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m-Isometric block Toeplitz operators

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Abstract

In this paper, we study m-isometric block Toeplitz operators with trigonometric symbols. In addition, we give a necessary and sufficient condition for block Toeplitz operators with trigonometric polynomial symbols to be m-contractive.

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1 Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. In the 1990s, Agler and Stankus [1] studied the following operator. For an operator $T \in \mathcal{L}(\mathcal{H})$ and a positive integer m, define

$$B^m(T) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} T^j.$$

We say that T is m-contractive (respectively, m-expansive and m-isometric) if $B^m(T) \geq 0$ (respectively, $B^m(T) \leq 0$ and $B^m(T) = 0$) for some positive integer m.

The m-isometric operators have been widely investigated in recent years. The theory of m-isometric operators was investigated especially by Agler and Stankus [1–3]. Agler and Stankus [1–3] developed a rich theory of m-isometric operators and highlighted its connections to Toeplitz operators and function theory. Agler [4] illustrated the connection between m-isometric operators and the classical disconjugacy theory. In recent years, there have been studies on products of m-isometric operators [5], and m-isometric composition operators [6]. In [7], the authors characterized m-isometric Toeplitz operators by properties of the rational symbols and gave some results for m-expansive and m-contractive Toeplitz operators with trigonometric polynomial symbols.

Let $L^2 \equiv L^2(\mathbb{T})$ be the set of square integrable measurable functions on $\mathbb{T}$ and $H^2 \equiv H^2(\mathbb{T})$ be the corresponding Hardy space. Let $L^\infty \equiv L^\infty(\mathbb{T})$ be the set of bounded measurable functions on $\mathbb{T}$ and let $H^\infty \equiv H^\infty(\mathbb{T}) := L^\infty \cap H^2$. We introduce the notion of block Toeplitz operators. Let $M_{n\times r}$ denote the set of all $n \times r$ complex matrices and write $M_n = M_{n\times n}$. We observe that $L^\infty_{M_n} = L^\infty \otimes M_n$ and $H^2_{\mathbb{C}^n} = H^2 \otimes \mathbb{C}^n$. For the matrix-valued function $\Phi \in L^\infty_{M_n}$, the *block Toeplitz operator with symbol* Φ is the operator T_Φ on the vector-valued Hardy space $H^2_{\mathbb{C}^n}$ of the unit disc defined by

$$T_\Phi f := P_n(\Phi f) \quad \bigl(f \in H^2_{\mathbb{C}^n}\bigr),$$

where P_n denotes the orthogonal projection of $L^2_{\mathbb{C}^n}$ $(= L^2 \otimes \mathbb{C}^n)$ onto $H^2_{\mathbb{C}^n}$. If we set $H^2_{\mathbb{C}^n} = H^2(\mathbb{T}) \oplus \cdots \oplus H^2(\mathbb{T})$, then we see that if

$$\Phi = \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ & \vdots & \\ \varphi_{n1} & \cdots & \varphi_{nn} \end{bmatrix}, \quad \text{then } T_\Phi = \begin{bmatrix} T_{\varphi_{11}} & \cdots & T_{\varphi_{1n}} \\ & \vdots & \\ T_{\varphi_{n1}} & \cdots & T_{\varphi_{nn}} \end{bmatrix}.$$

Gu, Hendricks, and Rutherford [8] studied the hyponormality of block Toeplitz operators and characterized the hyponormality of block Toeplitz operators in terms of their symbols. In particular, they showed that the hyponormality of the block Toeplitz operator T_Φ will force Φ to be normal, that is, $\Phi^*\Phi = \Phi\Phi^*$.

This paper is organized as follows. In Sect. 2, we present some preliminary knowledge of block Toeplitz operators and m-isometric operators. In Sect. 3, we give several results for the m-isometric (respectively, m-expansive and m-contractive) block Toeplitz operators with rational symbols. We give a concrete description of m-isometric block Toeplitz operators in terms of the coefficients of the matrix-valued rational symbols.

