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Extensions and demonstrations of Hölder's inequality

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Abstract

In this paper, we present some new extensions of Hölder's inequality and give a condition under which the equality holds. We also show that many existing inequalities related to the Hölder inequality are particular cases of the inequalities presented. At the same time, we further promote new inequalities based on the already introduced Hölder inequality.

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1 Introduction

Let $a_{ij} > 0$, $1 \leq i \leq m$, $1 \leq j \leq n$, $p_j > 1$, and $\sum_{j=1}^n p_j^{-1} = 1$. The following Hölder inequality is well known:

$$\sum_{i=1}^m \prod_{j=1}^n a_{ij} \leq \prod_{j=1}^n \left(\sum_{i=1}^m a_{ij}^{p_j} \right)^{\frac{1}{p_j}}. \quad (1)$$

The integral form of the Hölder inequality is

$$\int_a^b \left(\prod_{j=1}^n f_j(x) \right) dx \leq \prod_{j=1}^n \left(\int_a^b f_j^{p_j}(x) dx \right)^{1/p_j}.$$

In addition, from (1) the famous Cauchy inequality follows:

$$\left(\sum_{i=1}^n x_i y_i \right)^2 \leq \left(\sum_{i=1}^n x_i^2 \right) \left(\sum_{i=1}^n y_i^2 \right).$$

Analytic inequalities [1–12], especially Hölder's inequalities play an important role in mathematical analysis, harmonic analysis, functional analysis, and partial differential equations. It is precisely because of the importance of the Hölder inequality that more and more authors have invested in its research and have made a lot of optimization and advancement. In this paper, we describe the discrete and continuous forms of the Hölder inequality and its associated inferences.

Yang’s [13, 14] insights into inequalities have further led to several inferences. Qi’s [15, 16] several integral inequalities explain the integral form of the inequality and further elaboration of the integral form of the Hölder inequality. Tian [17–20], Tian and Ha [21, 22], Tian, Ha, and Wang [23], Tian and Pedrycz [24], Kwon and Bae [25], Tian, Zhu and Cheung [26], and Zhao and Cheung [27] give a series of meaningful improvements, generalization, properties, and applications of Hölder’s inequality. The papers in the references are of great guiding significance to the conception and promotion of this paper.

From the paper by Yang [14] we can get the following conclusions.

Let the function $h : (-\infty, +\infty) \rightarrow (0, +\infty)$ be defined as

$$h(t) = \prod_{k=1}^n \left[\sum_{i=1}^m \left(\prod_{j=1}^n a_{ij} \right)^{1-t} (a_{ik}^{p_k})^t \right]^{1/p_k}. \tag{2}$$

It is easy to see that $h \in C^\infty$ and (1) becomes

$$h(0) = \sum_{i=1}^m \prod_{j=1}^n a_{ij} \leq \prod_{j=1}^n \left(\sum_{i=1}^m a_{ij}^{p_j} \right)^{\frac{1}{p_j}} = h(1); \tag{3}$$

it is not difficult to see $h(0) = \sum_{i=1}^m \prod_{j=1}^n a_{ij}$, $h(1) = \prod_{j=1}^n (\sum_{i=1}^m a_{ij}^{p_j})^{1/p_j}$ under the conditions $p_k > 1, k = 1, 2, \dots, n$, and $\sum_{k=1}^n 1/p_k = 1$. From (2) it follows that

$$h'(t) \begin{cases} \geq 0, & t \geq 0, \\ \leq 0, & t \leq 0, \end{cases}$$

and the equation holds if and only if $t = 0$ or $a_{ik}^{p_k} / \prod_{j=1}^n a_{ij} = a_{jk}^{p_k} / \prod_{j=1}^n a_{ij}$ for $1 \leq i, j \leq m, k = 1, 2, \dots, n$.

From (2) and (3), it is natural to consider the function $h(t)$ more deeply. Similarly, we can define the function $g(t)$ related to the integral form of Hölder’s inequality:

$$g(t) = \prod_{k=1}^n \left[\int_a^b \left(\prod_{j=1}^n f_j(x) \right)^{1-t} f_k^{p_k t}(x) dx \right]^{1/p_k}, \quad t \in \mathbb{R}, \tag{4}$$

where $f_k(x) > 0, x \in [a, b], k = 1, 2, \dots, n$, and $f_k \in L^{p_k} [a, b]$.

