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A Hilbert-type integral inequality under configuring free power and its applications

Qiong Liu[1]*

*Correspondence:
liuqiongxx13@163.com
[1]College of Science, Shao yang
University, Shao yang, P.R. China

Abstract

By using the method of weight function, the technique of real analysis, and the theory of special functions, a multi-parameter Hilbert-type integral inequality and its equivalent form are established, and their constant factors are proved to be the best possible. The expressions of operator with norm are given. As an application, relevant results in the references and some new inequalities are obtained by assigning some parameter values.

MSC: 26D15

Keywords: Hilbert-type integral inequality; Weight function; Whittaker function; Operator; Application

1 Introduction

If $f, g : (0, \infty) \to \mathbb{R}$ are non-negative integrable functions, satisfying $0 < \int_0^\infty f^2(x)\,dx < \infty$, $0 < \int_0^\infty g^2(y)\,dy < \infty$, the celebrated Hilbert integral inequality is as follows (see [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\,dx\,dy < \pi \left\{ \int_0^\infty f^2(x)\,dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(y)\,dy \right\}^{\frac{1}{2}}, \tag{1}$$

where the constant factor π is the best possible. Inequality (1) is very important in harmonic analysis and theory of partial differential equations (see [1, 2]). During decades, inequality (1) has been extensively studied by numerous authors, evolved into a lot of meaningful results, which include the research of parametric quantization, mixed kernels, homogeneous kernels and non-homogeneous kernels, the extensions of fractal space, etc. (see [3–15]). In 2011, Yang gave an integral inequality of Hilbert type with exponential kernel as follows (see [16]):

$$\int_0^\infty \int_0^\infty e^{-xy} f(x)g(y)\,dx\,dy < \sqrt{\pi} \left\{ \int_0^\infty f^2(x)\,dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(y)\,dy \right\}^{\frac{1}{2}}, \tag{2}$$

where the constant factor $\sqrt{\pi}$ is the best possible.

In this paper, by using the method of weight function, the technique of real analysis, and the theory of special function, a Hilbert-type integral inequality and its equivalent form with the kernel as $\frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta}{e^{\gamma xy}}$ are given, and their optimum constant factor in relation to Whittaker function and the application of the obtained results are briefly discussed.

We configured with power parameters for each factor of the integral kernel. Besides, we introduce a free parameter θ (it can take any real number) when using the weight function method based on "Hardy interpolation problem". In practical applications, the conditions $0 < \int_0^\infty x^{p\theta-1} f^p(x)\,dx < \infty$ and $0 < \int_0^\infty y^{q\theta-1} g^q(y)\,dy < \infty$ in the obtained results are easily met by selecting the parameter θ as needed. On the other hand, a series of Hilbert-type integral inequalities with single kernels, mixed kernels, and compound kernels can be obtained by selecting appropriate parameter θ and other parameter values, so that the obtained results can be used more widely.

2 Preliminaries

Some special functions are required in the following deduction (see [17]).

(1) Suppose that $\mathrm{Re}(s) > 0$, then gamma function $\Gamma(s)$ and incomplete gamma function $\Gamma(s, a)$ $(a > 0)$ are defined by the expressions

$$\Gamma(s) := \int_0^\infty e^{-t} t^{s-1}\,dt, \tag{3}$$

$$\Gamma(s, a) := \int_a^\infty e^{-t} t^{s-1}\,dt. \tag{4}$$

(2) Beta function $B(u, v)$ $(u, v > 0)$ is defined by the expression

$$B(u, v) := \int_0^1 (1 - t)^{u-1} t^{v-1}\,dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u + v)}. \tag{5}$$

(3) Confluent hypergeometric function (also called Kummer function) ${}_1F_1(\lambda, \mu, z)$ $(\lambda, \mu, z > 0)$ is defined by the expression

$$ {}_1F_1(\lambda, \mu, z) := \sum_{n=0}^\infty \frac{(\lambda)_n}{n!(\mu)_n} z^n = \frac{\Gamma(\mu)}{\Gamma(\lambda)} \sum_{n=0}^\infty \frac{z^n \Gamma(n + \lambda)}{n! \Gamma(n + \mu)}, \tag{6}$$

here, the mark $(x)_n = x(x + 1)\cdots(x + n - 1) = \frac{\Gamma(n+x)}{\Gamma(x)}$ $(x > 0)$. If $\gamma > 0$, $\alpha > \theta - 1$, by (6), we obtain

$$ {}_1F_1(1, 2 + \alpha - \theta, \gamma) = \Gamma(2 + \alpha - \theta) \sum_{n=0}^\infty \frac{\gamma^n}{\Gamma(n + 2 + \alpha - \theta)}. \tag{7}$$