2 Preliminary

We first review the results of m-isometric Toeplitz operators with rational symbols. The properties of Toeplitz operators enable us to establish several consequences of m-isometric operators. Given a positive integer m, it follows from the definition that an operator $T \in \mathcal{L}(\mathcal{H})$ is an m-isometry if and only if

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \bigl\| T^j x \bigr\|^2 = 0 \quad \text{for all } x \in \mathcal{H}. \tag{2.1}$$

Using the identity (2.1) and the block Toeplitz operator with matrix-valued trigonometric polynomial symbol Φ, we consider the following equation:

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \bigl\| T^j_\Phi K \bigr\|^2 = 0 \tag{2.2}$$

for all $K \in H^2_{\mathbb{C}^n}$. A matrix function $\Theta \in H^\infty_{M_{m\times n}}$ is called *inner* if $\Theta^*\Theta = I_n$ a.e. on $\mathbb{T}$. From the well-known results for Toeplitz operators and isometric operators, we can extend the results to block versions. As the idea of the proof is completely the same as in that of [9], we omit the proof of the following lemma.

Lemma 2.1 *A necessary and sufficient condition that a Toeplitz operator T_Φ be an isometric operator is that Φ is inner.*

For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, write $\widetilde{\Phi}(z) := \Phi^*(\overline{z})$. We say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some

$A \in H^2_{M_{m\times r}}$ ($m \le n$). We also say that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of Φ and Ψ is a unitary constant, and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{m\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Two matrix functions Φ and Ψ in $H^2_{M_n}$ are *coprime* if they are both left and right coprime.

For $\Phi \in L^\infty_{M_n}$, we write

$$\Phi_+ := P_n\Phi \in H^2_{M_n} \quad \text{and} \quad \Phi_- := \left(P_n^\perp \Phi\right)^* \in H^2_{M_n}.$$

Thus we can write $\Phi = \Phi_+ + \Phi_-^*$. Recall that a function $\varphi \in L^\infty$ is said to be of *bounded type* (or in the Nevanlinna class) if there are analytic functions $\psi_1, \psi_2 \in H^\infty$ such that

$$\varphi(z) = \frac{\psi_1(z)}{\psi_2(z)} \quad \text{for almost all } z \in \mathbb{T}.$$

For a matrix-valued function $\Phi = [\varphi_{ij}] \in L^\infty_{M_n}$, we say that Φ is of *bounded type* if each entry φ_{ij} is of bounded type and that Φ is *rational* if each entry φ_{ij} is a rational function. Suppose $\Phi = [\varphi_{ij}] \in L^\infty_{M_n}$ is such that Φ^* is of bounded type. Then we can write $\varphi_{ij} = \theta_{ij}\overline{b}_{ij}$, where θ_{ij} is inner and θ_{ij} and b_{ij} are coprime. Thus if θ is the least common multiple of the θ_{ij}, then we write

$$\Phi = [\varphi_{ij}] = [\theta_{ij}\overline{b}_{ij}] = [\theta\overline{a}_{ij}] = \Theta A^* \tag{2.3}$$

where $\Theta = \theta I_n$ and $A \in H^2_{M_n}$. We note that Eq. (2.3) is minimal, in the sense that if ωI_n (ω is inner) is a common inner divisor of Θ and A, then ω is constant. Let $\Phi \equiv \Phi_+ + \Phi_-^* \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Then we can write

$$\Phi_+ = \Theta_0 A^* \quad \text{and} \quad \Phi_- = \Theta_1 B^*,$$

where $\Theta_i = \theta_i I_n$ with an inner function θ_i for $i = 0, 1$ and $A, B \in H^2_{M_n}$. In particular, if $\Phi \in L^\infty_{M_n}$ is rational then the θ_i can be chosen as finite Blaschke products. Let Ω be the greatest common left inner divisor of A and Θ_0. Then $A = \Omega A_\ell$ and $\Theta_0 = \Omega\Omega_0$ for some $A_\ell \in H^2_{M_n}$ and some inner matrix Ω_0. Therefore we can write

$$\Phi_+ = A_\ell^*\Omega_0$$

where Ω_0 and A_ℓ are left coprime. In this case $\Omega_0 A_\ell^*$ is called a *left coprime factorization* of Φ_+. Similarly,

$$\Phi_+ = \Delta_0 A_r^*$$

where Δ_0 and A_r are right coprime. In this case $\Delta_0 A_r^*$ is called a *right coprime factorization* of Φ_+.