The integral expression of Hölder’s inequality now becomes

$$g(0) = \int_a^b \left(\prod_{k=1}^n f_k(x) \right) dx \leq \prod_{k=1}^n \left(\int_a^b f_k^{p_k}(x) dx \right)^{1/p_k} = g(1). \tag{5}$$

As can be seen from the paper of Yang [14], (2) and (4) are generalized to the case $p_j > 1, \sum_{k=1}^n 1/p_k = r$. We will show that $h_r(t)$ and $g_r(t)$ are concave functions on \mathbb{R} and that $\min_{t \in \mathbb{R}} h_r(t) = h_r(0)$ and $\min_{t \in \mathbb{R}} g_r(t) = g_r(0)$; moreover, $h_r'(t)t \geq 0$ and $g_r'(t)t \geq 0$ for $t \in \mathbb{R}$. Therefore, we obtain refinements of (3) and (5):

$$h_r(0) = h_r(1) - h_r'(\tau), \quad \tau \in (0, 1),$$

$$g_r(0) = g_r(1) - g_r'(s), \quad s \in (0, 1),$$

where $0 \leq h'_r(\tau) \leq h'_r(1)$ and $0 \leq g'_r(s) \leq g'_r(1)$. We also obtain conditions at which the equalities hold.

2 Main results

We begin this section with three lemmas, which will be used in the sequel.

From Yang [13] we get the following lemma.

Lemma 2.1 ([13]) *If A and B are positive numbers, then*

$$(\ln A - \ln B)(A^t - B^t) \begin{cases} \geq 0, & t \geq 0, \\ \leq 0, & t \leq 0, \end{cases}$$

and the equality holds if and only if $(A - B)t = 0$.

Lemma 2.2 ([14]) *Let $rp_k > 1$, $\sum_{k=1}^n \frac{1}{p_k} = r$, $a_{ij} > 0$, $1 \leq i \leq m$, $1 \leq j \leq n$, $b_i = (\prod_{j=1}^n a_{ij})^{\frac{1}{r}}$, $d_{ik} = a_{ik}^{rp_k} / \prod_{j=1}^n a_{ij}$, $1 \leq i \leq m$, $1 \leq k \leq n$. Then*

$$\sum_{k=1}^n \frac{1}{p_k} \left(\sum_{i=1}^m b_i \ln d_{ik} \right) = 0.$$

Proof By the definition of b_i and d_{ik} we have for $i = 1, 2, \dots, m$:

$$\begin{aligned} \sum_{k=1}^n \frac{1}{p_k} \ln d_{ik} &= \sum_{k=1}^n \frac{1}{p_k} \ln \left(\frac{a_{ik}^{rp_k}}{b_i^r} \right) \\ &= \sum_{k=1}^n r \ln a_{ik} - \sum_{k=1}^n \frac{r}{p_k} \ln b_i \\ &= r \left(\ln \left(\prod_{k=1}^n \frac{a_{ik}}{b_i^r} \right) \right) \\ &= 0. \end{aligned}$$

Therefore

$$\sum_{k=1}^n \frac{1}{p_k} \left(\sum_{i=1}^m b_i^r \ln d_{ik} \right) = \sum_{i=1}^m b_i^r \left(\sum_{k=1}^n \frac{1}{p_k} \ln d_{ik} \right) = 0.$$

Thus the proof of the lemma is completed. □

Lemma 2.3 ([14]) *Let $rp_k > 1$, $\sum_{k=1}^n \frac{1}{p_k} = r$, $f_k(x) > 0$, $F(x) = (\prod_{j=1}^n f_j(x))^{\frac{1}{r}}$, $g_k(x) = f_k^{rp_k} / F^r(x)$, $x \in [a, b]$, $k = 1, 2, \dots, n$. Then*

$$\sum_{k=1}^n p_k^{-1} \ln g_k(x) = 0, \quad x \in [a, b].$$

Proof After a simple operation, we have

$$\begin{aligned} \sum_{k=1}^n p_k^{-1} \ln g_k(x) &= \sum_{k=1}^n \frac{1}{p_k} \ln \frac{f_k^{rp_k}(x)}{F^r(x)} \\ &= \sum_{k=1}^n \frac{1}{p_k} [rp_k \ln f_k(x) - r \ln F(x)] \\ &= \sum_{k=1}^n \ln f_k(x) - \sum_{k=1}^n \frac{r}{p_k} \ln F(x) \\ &= r \ln \prod_{k=1}^n f_k(x) - r^2 \ln F(x) \\ &= r \ln \frac{\prod_{k=1}^n f_k(x)}{F^r(x)} \\ &= 0. \end{aligned}$$

Thus the proof of the lemma is completed. □

The main result of this paper is the following theorem.

