\\ \|x_k-L\|\ge\frac{1}{m}}}^{n} 1\right) \tag{4}$$

$$= \frac{1}{m} f\left(\#\left\{k \le n : \|x - L\| > \frac{1}{m}\right\}\right). \tag{5}$$

Since (x_n) is f-strongly Cesàro convergent to L, we have

$$\lim_{n\to\infty} \frac{f(\sum_{k=1}^{n} \|x_k - L\|)}{f(n)} = 0,$$

therefore dividing Eq. (3) by $f(n)$, and taking the limit as $n \to \infty$ we obtain for each $m \in \mathbb{N}$

$$\lim_{n\to\infty} \frac{f(\#\{k \le n : \|x_k - L\| > \frac{1}{m}\})}{f(n)} = 0,$$

which implies that (x_n) is f-statistically convergent to L.

On the other hand, since (x_n) is f-strongly Cesàro convergent to L then by applying Proposition 2.3 and Connor–Khan–Orhan's result, we find that (x_n) is uniformly integrable as we desired. □

Let us denote by $c_0(X)$ the sequences on X which are convergent to 0, and $\ell_1(X)$ the sequences $(x_n) \subseteq X$ such that $\sum_n \|x_n\| < \infty$. From the theorem above we deduce that if $(x_n) \subseteq X$, and

(1) for every f modulus (x_n) is f-strongly Cesáro convergent to L

then

(2) for every f modulus (x_n) is f-statistically convergent to L.

Moreover in [1] it was proved that the statement (2) is equivalent to $(x_n - L) \in c_0(X)$, i.e., the sequence converges to zero. Analogously we have the following.

Theorem 3.3 *A sequence* $(x_n) \subseteq X$ *satisfies* (1) *if and only if the sequence* $(x_n - L)$ *belongs to* $\ell_1(X)$.

Proof It is trivial to see that if $\sum_{n\in\mathbb{N}} \|x_n - L\| < +\infty$ then for every f modulus (x_n) is f-strongly Cesáro convergent to L.

Conversely, let us suppose that for every f modulus the sequence (x_n) is f-strongly Cesáro convergent to L but $(x_n - L) \notin \ell_1(X)$.

We consider the set of natural numbers A defined by

$$\#\{i \in A : i \leq n\} = \min\left\{n, \left\lfloor \sum_{k=1}^{n} \|x_n - L\| \right\rfloor\right\},$$

(where $\lfloor x \rfloor$ means the greatest integer smaller than x) it is clear that A is infinite, so using Lemma 3.4 in [1] there exists g an unbounded modulus such that

$$\lim_{n\to\infty} \frac{g(\#\{i \in A : i \leq n\})}{g(n)} = 1,$$

but this is a contradiction. □

Theorem 3.4 *Let us suppose that f is a compatible modulus function and (x_n) is a uniformly integrable sequence. Then if (x_n) is f-statistically convergent to L then (x_n) is f-strongly Cesàro convergent to L. Moreover, if f is a modulus such that all uniformly integrable and f-statistically convergent sequences (x_n) are f-strongly Cesàro convergent, then the modulus f must be compatible.*

Proof Let (x_n) be a bounded sequence such that (x_n) is f-statistically convergent to L and uniformly integrable.

Let us consider $\varepsilon > 0$. Since f is compatible there exists $\varepsilon' > 0$ such that

$$\frac{f(n\varepsilon')}{f(n)} < \frac{\varepsilon}{3} \tag{6}$$

for all $n \geq n_0(\varepsilon)$.

Now, since (x_n) is uniformly integrable, there exists a natural number $M > 0$ large enough satisfying $\frac{1}{M} < \varepsilon'$ and for all $n \in \mathbb{N}$

$$\frac{1}{n} \sum_{\substack{k=1 \\ \|x_k - L\| \geq M}}^{n} \|x_k - L\| < \varepsilon'. \tag{7}$$

And since (x_n) is f-statistically convergent to L, there exists a natural number, which we abusively denote by $n_0(\varepsilon)$, such that for all $n \geq n_0(\varepsilon)$

$$\frac{1}{f(n)} f\bigl(\#\bigl\{k : \|x_k - L\| > \varepsilon'\bigr\}\bigr) < \frac{\varepsilon}{3M}. \tag{8}$$

Therefore

$$\begin{aligned}\frac{f(\sum_{k=1}^{n} \|x_k - L\|)}{f(n)} \leq{}& \frac{1}{f(n)} f\Biggl(\sum_{\substack{k=1\\ M>\|x_k-L\|\geq\varepsilon'}}^{n} \|x_k - L\| \Biggr) \\ &+ \frac{1}{f(n)} f\Biggl(\sum_{\substack{k=1\\ \|x_k-L\|\geq M}}^{n} \|x_k - L\| \Biggr) \\ &+ \frac{1}{f(n)} f\Biggl(\sum_{\substack{k=1\\ \|x_k-L\|<\varepsilon'}}^{n} \|x_k - L\| \Biggr). \end{aligned}\tag{9}$$