We write $\mathcal{Z}(\theta)$ is the set of zeros of an inner function θ. The following lemma will be useful in the sequel.

Lemma 2.2 ([10]) *Let $B \in H^2_{M_n}$ and $\Theta := \theta I_n$ with a finite Blaschke product θ. Then the following statements are equivalent*:

(i) *$B(\alpha)$ is invertible for each $\alpha \in \mathcal{Z}(\theta)$*;

(ii) *B and Θ are right coprime*;

(iii) *B and Θ are left coprime.*

3 Main results

In this section, we give several results for m-isometric block Toeplitz operators with trigonometric polynomial symbols.

Theorem 3.1 *Let $\Phi := \Phi_+ + \Phi_-^*$ is a rational function of the form*

$$\Phi_+ = \Theta_0 A^* \quad \text{and} \quad \Phi_- = \Theta_1 B^* \quad (\text{coprime factorization}),$$

where $\Theta_i = \theta_i I_n$ with a finite Blaschke product θ_i for $i = 0, 1$ and $A, B \in H^2_{M_n}$ and $\mathcal{Z}(\theta_0) \cap \mathcal{Z}(\theta_1) \neq \emptyset$. If T_Φ is an m-isometry, then Φ is analytic.

Proof Since T_Φ is an m-isometry,

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_\Phi^{*j} T_\Phi^j K = 0$$

holds for all $K \in H^2_{\mathbb{C}^n}$. Put $K = \Theta_0^m \Theta_1^m$. Then

$$\begin{aligned} 0 &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_\Phi^{*j} T_\Phi^j K = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \Phi^{*j} \Phi^j \Theta_0^m \Theta_1^m \\ &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \left(\Phi_+^* + \Phi_-\right)^j \left(\Phi_+ + \Phi_-^*\right)^j \Theta_0^m \Theta_1^m \\ &= A^m B^m + H, \end{aligned}$$

for some $H \in \theta H^\infty_{M_n}$ where $\theta \in \mathcal{Z}(\theta_0) \cap \mathcal{Z}(\theta_1)$. Therefore $A^m(\alpha) B^m(\alpha) = 0$ for $\alpha \in \mathcal{Z}(\theta)$. By Lemma 2.2, $A(\alpha)$ and $B(\alpha)$ are invertible, a contradiction. So either Φ_+ or Φ_- is zero. If $\Phi_+ = 0$, i.e., $\Phi = \Phi_-^*$, then, for $\Theta_1^m \in H^2_{\mathbb{C}^n}$,

$$\begin{aligned} 0 &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_\Phi^{*j} T_\Phi^j \Theta_1^m = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_{\Phi_-}^j T_{\Phi_-^*}^j \Theta_1^m \\ &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \Phi_-^j P_n\left(\Phi_-^{*j} \Theta_1^m\right) = (-1)^m \Theta_1^m + KB \end{aligned}$$

for some nonzero $K \in H^\infty_{M_n}$. Since B and Θ_1 are coprime, we have a contradiction. Therefore $\Phi_- = 0$ and hence Φ is analytic. This completes the proof. □

Example 3.2 Suppose that

$$\Phi(z) = \begin{bmatrix} \frac{2z-1}{1-\frac{1}{2}z} & 0 \\ 0 & \frac{z-\frac{1}{3}}{1-\frac{1}{3}z} \end{bmatrix}.$$

Then Φ is analytic, but

$$T_{\Phi^*}^2 T_\Phi^2 - 2T_{\Phi^*}T_\Phi + I = \begin{bmatrix} 9I & 0 \\ 0 & I \end{bmatrix} \neq 0.$$

Hence T_Φ is not 2-isometric.

Next, we show that every m-isometric block Toeplitz operators with trigonometric polynomial symbol is an isometry.

Theorem 3.3 *Let $\Phi := \Phi_+ + \Phi_-^*$ be normal and rational of the form*

$$\Phi_+ = \Theta_0 A^* \quad \textit{and} \quad \Phi_- = \Theta_1 B^* \quad (\textit{coprime factorization}),$$

where $\Theta_i = \theta_i I_n$ with a finite Blaschke product θ_i for $i = 0, 1$ and $A, B \in H^2_{M_n}$ and $\mathcal{Z}(\theta_0) \cap \mathcal{Z}(\theta_1) \neq \emptyset$. Then the Toeplitz operator T_Φ is an m-isometry if and only if Φ is inner.