As can be seen from Kwon and Bae [25], we can get the continuous function $h_r(t)$ proposed by Theorem 2.4. In this paper, we consider continuous variables $f(x, y)$ instead of discrete variables a_{ij} . At the same time, we will introduce the discrete form of $h_r(t)$ used in this paper by the continuous function.

Theorem 2.4 *Let $X = (X, \mu)$ and (Y, ν) be positive measure spaces with $\mu(X) = 1$ and $0 < \nu(Y) < \infty$. Let $f(x, y)$ be a real-valued bounded measurable function on $X \times Y$. Define the function $h_r(t) : \mathbb{R} \rightarrow \mathbb{R}^+$ by*

$$h_r(t) = \exp \left[\int_X \ln \left(\int_Y G(y) e^{tH(x,y)} d\nu(y) \right) d\mu(x) \right], \tag{6}$$

where $G(y) = e^{\int_X f(x,y) d\mu(x)}$ and $H(x, y) = rf(x, y) - \int_X f(x, y) d\mu(x)$.

Then the discrete form of $h_r(t)$ can be further pushed out:

$$h_r(t) = \prod_{k=1}^n \left[\sum_{i=1}^n \left(\prod_{j=1}^m X_{ij} \right)^{1-t} (X_{ik}^{rp_k})^t \right]^{\frac{1}{p_k}}, \tag{7}$$

where $rp_k > 1$, $\sum_{k=1}^n \frac{1}{p_k} = r$, $X_{ij} > 0$ ($1 \leq i \leq m, 1 \leq j \leq n$), and X_{ij} are measurable.

Proof Take X, Y as two discrete spaces $X = 1, 2, 3, \dots, n, Y = 1, 2, 3, \dots, m$, and take a positive sequence p_j satisfying $\sum_{j=1}^n \frac{1}{p_j} = r$. Set μ, ν as

$$\begin{aligned} \mu &= \sum_{j=1}^n \frac{1}{p_j} \delta_j, \\ \nu &= \sum_{i=1}^m \delta_i, \end{aligned}$$

where δ_j and δ_i denote the unit mass concentrated at j and i , respectively. Then $(X, 2^X, \mu)$ and $(Y, 2^Y, \nu)$ are positive measure spaces with $\mu(X) = 1$ and $0 < \nu(Y) = m < \infty$. Here $2^{\{\cdot\}}$ means the set of all subsets of $\{\cdot\}$. Let

$$e^{f(j,i)} = X_{ij}^{p_j}.$$

Then

$$\begin{aligned} \exp\left(\int_X f(x,y) d\mu(x)\right) &= \exp\left(\sum_{j=1}^n \frac{1}{p_j} \ln X_{ij}^{p_j}\right) \\ &= \exp\left(\sum_{j=1}^n \ln X_{ij}\right) \\ &= \exp\left(\ln \prod_{j=1}^n X_{ij}\right) = \prod_{j=1}^n X_{ij}. \end{aligned}$$