(4) Whittaker function $M(k, m, z)$ is defined as

$$M(k, m, z) := z^{m+\frac{1}{2}} e^{-\frac{z}{2}} {}_1F_1\left(m - k + \frac{1}{2}, 2m + 1, z\right). \tag{8}$$

By (7) and (8), we have

$$M\left(\frac{\alpha - \theta}{2}, \frac{\alpha - \theta}{2} + \frac{1}{2}, \gamma\right) = \gamma^{\frac{\alpha-\theta}{2}+1} e^{-\frac{\gamma}{2}} \Gamma(2 + \alpha - \theta) \sum_{n=0}^\infty \frac{\gamma^n}{\Gamma(n + 2 + \alpha - \theta)}. \tag{9}$$

When $\gamma > 0$, $\alpha > \theta - 1$, by (5) and (9), we find

$$I_1 = \int_0^1 e^{-\gamma t} t^{\alpha - \theta}\,dt$$

$$= e^{-\gamma} \int_0^1 e^{\gamma(1-t)} t^{\alpha-\theta} \, dt$$

$$= e^{-\gamma} \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} \int_0^1 (1-t)^n t^{\alpha-\theta} \, dt$$

$$= e^{-\gamma} \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} \frac{\Gamma(n+1)\Gamma(1+\alpha-\theta)}{\Gamma(n+2+\alpha-\theta)}$$

$$= \frac{\gamma^{\frac{\theta-\alpha}{2}-1} e^{-\frac{\gamma}{2}} \Gamma(1+\alpha-\theta)}{\Gamma(2+\alpha-\theta)} \left[\gamma^{\frac{\alpha-\theta}{2}+1} e^{-\frac{\gamma}{2}} \Gamma(2+\alpha-\theta) \sum_{n=0}^{\infty} \frac{\gamma^n}{\Gamma(n+2+\alpha-\theta)} \right]$$

$$= \frac{\gamma^{\frac{\theta-\alpha}{2}-1} e^{-\frac{\gamma}{2}}}{1+\alpha-\theta} M\left(\frac{\alpha-\theta}{2}, \frac{\alpha-\theta}{2}+\frac{1}{2}, \gamma \right). \tag{10}$$

Furthermore, setting $\gamma t = u$, when $\gamma > 0$, $\beta > \theta - 1$, by (4), we find

$$I_2 = \int_1^{\infty} e^{-\gamma t} t^{\beta-\theta} \, dt = \gamma^{\theta-\beta-1} \int_{\gamma}^{\infty} e^{-u} u^{\beta-\theta} \, du = \gamma^{\theta-\beta-1} \Gamma(1+\beta-\theta,\gamma). \tag{11}$$

Lemma 1 *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\theta \in \mathbb{R}$, $\gamma \geq 0$, when $\gamma > 0$, $\alpha > \theta - 1$, θ is an arbitrary real number. The weight functions are defined by the following expressions*:

$$\omega(\alpha,\beta,\gamma,\theta,x) := \int_0^{\infty} \frac{(\min\{1,xy\})^{\alpha} (\max\{1,xy\})^{\beta}}{e^{\gamma xy}} \frac{y^{-\theta}}{x^{\theta-1}} \, dy, \quad x \in (0,+\infty),$$

$$\omega(\alpha,\beta,\gamma,\theta,y) := \int_0^{\infty} \frac{(\min\{1,xy\})^{\alpha} (\max\{1,xy\})^{\beta}}{e^{\gamma xy}} \frac{x^{-\theta}}{y^{\theta-1}} \, dx, \quad y \in (0,+\infty),$$

then we have

$$\omega(\alpha,\beta,\gamma,\theta,x)$$
$$= \omega(\alpha,\beta,\gamma,\theta,y) = C(\alpha,\beta,\gamma,\theta)$$
$$= \begin{cases} \frac{1}{\alpha-\theta+1} + \frac{1}{\beta-\theta+1}, & \gamma = 0, \alpha > \theta, \beta < \theta - 1, \\ \frac{\gamma^{\frac{\theta-\alpha}{2}-1} e^{-\frac{\gamma}{2}}}{1+\alpha-\theta} M(\frac{\alpha-\theta}{2}, \frac{\alpha-\theta}{2}+\frac{1}{2}, \gamma) + \gamma^{\theta-\beta-1} \Gamma(1+\beta-\theta,\gamma), & \gamma > 0, \alpha, \beta > \theta - 1. \end{cases} \tag{12}$$