Proof Sufficiency is obvious. To prove necessity, if T_Φ is an m-isometry, then

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_\Phi^{*j} T_\Phi^j K = 0$$

holds for all $K \in H^2_{\mathbb{C}^n}$. Put $K = \Theta_0^m \Theta_1^m$. Since Φ is normal

$$\begin{aligned} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T_\Phi^{*j} T_\Phi^j \Theta_0^m \Theta_1^m &= \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} \Phi^{*j} \Phi^j \Theta_0^m \Theta_1^m \\ &= \left(\Phi^*\Phi - I\right)^m \Theta_0^m \Theta_1^m. \end{aligned}$$

Thus $(\Phi^*\Phi - I)^m = 0$ and so, $\Phi^*\Phi - I$ is nilpotent of order m. Hence the spectrum $\sigma(\Phi^*\Phi) = \{1\}$. Thus $\Phi^*\Phi = I$. From Lemma 2.1 and Theorem 3.1, Φ is inner. This completes the proof. □

Corollary 3.4 *Suppose that Φ, Ψ are normal and rational of the form*

$$\Phi_+ = \Theta_0 A^* \quad \textit{and} \quad \Phi_- = \Theta_1 B^* \quad (\textit{coprime factorization})$$

and

$$\Psi_+ = \Theta_2 C^* \quad \textit{and} \quad \Psi_- = \Theta_3 D^* \quad (\textit{coprime factorization}),$$

where $\Theta_i = \theta_i I_n$ with a finite Blaschke product θ_i for $i = 0, 1, 2, 3$ and $A, B, C, D \in H^2_{M_n}$ and $\mathcal{Z}(\theta_0) \cap \mathcal{Z}(\theta_1) \neq \emptyset$ and $\mathcal{Z}(\theta_2) \cap \mathcal{Z}(\theta_3) \neq \emptyset$. Then the following hold.

(i) *If T_Φ and T_Ψ are m-isometric operators, then $T_\Phi T_\Psi$ and $T_\Psi T_\Phi$ are m-isometric operators.*

(ii) *If T_Φ and T_Ψ are m-isometric operators, then $T_\Phi - T_\Psi$ is an m-isometric operator if and only if $\Phi^*\Psi + \Psi^*\Phi = I$.*

Proof By Theorem 3.3, T_Φ and T_Ψ are isometric operators. Therefore $T_\Psi^* T_\Phi^* T_\Phi T_\Psi = I$, i.e., $T_\Phi T_\Psi$ is isometric and so $\Phi\Psi$ is normal. Hence $T_\Phi T_\Psi$ is an m-isometric operator. Similarly, $T_\Psi T_\Phi$ is also an m-isometric operator. For (ii), suppose that $T_\Phi - T_\Psi$ is an m-isometric operator. By Theorem 3.3, $T_\Phi - T_\Psi$ is an isometry. Hence

$$\begin{aligned} 0 &= \left(T_\Phi^* - T_\Psi^*\right)(T_\Phi - T_\Psi) - I = T_\Phi^* T_\Phi - T_\Phi^* T_\Psi - T_\Psi^* T_\Phi + T_\Psi^* T_\Psi - I \\ &= I - T_{\Phi^*\Psi+\Psi^*\Phi}. \end{aligned}$$

Thus, we have $T_{\Phi^*\Psi+\Psi^*\Phi} = I$ or equivalently, $\Phi^*\Psi + \Psi^*\Phi = I$. Conversely, if $\Phi^*\Psi + \Psi^*\Phi = I$, then $T_\Phi - T_\Psi$ is isometry. Hence $T_\Phi - T_\Psi$ is an m-isometric operator. □

Next, we study contractive and expansive Toeplitz operators. We consider m-contractive Toeplitz operators with trigonometric polynomial symbols when $m = 1$ and 2.