Therefore $h_r(t)$ has the expression

$$\begin{aligned} h_r(t) &= \exp\left[\int_X \ln\left(\int_Y G(y)e^{tH(x,y)} d\nu(y)\right) d\mu(x)\right] \\ &= \exp\left[\int_X \ln\left(\int_Y e^{\int_X f(x,y) d\mu(x)} e^{rtf(x,y)-t\int_X f(x,y) d\mu(x)} d\nu(y)\right) d\mu(x)\right] \\ &= \exp\left[\int_X \ln\left(\int_Y (e^{\int_X f(x,y) d\mu(x)})^{1-t} (e^{f(x,y)})^{rt} d\nu(y)\right) d\mu(x)\right] \\ &= \exp\left[\int_X \ln\left(\int_Y \left(\prod_{j=1}^n X_{ij}\right)^{1-t} (X_{ik}^{rp_k})^t d\nu(y)\right) d\mu(x)\right] \\ &= \exp\left[\int_X \ln\left(\sum_{i=1}^m \left(\prod_{j=1}^n X_{ij}\right)^{1-t} (X_{ik}^{rp_k})^t\right) d\mu(x)\right] \\ &= \exp\left[\sum_{k=1}^n \frac{1}{p_k} \ln\left(\sum_{i=1}^m \left(\prod_{j=1}^n X_{ij}\right)^{1-t} (X_{ik}^{rp_k})^t\right)\right] \\ &= \prod_{k=1}^n \left[\sum_{i=1}^m \left(\prod_{j=1}^n X_{ij}\right)^{1-t} (X_{ik}^{rp_k})^t\right]^{\frac{1}{p_k}}. \end{aligned}$$

Thus the proof of the theorem is completed. □

Theorem 2.5 Let $a_{ij} > 0$ ($1 \leq i \leq m, 1 \leq j \leq n$), $rp_k > 1$, $\sum_{k=1}^n 1/p_k = r$. Define the positive function

$$h_r(t) = \prod_{k=1}^n \left[\sum_{i=1}^m \left(\prod_{j=1}^n a_{ij}\right)^{1-t} (a_{ik}^{rp_k})^t\right]^{1/p_k}, \quad t \in \mathbb{R}. \tag{8}$$

Then the function $h_r(t)$ defined by (8) is concave, that is, $h_r''(t) \geq 0$ for $t \in \mathbb{R}$, and the equality holds if and only if

$$\frac{a_{ik}^{rp_k}}{\prod_{j=1}^n a_{ij}} = \frac{a_{jk}^{rp_k}}{\prod_{l=1}^n a_{jl}}, \quad 1 \leq i, j \leq m, k = 1, 2, \dots, n. \tag{9}$$

In this case, $h_r(t)$ is a constant.

Proof It is easy to see from (2) that $h_r(t)$ can be expressed in the following equivalent form:

Let $b_i = (\prod_{j=1}^n a_{ij})^{\frac{1}{r}}$, $d_{ik} = a_{ik}^{rp_k} / b_i^r$,

$$h_r(t) = \prod_{k=1}^n \left[\sum_{i=1}^m b_i^r \left(\frac{a_{ik}^{rp_k}}{b_i^r} \right)^t \right]^{\frac{1}{p_k}}.$$

Let $H_r(t) = \ln h_r(t)$,

$$H_r(t) = \sum_{k=1}^n \frac{1}{p_k} \ln \left(\sum_{i=1}^m b_i^r d_{ik}^t \right).$$

Then by Lemmas 2.1 and 2.2 we obtain

$$\begin{aligned} H_r'(t) &= \frac{h_r'(t)}{h_r(t)} = \sum_{k=1}^n \frac{1}{p_k} \frac{\sum_{i=1}^m b_i^r d_{ik}^t \ln d_{ik}}{\sum_{i=1}^m b_i^r d_{ik}^t} \\ &= \sum_{k=1}^n \frac{1}{p_k} \frac{\sum_{i=1}^m b_i^r d_{ik}^t \ln d_{ik}}{\sum_{i=1}^m b_i^r d_{ik}^t} - \sum_{k=1}^n \frac{1}{p_k} \frac{\sum_{i=1}^m b_i^r \ln d_{ik}}{\sum_{i=1}^m b_i^r} \\ &= \sum_{k=1}^n \frac{1}{p_k} \frac{\sum_{1 \leq i < j \leq m} b_i^r b_j^r (\ln d_{ik} - \ln d_{jk})(d_{ik}^t - d_{jk}^t)}{(\sum_{i=1}^m b_i^r)(\sum_{i=1}^m b_i^r d_{ik}^t)} \quad \begin{cases} \geq 0, & t \geq 0, \\ \leq 0, & t \leq 0. \end{cases} \end{aligned}$$