Since f is increasing, according to (8) we find that for all $n \geq n_0(\varepsilon)$ the first term of (9) is

$$\begin{aligned}\frac{1}{f(n)} f\Biggl(\sum_{\substack{k=1\\ M>\|x_k-L\|\geq\varepsilon'}}^{n} \|x_k - L\| \Biggr) &< \frac{f(\#\{k \leq n : \|x_k - L\| > \varepsilon'\} \cdot M)}{f(n)} \\ &\leq M \frac{1}{f(n)} f\bigl(\#\bigl\{k \leq n : \|x_k - L\| > \varepsilon'\bigr\}\bigr) \\ &< M \frac{\varepsilon}{3M} = \frac{\varepsilon}{3}. \end{aligned}\tag{10}$$

On the other hand, let us estimate the second summand of the inequality (9). Using that f is increasing and by applying firstly the inequality (7) and later inequality (6) we have for $n \geq n_0(\varepsilon)$

$$\begin{aligned}\frac{1}{f(n)} f\Biggl(\sum_{\substack{k=1\\ \|x_k-L\|\geq M}}^{n} \|x_k - L\| \Biggr) &= \frac{1}{f(n)} f\Biggl(n\frac{1}{n} \sum_{\substack{k=1\\ \|x_k-L\|\geq M}}^{n} \|x_k - L\| \Biggr) \\ &\leq \frac{1}{f(n)} f\bigl(n\varepsilon'\bigr) \leq \frac{\varepsilon}{3}. \end{aligned}\tag{11}$$

Finally, for the third summand in (9) by applying inequality (6) we obtain if $n \geq n_0(\varepsilon)$

$$\frac{1}{f(n)} f\Biggl(\sum_{\substack{k=1\\ \|x_k-L\|\leq\varepsilon'}}^{n} \|x_k - L\| \Biggr) \leq \frac{1}{f(n)} f\biggl(n\frac{1}{M} \biggr) < \frac{\varepsilon}{3}. \tag{12}$$

Thus, by using inequalities (10), (11), and (12) into inequality (9) we obtain if $n \geq n_0(\varepsilon)$

$$\frac{f(\sum_{k=1}^{n} \|x_k - L\|)}{f(n)} \leq \varepsilon,$$

that is, (x_n) is f-strongly Cesàro convergent to L as we desired. Assume that f is not compatible. Thus, as in the proof in Proposition 2.9 we can construct sequences (ε_k), (m_k) such that $f(m_k\varepsilon_k) \geq cf(m_k)$ for some $c > 0$. Moreover, we can construct (m_k) inductively, such that the sequence

$$r_k = \frac{m_{k+1}\varepsilon_{k+1} - m_k\varepsilon_k}{m_{k+1} - m_k}$$

is decreasing and converging to 0. Let us consider $x_n = \sum_{k=0}^{\infty} r_{k+1}\chi_{(m_k,m_{k+1}]}(n)$. Since (x_n) is decreasing, (x_n) if f-statistically convergent to 0. On the other hand $f(\sum_{l=1}^{m_k} |x_l|) = f(m_k\varepsilon_k) \geq cf(m_k)$, which gives that (x_n) is not f-strong Cesàro convergent, as we desired. □

Remark 3.5 Let us observe that uniform integrability in Theorem 3.4 is necessary. Set $n_j = j^2$ and let us define

$$x_k = \begin{cases} 0, & k \neq j^2 \text{ for all } j, \\ j^2, & k = j^2 \text{ for some } j. \end{cases}$$

The sequence (x_k) is not uniformly integrable, it is statistically convergent to 0, and it is not strongly Cesàro summable.

Remark 3.6 Let us observe that the first part of Theorem 3.4 can be obtained directly by using several results. Namely, the converse of Proposition 2.7, Connor–Khan–Orhan's result and Proposition 2.8. Indeed, if (x_n) is f-statistically convergent, then (x_n) is statistically convergent. By Connor–Khan–Orhan's result, since (x_n) is uniformly bounded, we find that (x_n) is strongly Cesàro convergent. And finally, since f is a compatible modulus, we find that (x_n) is f-strongly Cesàro convergent.

4 Concluding remarks and open questions

Given a modulus function f, if $A \subset \mathbb{N}$, the f-density of A; $d_f(A)$ is defined by

$$d_f(A) = \lim_{n\to\infty} \frac{f(\#\{k \leq n : k \in A\})}{f(n)}$$

whenever this limit exists. When $f(x) = x$ then we have the usual density of natural numbers. It is well known that if $d_f(A) = 0$ then $d(A) = 0$; the converse in general is not true. Using Proposition 2.7 we can get the following result.

Corollary 4.1 *If f is an unbounded modulus, the following conditions are equivalent*:

(1) *For every $(x_n) \subset X$ and $x \in X$, if (x_n) is statistically convergent to x then it is also f-statistically convergent to the same x.*
(2) *For every $A \subset \mathbb{N}$, if $d(A) = 0$ then $d_f(A) = 0$.*
(3) *f is compatible.*

While the usual density works well with complements, that is, $d(\mathbb{N} \setminus A) = 1 - d(A)$, however, this property fails for f-density. The importance of this property is that it allows us to define statistically convergence by looking at the complement. For this reason, it will be interesting to characterize the modulus function f for which the following property is satisfied: For any $A \subset N$ if $d_f(\mathbb{N} \setminus A) = 1$ then $d_f(A) = 0$. As was pointed out in [1] if $f(x) = \log(1 + x)$ then the last property is not satisfied. Is it possible to find a counterexample for a compatible module function?

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally and significantly in writing this paper. All the authors read and approved the final manuscript.

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