The next lemma plays a key role in finding necessary and sufficient conditions for m-expansive and m-contractive Toeplitz operators.

Lemma 3.5 ([11]) *Let A, B, C be complex matrices where A and C are square matrices. Then*

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0 \quad \Leftrightarrow \quad \begin{cases} A \geq 0, \\ B = AW \quad (\textit{for some } W), \\ C \geq W^*AW. \end{cases}$$

Moreover, $\operatorname{rank}\left[\begin{smallmatrix} A & B \\ B^* & C \end{smallmatrix}\right] = \operatorname{rank} A \Leftrightarrow C = W^*AW$.

First, we consider the 1-contractive Toeplitz operators.

Theorem 3.6 *Suppose that Φ is a matrix-valued function in $L^\infty_{\mathbb{C}^2}$ of the form*

$$\Phi(z) = \begin{bmatrix} a(z) & b(z) \\ c(z) & d(z) \end{bmatrix}. \tag{3.1}$$

(i) *If $\|a\|_\infty + \|c\|_\infty > 1$, then T_Φ is contractive if and only if*

$$T_{b,d}^* T_{b,d} - I \geq T_{b,d}^* T_{a,c}\left(T_{a,c}^* T_{a,c} - I\right)^{-1} T_{a,c}^* T_{b,d},$$

where $T_{x,y} = \left[\begin{smallmatrix} T_x \\ T_y \end{smallmatrix}\right]$.

(ii) *If $\|a\|_\infty + \|c\|_\infty = 1$, then T_Φ is contractive if and only if $\|b\|_\infty + \|d\|_\infty \geq 1$.*

(iii) *If $\|a\|_\infty + \|c\|_\infty < 1$, then T_Φ is never contractive.*

Proof From the definition, T_Φ is contractive if and only if $T_{\Phi^*} T_\Phi - I \geq 0$ or equivalently,

$$\begin{bmatrix} T_{\overline{a}} T_a + T_{\overline{c}} T_c - I & T_{\overline{a}} T_b + T_{\overline{c}} T_d \\ T_{\overline{b}} T_a + T_{\overline{d}} T_c & T_{\overline{b}} T_b + T_{\overline{d}} T_d - I \end{bmatrix} \geq 0.$$

By Lemma 3.5, T_Φ is contractive if and only if

$$T_{\overline{a}} T_a + T_{\overline{c}} T_c \geq I, \qquad T_{\overline{a}} T_b + T_{\overline{c}} T_d = (T_{\overline{a}} T_a + T_{\overline{c}} T_c - I)W \tag{3.2}$$

and

$$T_{\overline{b}}T_b + T_{\overline{d}}T_d - I \geq W^*(T_{\overline{a}}T_b + T_{\overline{c}}T_d) \tag{3.3}$$

for some W. If $\|a\|_\infty + \|c\|_\infty > 1$ then $T_{\overline{a}}T_a + T_{\overline{c}}T_c - I$ is invertible. From Eq. (3.2), we deduce that

$$W = (T_{\overline{a}}T_a + T_{\overline{c}}T_c - I)^{-1}(T_{\overline{a}}T_b + T_{\overline{c}}T_d).$$

Therefore, by the inequality (3.3),

$$T_{\overline{b}}T_b + T_{\overline{d}}T_d - I \geq \left(T_b^*T_{\overline{a}}^* + T_d^*T_{\overline{c}}^*\right)(T_{\overline{a}}T_a + T_{\overline{c}}T_c - I)^{-1}(T_{\overline{a}}T_b + T_{\overline{c}}T_d)$$

or equivalently,

$$T_{b,d}^*T_{b,d} - I \geq T_{b,d}^*T_{a,c}\left(T_{a,c}^*T_{a,c} - I\right)^{-1}T_{a,c}^*T_{b,d},$$

where $T_{x,y} = \left[\begin{smallmatrix} T_x \\ T_y \end{smallmatrix}\right]$. If $\|a\|_\infty + \|c\|_\infty = 1$, then $T_{\overline{a}}T_a + T_{\overline{c}}T_c = I$ and from Eqs. (3.2) and (3.3),

$$T_{\overline{b}}T_b + T_{\overline{d}}T_d - I \geq 0.$$

If $\|a\|_\infty + \|c\|_\infty < 1$, then $T_{\overline{a}}T_a + T_{\overline{c}}T_c < I$. By Eq. (3.2), T_Φ is never contractive. This completes the proof. □

Using the same arguments of Theorem 3.6 that we can check the following consequence.