Therefore $H_r'(t) = 0$ if and only if $t = 0$ or

$$d_{ik} = d_{jk}, \quad 1 \leq i < j \leq m, k = 1, 2, \dots, n.$$

Similarly, by Lemmas 2.1 and 2.2 we obtain

$$H_r''(t) = \sum_{k=1}^n \frac{1}{p_k} \frac{(\sum_{i=1}^n b_i^r d_{ik}^t)(\sum_{i=1}^m b_i^r d_{ik}^t (\ln d_{ik})^2) - (\sum_{i=1}^m b_i^r d_{ik}^t \ln d_{ik})^2}{(\sum_{i=1}^m b_i^r d_{ik}^t)^2}.$$

It is easy to see that $H_r'' \geq 0$ and the equality holds at some t_0 if and only if

$$d_{ik} = d_{jk}, \quad 1 \leq i < j \leq m, k = 1, \dots, n.$$

Since $h_r'(t) = h_r(t)H_r'(t)$, we have $h_r''(t) = h_r(t)[(H_r'(t))^2 + H_r''(t)] \geq 0$, and $h_r''(t) = 0$ if and only if $H_r'(t) = 0$ and $H_r''(t) = 0$.

The proof of the theorem is completed. □

From our discussion we see that $h_r''(t) \geq 0$ and the equality holds at some $t_0 \in \mathbb{R}$ if and only if (9) holds. In this case, $h_r''(t) \equiv 0$ and $h_r'(t) = h_r'(0) = \text{const}$. Since, by the proof, $h_r'(t) = h_r(0)H_r'(0) = 0$, which yields $h_r(t) = h_r(0) = \text{const}$, $t \in \mathbb{R}$.

Corollary 2.6 *If (9) holds, then $h_r(t) = \text{const}$. Otherwise, $h_r''(t) > 0$, $t \in \mathbb{R}$, and $h_r'(t)t > 0$ for $t \neq 0$; in particular, for $0 = t_1 < t_2 < \dots < t_N = 1$, we have*

$$\begin{aligned}
 h_r(0) &= h_r(t_1) < h_r(t_2) < \dots < h_r(t_N) = h_r(1), \\
 h_r(0) &= \min_{t \in \mathbb{R}} h_r(t) = \prod_{k=1}^n \left[\sum_{i=1}^m b_i^r \right]^{\frac{1}{p_k}} = \left(\sum_{i=1}^m b_i^r \right)^r \\
 &= \left(\sum_{i=1}^m \prod_{j=1}^n a_{ij} \right)^r \leq h_r(1) = \prod_{k=1}^n \left[\sum_{i=1}^m a_{ik}^{rp_k} \right]^{\frac{1}{p_k}}.
 \end{aligned}$$

If $r = 1$, then we get a refinement of (1),

$$\sum_{i=1}^m \prod_{j=1}^n a_{ij} \leq \prod_{j=1}^n \left(\sum_{i=1}^m a_{ij}^{p_j} \right)^{\frac{1}{p_j}}.$$

Corollary 2.7 *If (9) holds, then $h_r(t) = \text{const}$. Otherwise,*

$$h_r(0) = h_r(1) - h_r'(\tau), \quad \tau \in (0, 1),$$

and $0 < h_r'(\tau) < h_r'(1)$, where

$$h_r'(1) = H_r'(1)h_r(1) = \left(\sum_{k=1}^n \frac{1}{p_k} \frac{\sum_{1 \leq i < j \leq m} b_i^r b_j^r (\ln d_{ik} - \ln d_{jk})(d_{ik} - d_{jk})}{(\sum_{i=1}^m b_i^r)(\sum_{i=1}^m b_i^r d_{ik})} \right) \prod_{j=1}^n \left(\sum_{i=1}^m a_{ij}^{p_j} \right)^{\frac{1}{p_j}}.$$

Theorem 2.8 *Let $f_k(x) > 0$, $x \in [a, b]$, $k = 1, 2, \dots, n$, and $f_k \in L^{p_k}[a, b]$. Define the positive function*

$$g_r(t) = \prod_{k=1}^n \left[\int_a^b \left(\prod_{j=1}^n f_j(x) \right)^{1-t} f_k^{rp_k t}(x) dx \right]^{\frac{1}{p_k}}, \quad t \in \mathbb{R}. \tag{10}$$