Proof Setting $xy = t$, when $\gamma = 0$, $\alpha > \theta$, $\beta < \theta - 1$, we have

$$\omega(\alpha,\beta,\gamma,\theta,x) := \int_0^{\infty} \frac{(\min\{1,xy\})^{\alpha} (\max\{1,xy\})^{\beta}}{e^{\gamma xy}} \frac{y^{-\theta}}{x^{\theta-1}} \, dy$$

$$= \int_0^1 t^{\alpha-\theta} \, dt + \int_1^{\infty} t^{\beta-\theta} \, dt$$

$$= \frac{1}{\alpha-\theta+1} + \frac{1}{\beta-\theta+1}.$$

When $\gamma > 0$, $\alpha, \beta > \theta - 1$, we have

$$\omega(\alpha,\beta,\gamma,\theta,x) := \int_0^{\infty} \frac{(\min\{1,xy\})^{\alpha} (\max\{1,xy\})^{\beta}}{e^{\gamma xy}} \frac{y^{-\theta}}{x^{\theta-1}} \, dy$$

$$= \int_0^\infty \frac{(\min\{1,t\})^\alpha (\max\{1,t\})^\beta}{e^{\gamma t}} t^{-\theta} \, dt$$

$$= \int_0^1 e^{-\gamma t} t^{\alpha-\theta} \, dt + \int_1^\infty e^{-\gamma t} t^{\beta-\theta} \, dt = I_1 + I_2$$

$$= \frac{\gamma^{\frac{\theta-\alpha}{2}-1} e^{-\frac{\gamma}{2}}}{1+\alpha-\theta} M\left(\frac{\alpha-\theta}{2}, \frac{\alpha-\theta}{2} + \frac{1}{2}, \gamma\right) + \gamma^{\theta-\beta-1} \Gamma(1+\beta-\theta,\gamma).$$

Similarly, we can get $\omega(\alpha,\beta,\gamma,\theta,y) = C(\alpha,\beta,\gamma,\theta)$. $\square$

Lemma 2 *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\theta \in \mathbb{R}$, $\gamma \geq 0$, when $\gamma = 0$, $\alpha > \theta$, $\beta < \theta - 1$, and when $\gamma > 0$, $\alpha, \beta > \theta - 1$, ε is a sufficiently small positive number, both real functions $\tilde{f}(x), \tilde{g}(y)$ are defined as*

$$\tilde{f}(x) = \begin{cases} 0, & x \in (0,1), \\ x^{-\theta-\frac{\varepsilon}{p}}, & x \in [1,\infty), \end{cases} \qquad \tilde{g}(y) = \begin{cases} 0, & x \in (1,\infty), \\ y^{-\theta+\frac{\varepsilon}{q}}, & y \in (0,1], \end{cases}$$

then we have

$$\tilde{J} \cdot \varepsilon = \left[\int_0^\infty x^{p\theta-1} \tilde{f}^p(x) \, dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q\theta-1} \tilde{g}^p(y) \, dy\right]^{\frac{1}{q}} \cdot \varepsilon = 1, \tag{13}$$

$$\tilde{h} \cdot \varepsilon = \varepsilon \int_0^\infty \int_0^\infty \frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta \tilde{f}(x) \tilde{g}(y)}{e^{\gamma xy}} \, dx \, dy$$

$$> C(\alpha,\beta,\gamma,\theta)(1 - o(1)) \quad (\varepsilon \to 0^+). \tag{14}$$

Proof With the defined functions above, we can easily get

$$\tilde{J} \cdot \varepsilon = \left[\int_0^\infty x^{p\theta-1} \tilde{f}^p(x) \, dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q\theta-1} \tilde{g}^p(y) \, dy\right]^{\frac{1}{q}} \cdot \varepsilon$$

$$= \left[\int_1^\infty x^{-1-\varepsilon} \, dx\right]^{\frac{1}{p}} \left[\int_0^1 y^{-1+\varepsilon} \, dy\right]^{\frac{1}{q}} \cdot \varepsilon = 1.$$