Corollary 3.7 *Suppose that Φ is a matrix-valued function in $L^\infty_{\mathbb{C}^2}$ of the form* (3.1).

(i) *If $T_{\overline{a}}T_a + T_{\overline{c}}T_c < I$, then T_Φ is expansive if and only if*

$$T_{\overline{b}}T_b + T_{\overline{d}}T_d - I \leq \left(T_b^*T_{\overline{a}}^* + T_d^*T_{\overline{c}}^*\right)(T_{\overline{a}}T_a + T_{\overline{c}}T_c - I)^{-1}(T_{\overline{a}}T_b + T_{\overline{c}}T_d).$$

(ii) *If $T_{\overline{a}}T_a + T_{\overline{c}}T_c = I$, then T_Φ is expansive if and only if $T_{\overline{b}}T_b + T_{\overline{d}}T_d - I \leq 0$.*

(iii) *If $T_{\overline{a}}T_a + T_{\overline{c}}T_c > I$, then T_Φ is never expansive.*

Corollary 3.8 *Suppose that Φ is a matrix-valued function in $L^\infty_{\mathbb{C}^2}$ of the form* (3.1). *Then the following statements hold.*

(i) *If $b = 0$ and $c = 0$, then T_Φ is contractive if and only if T_a and T_d are contractive.*

(ii) *If $a = 0$ and $d = 0$, then T_Φ is contractive if and only if T_b and T_c are contractive.*

Proof If $b = c = 0$, then from Eqs. (3.2) and (3.3), T_Φ is contractive if and only if $T_{\overline{a}}T_a \geq I$ and $T_{\overline{d}}T_d \geq I$. □

Corollary 3.9 *Suppose that Φ is a matrix-valued function in $L^\infty_{\mathbb{C}^2}$ of the form* (3.1) *where T_a, T_b, T_c, and T_d are isometric. Then T_Φ is contractive if and only if $T_{\overline{a}b+\overline{c}d}$ is expansive.*

Proof From Lemma 2.1 and Theorem 3.6, we have $T_{a\overline{b}+c\overline{d}}T_{\overline{a}b+\overline{c}d} \leq I$ or equivalently, $T_{\overline{a}b+\overline{c}d}$ is expansive. □

Next, we study 2-contractive Toeplitz operators. Suppose that Φ is a matrix-valued function in $L^\infty_{\mathbb{C}^n}$ of the form

$$\Phi(z) = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix}.$$

Then by a direct calculation we deduce that

$$B^2(T_\Phi) = T^2_{\Phi^*}T^2_\Phi - 2T_{\Phi^*}T_\Phi + I = \begin{bmatrix} M_1 & M_2 \\ M_2^* & M_3 \end{bmatrix}, \tag{3.4}$$

where

$$\begin{aligned} M_1 &= T^2_{A^*}T^2_A + T_{A^*}T_{C^*}T_CT_A + T_{C^*}T_{B^*}T^2_A + T_{C^*}T_{D^*}T_CT_A + T^2_{A^*}T_BT_C \\ &\quad + T_{A^*}T_{C^*}T_DT_C + T_{C^*}T_{B^*}T_BT_C + T_{C^*}T_{D^*}T_DT_C - 2T_{A^*}T_A - 2T_{C^*}T_C + I, \\ M_2 &= T^2_{A^*}T_AT_B + T_{A^*}T_{C^*}T_CT_B + T_{C^*}T_{B^*}T_AT_B + T_{C^*}T_{D^*}T_CT_B + T^2_{A^*}T_BT_D \\ &\quad + T_{A^*}T_{C^*}T^2_D + T_{C^*}T_{B^*}T_BT_D + T_{C^*}T_{D^*}T^2_D - 2T_{A^*}T_B - 2T_{C^*}T_D, \end{aligned}$$

and

$$\begin{aligned} M_3 &= T_{B^*}T_{A^*}T_AT_B + T_{B^*}T_{C^*}T_CT_B + T_{D^*}T_{B^*}T_AT_B + T^2_{D^*}T_CT_B + T_{B^*}T_{A^*}T_BT_D \\ &\quad + T_{B^*}T_{C^*}T^2_D + T_{D^*}T_{B^*}T_BT_D + T^2_{D^*}T^2_D - 2T_{B^*}T_B - 2T_{D^*}T_D + I. \end{aligned}$$