The function $g_r(t)$ defined by (10) is concave, that is, $g_r''(t) \geq 0$ for $t \in \mathbb{R}$, and the equality holds if and only if

$$\frac{f_k^{rp_k}(x)}{\prod_{j=1}^n f_j(x)} = c_k = \text{const}. \tag{11}$$

Proof It is known from Lemma 2.3 that if $F(x) = (\prod_{j=1}^n f_j(x))^{\frac{1}{r}}$ and $g_k(x) = f_k^{rp_k}(x)/F^r(x)$, then

$$g_r(t) = \prod_{k=1}^n \left[\int_a^b F^r(x) g_k^t(x) dx \right]^{\frac{1}{p_k}}.$$

Let $G_r(t) = \ln g_r(t)$. Then

$$G_r(t) = \sum_{k=1}^n \frac{1}{p_k} \ln \left(\int_a^b F^r(x) g_k^t(x) dx \right).$$

Then by Lemmas 2.1 and 2.3 we obtain

$$\begin{aligned} G'_r(t) &= \sum_{k=1}^n \frac{1}{p_k} \frac{\int_a^b F^r(x) g_k^t(x) \ln g_k(x) dx}{\int_a^b F^r(x) g_k^t(x) dx} \\ &= \sum_{k=1}^n \frac{1}{p_k} \frac{\int_a^b F^r(x) g_k^t(x) \ln g_k(x) dx}{\int_a^b F^r(x) g_k^t(x) dx} - \frac{\int_a^b F^r(x) (\sum_{k=1}^n \frac{1}{p_k} \ln g_k(x)) dx}{\int_a^b F^r(x) dx} \\ &= \sum_{k=1}^n \frac{1}{p_k} \left[\frac{\int_a^b F^r(x) g_k^t(x) \ln g_k(x) dx}{\int_a^b F^r(x) g_k^t(x) dx} - \frac{\int_a^b F^r(x) \ln g_k(x) dx}{\int_a^b F^r(x) dx} \right] \\ &= \sum_{k=1}^n \frac{1}{2p_k} \frac{\int_a^b \int_a^b F^r(x) F^r(y) (g_k^t(x) - g_k^t(y)) (\ln g_k(x) - \ln g_k(y)) dx dy}{\int_a^b F^r(x) g_k^t(x) dx \int_a^b F^r(x) dx} \begin{cases} \geq 0, & t \geq 0, \\ \leq 0, & t \leq 0. \end{cases} \end{aligned}$$

Therefore $G'_r(t) = 0$ if and only if

$$\frac{f_k^{r p_k}(x)}{\prod_{j=1}^n f_j(x)} = c_k = \text{const.}$$

Similarly, by Lemmas 2.1 and 2.3 we obtain

$$G''_r(t) = \sum_{k=1}^n \frac{1}{p_k} \frac{\int_a^b \int_a^b F^r(x) F^r(y) g_k^t(x) g_k^t(y) (\ln g_k(x) - \ln g_k(y))^2 dx dy}{2(\int_a^b F^r(x) g_k^t(x) dx)^2} \geq 0.$$

It is easy to see $G''_r(t) \geq 0$ and the equality holds at some t_0 if and only if

$$\frac{f_k^{r p_k}(x)}{\prod_{j=1}^n f_j(x)} = c_k = \text{const.}$$

Since $g'_r(t) = g_r(t) G'_r(t)$, we have $g''_r(t) = g_r(t) [(G'_r(t))^2 + G''_r(t)] \geq 0$, and $g''_r(t) = 0$ if and only if $G'_r(t) = 0$ and $G''_r(t) = 0$.

Thus the proof of the theorem is completed. □

Corollary 2.9 *If $g_k(x) = \text{const}$, then $g_r(t) = \text{const}$. Otherwise, $g''_r(t) > 0$, $t \in \mathbb{R}$, and $g'_r(t)t > 0$ for $t \neq 0$; in particular, for $0 = t_1 < t_2 < \dots < t_N = 1$, we have*

$$\begin{aligned} g_r(0) &= g_r(t_1) < g_r(t_2) < \dots < g_r(t_N) = g_r(1), \\ g_r(0) &= \prod_{k=1}^n \left[\int_a^b F^r(x) dx \right]^{\frac{1}{p_k}} \leq g_r(1) = \prod_{k=1}^n \left[\int_a^b F^r(x) g_k(x) dx \right]^{\frac{1}{p_k}}. \end{aligned}$$