Since the induced relations are very complicated, in order to consider the 2-contractive or 2-expansive Toeplitz operators, we consider the case with a simple symbol. Recently, m-isometric Toeplitz operators with single-valued symbols were studied in [7].

Lemma 3.10 ([7]) *Let φ be a rational function. A Toeplitz operator T_φ is an m-isometry if and only if T_φ is an isometry.*

Theorem 3.11 *Suppose that Φ is a matrix-valued rational function in $L^\infty_{\mathbb{C}^n}$ of the form*

$$\Phi(z) = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix},$$

where T_A, T_B, T_C, and T_D are m-isometric operators. Then T_Φ is 2-contractive if and only if

$$M_1 \geq 0, \qquad M_2 = M_1W, \quad \text{and} \quad M_3 \geq W^*M_2 \tag{3.5}$$

for some W where

$$\begin{aligned} M_1 &= T_{C^*B^*A^2} + T_{C^*D^*CA} + T_{A^{*2}BC} + T_{A^*C^*DC} + I, \\ M_2 &= T_{C^*B^*AB} + T_{C^*D^*CB} + T_{A^{*2}BD} + T_{A^*C^*D^2}, \end{aligned}$$

$$M_3 = T_{D^*B^*AB} + T_{D^{*2}CB} + T_{B^*A^*BD} + T_{B^*C^*D^2} + I.$$

In particular, suppose that A, B, C, and D are single-valued trigonometric polynomials. If T_Φ is 2-contractive then

$$\deg A = \deg D, \qquad 2\deg A = \deg B + \deg C. \tag{3.6}$$

Proof From Lemmas 2.1, 3.10, and Eq. (3.4), if follows that T_Φ is 2-contractive if and only if the relations (3.5) hold for some W. In particular, if A, B, C, and D are single-valued trigonometric polynomials, then from Lemma 3.10, we can set $A(z) = \lambda_1 z^{k_a}$, $B(z) = \lambda_2 z^{k_b}$, $C(z) = \lambda_3 z^{k_c}$, and $D(z) = \lambda_4 z^{k_d}$ with $|\lambda_1| = |\lambda_2| = |\lambda_3| = |\lambda_4| = 1$. Hence

$$M_1 = T_{\lambda_1^2\overline{\lambda_2\lambda_3}z^{2k_a-k_b-k_c}} + T_{\lambda_1\overline{\lambda_4}z^{k_a-k_d}} + T_{\overline{\lambda_1}^2\lambda_2\lambda_3 z^{k_b+k_c-2k_a}} + T_{\overline{\lambda_1}\lambda_4 z^{k_d-k_a}} + I.$$

From Eq. (3.5),

$$\begin{aligned} M_1 \geq 0 \quad &\Longleftrightarrow \quad T_{\lambda_1^2\overline{\lambda_2\lambda_3}z^{2k_a-k_b-k_c}+\lambda_1\overline{\lambda_4}z^{k_a-k_d}+\overline{\lambda_1}^2\lambda_2\lambda_3 z^{k_b+k_c-2k_a}+\overline{\lambda_1}\lambda_4 z^{k_d-k_a}} + I \geq 0 \\ &\Longleftrightarrow \quad T_{2\,\mathrm{Re}\{\lambda_1^2\overline{\lambda_2\lambda_3}z^{2k_a-k_b-k_c}+\lambda_1\overline{\lambda_4}z^{k_a-k_d}\}+1} \geq 0, \end{aligned}$$

and hence $k_a = k_d$ and $2k_a = k_b + k_c$. This completes the proof. □

The next result is a necessary and sufficient condition for T_Φ to be 2-contractive, under the same hypotheses as Theorem 3.11 but with the additional condition that each entry in the matrix is a single-valued function.