Corollary 2.10 *If (11) holds, then $g_r(t) = \text{const}$. Otherwise,*

$$g_r(0) = g_r(1) - g'_r(s), \quad s \in (0, 1),$$

and $0 < g'_r(s) < g'_r(1)$, where

$$\begin{aligned} g'_r(1) &= G'_r(1)g_r(1) \\ &= \sum_{k=1}^n \frac{1}{2p_k} \frac{\int_a^b \int_a^b F^r(x)F^r(y)(g_k(x) - g_k(y))(\ln g_k(x) - \ln g_k(y)) \, dx \, dy}{\int_a^b F^r(x)g_k(x) \, dx \int_a^b F^r(x) \, dx} \\ &\quad \times \prod_{k=1}^n \left(\int_a^b f_k^{r p_k}(x) \, dx \right)^{\frac{1}{p_k}}. \end{aligned}$$

Next, we use the existing inequalities to derive the inequalities obtained in Kwon and Bae [25].

Theorem 2.11 *Let $a_{ij} > 0$, $p_k > 1$, $\alpha_{kj} \in \mathbb{R}$, $1 \leq i \leq m$, $1 \leq j, k \leq n$, $\sum_{k=1}^n \frac{1}{p_k} = 1$, and $\sum_{k=1}^n \alpha_{kj} = 0$. Then*

$$\sum_{i=1}^m \prod_{j=1}^n a_{ij} \leq \prod_{k=1}^n \left(\sum_{i=1}^m \left(\prod_{j=1}^n a_{ij}^{1+p_k \alpha_{kj}} \right) \right)^{\frac{1}{p_k}}. \tag{12}$$

Moreover, for the integral form of this inequality, if $f_j(x) > 0$ ($j = 1, 2, \dots, n$), $x \in [a, b]$, $-\infty < a < b < +\infty$, and $f_j(x) \in C[a, b]$, then

$$\int_a^b \left(\prod_{j=1}^n f_j(x) \right) \, dx \leq \prod_{k=1}^n \left(\int_a^b \prod_{j=1}^n f_j^{1+p_k \alpha_{kj}}(x) \, dx \right)^{\frac{1}{p_k}}. \tag{13}$$

Proof From (2) we can see that $h(0)$ is the minimum point of $h(t)$.

Thus $h(0) \leq h(t)$, that is,

$$\sum_{i=1}^m \prod_{j=1}^n a_{ij} \leq \prod_{k=1}^n \left[\sum_{i=1}^m \left(\prod_{j=1}^n a_{ij} \right)^{1-t} \left(a_{ik}^{p_k} \right)^t \right]^{\frac{1}{p_k}}.$$

Under the assumptions of (2), taking $t = -p_k \alpha_{kj}$ for $j \neq k$ and $t = \alpha_{kk}/(1 - \frac{1}{p_k})$ for $j = k$ with $\sum_{k=1}^n \alpha_{kj} = 0$, we have

$$\sum_{i=1}^m \prod_{j=1}^n a_{ij} \leq \prod_{k=1}^n \left(\sum_{i=1}^m \left(\prod_{j=1}^n a_{ij}^{1+p_k \alpha_{kj}} \right) \right)^{\frac{1}{p_k}}.$$

Similarly, the integral expression of Hölder’s inequality now becomes $g(0) \leq g(t)$, that is,

$$\int_a^b \left(\prod_{k=1}^n f_k(x) \right) \, dx \leq \prod_{k=1}^n \left(\int_a^b \left(\prod_{j=1}^n f_j(x) \right)^{1-t} f_k^{p_k t}(x) \, dx \right)^{\frac{1}{p_k}}, \quad t \in \mathbb{R}.$$

Next, we make some changes to $g(t)$. Taking $t = -p_k\alpha_{kj}$ for $j \neq k$ and $t = \alpha_{kk}/(1 - \frac{1}{p_k})$ for $j = k$ with $\sum_{k=1}^n \alpha_{kj} = 0$, we have

$$\int_a^b \left(\prod_{j=1}^n f_j(x) \right) dx \leq \prod_{k=1}^n \left(\int_a^b \prod_{j=1}^n f_j^{1+p_k\alpha_{kj}}(x) dx \right)^{\frac{1}{p_k}}.$$

Thus, the proof of the theorem is completed. □

Similarly, we also consider the case $p_j > 1, \sum_{k=1}^n \frac{1}{p_k} = r$.