Corollary 3.12 *Suppose that Φ is a matrix-valued function in $L^\infty_{\mathbb{C}^2}$ of the form*

$$\Phi(z) = \begin{bmatrix} az^m & bz^l \\ cz^{2m-l} & dz^m \end{bmatrix}$$

where $|a| = |b| = |c| = |d| = 1$. Then T_Φ is 2-contractive if and only if

$$\begin{cases} (2\,\mathrm{Re}\{\overline{a}^2bc + \overline{a}d\} + 1)(2\,\mathrm{Re}\{\overline{bc}d^2 + a\overline{d}\} + 1) \geq |a\overline{c} + b\overline{d} + \overline{a}^2bd + \overline{ac}d^2|^2 \\ \quad \text{if } 2\,\mathrm{Re}\{a^2\overline{bc} + a\overline{d}\} + 1 > 0, \\ 2\,\mathrm{Re}\{\overline{bc}d^2 + a\overline{d}\} + 1 \geq 0 \quad \text{if } 2\,\mathrm{Re}\{a^2\overline{bc} + a\overline{d}\} + 1 = 0. \end{cases}$$

Furthermore, if $2\,\mathrm{Re}\{a^2\overline{bc} + a\overline{d}\} + 1 < 0$, then T_Φ is not 2-contractive.

Proof From the proof of Theorem 3.11, we deduced that T_Φ is 2-contractive if and only if

$$M_1 \geq 0, \qquad M_2 = M_1 W, \quad \text{and} \quad M_3 \geq W^* M_2 \tag{3.7}$$

for some W where

$$M_1 = T_{a^2\overline{bc}} + T_{a\overline{d}} + T_{\overline{a}^2bc} + T_{\overline{a}d} + I,$$

$$M_2 = T_{a\bar{c}z^{l-m}} + T_{b\bar{d}z^{l-m}} + T_{\bar{a}^2 bdz^{l-m}} + T_{\overline{ac}d^2 z^{l-m}},$$

$$M_3 = T_{a\bar{d}} + T_{bc\bar{d}^2} + T_{\bar{a}d} + T_{\overline{bc}d^2} + I.$$

From Eq. (3.7), $M_1 \geq 0$ if and only if $2\operatorname{Re}\{a^2\overline{bc} + a\overline{d}\} + 1 \geq 0$. There are two cases to consider. If $2\operatorname{Re}\{a^2\overline{bc} + a\overline{d}\} + 1 > 0$, then

$$W = \left(2\operatorname{Re}\left\{a^2\overline{bc} + a\overline{d}\right\} + 1\right)^{-1} T_{(a\bar{c}+b\bar{d}+\bar{a}^2 bd+\overline{ac}d^2)z^{l-m}}.$$

Hence $M_3 \geq W^* M_2$ if and only if

$$\left(2\operatorname{Re}\left\{a^2\overline{bc} + a\overline{d}\right\} + 1\right)\left(2\operatorname{Re}\left\{\overline{bc}d^2 + a\overline{d}\right\} + 1\right) \geq \left|a\bar{c} + b\bar{d} + \bar{a}^2 bd + \overline{ac}d^2\right|^2.$$

If $2\operatorname{Re}\{a^2\overline{bc} + a\overline{d}\} + 1 = 0$, then $M_1 = M_2 = 0$ and hence $M_3 \geq 0$ or equivalently, $2\operatorname{Re}\{\overline{bc}d^2 + a\overline{d}\} + 1 \geq 0$. This completes the proof. $\square$

Example 3.13 Suppose that $\Phi(z) = \left[\begin{smallmatrix} -z & -z \\ z & z \end{smallmatrix}\right]$. Then Eq. (3.6) holds, but $M_1 = -3I < 0$. Therefore T_Φ is not 2-contractive. Hence the converse of Theorem 3.11 does not hold.

4 Conclusion

In [7], m-isometric Toeplitz operators were studied for the case of single trigonometric polynomial symbols. In this paper, we study the properties of Toeplitz operators with matrix-valued trigonometric polynomial symbols and we obtain a necessary and sufficient condition for block Toeplitz operators with trigonometric polynomial symbols to be m-isometric or m-contractive.

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Competing interests
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