Theorem 2.12 *Let $a_{ij} > 0, rp_k > 1, \alpha_{kj} \in \mathbb{R}, 1 \leq i \leq m, 1 \leq j, k \leq n, \sum_{k=1}^n \frac{1}{p_k} = r$, and $\sum_{k=1}^n \alpha_{kj} = 0$. Then*

$$\sum_{i=1}^m \prod_{j=1}^n a_{ij} \leq \prod_{k=1}^n \left(\sum_{i=1}^m \left(\prod_{j=1}^n a_{ij}^{1+rp_k\alpha_{kj}} \right) \right)^{\frac{1}{p_k}}. \tag{14}$$

Moreover, for the integral form of this inequality, if $f_j(x) > 0 (j = 1, 2, \dots, n), x \in [a, b], -\infty < a < b < +\infty$, and $f_j(x) \in C[a, b]$, then

$$\int_a^b \left(\prod_{j=1}^n f_j(x) \right) dx \leq \prod_{k=1}^n \left(\int_a^b \prod_{j=1}^n f_j^{1+rp_k\alpha_{kj}}(x) dx \right)^{\frac{1}{p_k}}. \tag{15}$$

Proof From (8) we get that $h_r(0)$ is the minimum point of $h_r(t)$. Thus, $h_r(0) \leq h_r(t)$, that is,

$$\sum_{i=1}^m \prod_{j=1}^n a_{ij} \leq \prod_{k=1}^n \left[\sum_{i=1}^m \left(\prod_{j=1}^n a_{ij} \right)^{1-t} \left(a_{ik}^{rp_k} \right)^t \right]^{\frac{1}{p_k}}.$$

Under the assumptions of (8), taking $t = -rp_k\alpha_{kj}$ for $j \neq k$ and $t = \alpha_{kk}/(1 - \frac{1}{rp_k})$ for $j = k$ with $\sum_{k=1}^n \alpha_{kj} = 0$, we have

$$\sum_{i=1}^m \prod_{j=1}^n a_{ij} \leq \prod_{k=1}^n \left(\sum_{i=1}^m \left(\prod_{j=1}^n a_{ij}^{1+rp_k\alpha_{kj}} \right) \right)^{\frac{1}{p_k}}.$$

Similarly, the integral expression of Hölder’s inequality now becomes $g_r(0) \leq g_r(t)$, that is,

$$\int_a^b \left(\prod_{k=1}^n f_k(x) \right) dx \leq \prod_{k=1}^n \left(\int_a^b \left(\prod_{j=1}^n f_j(x) \right)^{1-t} f_k^{rp_k t}(x) dx \right)^{\frac{1}{p_k}}, \quad t \in \mathbb{R}.$$

Similarly to discrete $h_r(t)$, we also make some changes to the continuous $g_r(t)$. Taking $t = -rp_k\alpha_{kj}$ for $j \neq k$ and $t = \alpha_{kk}/(1 - \frac{1}{p_k})$ for $j = k$ with $\sum_{k=1}^n \alpha_{kj} = 0$, we have

$$\int_a^b \left(\prod_{j=1}^n f_j(x) \right) dx \leq \prod_{k=1}^n \left(\int_a^b \prod_{j=1}^n f_j^{1+rp_k\alpha_{kj}}(x) dx \right)^{\frac{1}{p_k}}.$$

Thus the proof of the theorem is completed. □

3 Conclusion

As is well known, the Hölder inequality and its derivative theorems play an important role in mathematical analysis. In this paper, we have presented a wider range of discrete function $h_r(t)$ and continuous function $g_r(t)$ based on the existing discrete function $h(t)$ and continuous function $g(t)$. In addition, we give a proof and application of $h_r(t)$ and $g_r(t)$, and, finally, by using the results obtained we get an extension of Hölder's inequality to the new inequality. In the future research, we will continue to explore other derived inequalities of Hölder's inequality.

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Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors read and approved the final manuscript.

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