Setting $xy = t$, when $\gamma = 0$, notice the condition as $\alpha > \theta$, $\beta < \theta - 1$. By Fubini's theorem of commutative integral order (see [18]), we obtain

$$\tilde{h} \cdot \varepsilon = \varepsilon \int_0^\infty \int_0^\infty (\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta \tilde{f}(x) \tilde{g}(y) \, dx \, dy$$

$$= \varepsilon \int_1^\infty x^{-1-\varepsilon} \, dx \int_0^x (\min\{1,t\})^\alpha (\max\{1,t\})^\beta t^{-\theta+\frac{\varepsilon}{q}} \, dt$$

$$= \varepsilon \int_1^\infty x^{-1-\varepsilon} \, dx \left[\int_0^1 t^{\alpha-\theta+\frac{\varepsilon}{q}} \, dt + \int_1^x t^{\beta-\theta+\frac{\varepsilon}{q}} \, dt\right]$$

$$= \varepsilon \int_1^\infty x^{-1-\varepsilon} \, dx \left[\int_0^1 t^{\alpha-\theta+\frac{\varepsilon}{q}} \, dt + \int_1^\infty t^{\beta-\theta+\frac{\varepsilon}{q}} \, dt - \int_x^\infty t^{\beta-\theta+\frac{\varepsilon}{q}} \, dt\right]$$

$$= \frac{1}{\alpha-\theta+1+\frac{\varepsilon}{q}} + \frac{1}{\beta-\theta+1+\frac{\varepsilon}{q}} - \varepsilon \int_1^\infty x^{-1-\varepsilon} \, dx \int_x^\infty t^{\beta-\theta+\frac{\varepsilon}{q}} \, dt$$

$$> \frac{1}{\alpha-\theta+1+\frac{\varepsilon}{q}} + \frac{1}{\beta-\theta+1+\frac{\varepsilon}{q}} - \varepsilon \int_1^\infty x^{-1} \, dx \int_x^\infty t^{\beta-\theta+\frac{\varepsilon}{q}} \, dt$$

$$= \frac{1}{\alpha - \theta + 1 + \frac{\varepsilon}{q}} + \frac{1}{\beta - \theta + 1 + \frac{\varepsilon}{q}} - \frac{\varepsilon}{(\beta - \theta + 1 + \frac{\varepsilon}{q})^2}$$

$$= \frac{1}{\alpha - \theta + 1} + \frac{1}{\beta - \theta + 1} - o_1(1).$$

In addition, when $\gamma > 0$, $\alpha, \beta > \theta - 1$, notice the fact $e^{-\gamma t} < 1$, $t \in [x, \infty)$ $(x \geq 1)$. Making use of (10) and (11), we also obtain

$$\tilde{h} \cdot \varepsilon = \varepsilon \int_0^\infty \int_0^\infty \frac{(\min\{1, xy\})^\alpha (\max\{1, xy\})^\beta \tilde{f}(x) \tilde{g}(y)}{e^{\gamma xy}} \, dx \, dy$$

$$= \varepsilon \int_1^\infty x^{-1-\varepsilon} \, dx \int_0^x \frac{(\min\{1, t\})^\alpha (\max\{1, t\})^\beta}{e^{\gamma t}} t^{-\theta + \frac{\varepsilon}{q}} \, dt$$

$$= \varepsilon \int_1^\infty x^{-1-\varepsilon} \, dx \left[\int_0^1 e^{-\gamma t} t^{\alpha - \theta + \frac{\varepsilon}{q}} \, dt + \int_1^x e^{-\gamma t} t^{\beta - \theta + \frac{\varepsilon}{q}} \, dt \right]$$

$$= \varepsilon \int_1^\infty x^{-1-\varepsilon} \, dx \left[\int_0^1 e^{-\gamma t} t^{\alpha - \theta + \frac{\varepsilon}{q}} \, dt + \int_1^\infty e^{-\gamma t} t^{\beta - \theta + \frac{\varepsilon}{q}} \, dt - \int_x^\infty e^{-\gamma t} t^{\beta - \theta + \frac{\varepsilon}{q}} \, dt \right]$$

$$= \frac{\gamma^{\frac{\theta - \alpha}{2} - 1 - \frac{\varepsilon}{2q}} e^{-\frac{\gamma}{2}}}{1 + \alpha - \theta + \frac{\varepsilon}{q}} M\left(\frac{\alpha - \theta}{2} + \frac{\varepsilon}{2q}, \frac{\alpha - \theta}{2} + \frac{1}{2} + \frac{\varepsilon}{2q}, \gamma \right)$$

$$\quad + \gamma^{\theta - \beta - 1 - \frac{\varepsilon}{q}} \Gamma\left(1 + \beta - \theta + \frac{\varepsilon}{q}, \gamma \right)$$

$$\quad - \varepsilon \int_1^\infty x^{-1-\varepsilon} \, dx \int_x^\infty e^{-\gamma t} t^{\beta - \theta + \frac{\varepsilon}{q}} \, dt$$

$$> \frac{\gamma^{\frac{\theta - \alpha}{2} - 1 - \frac{\varepsilon}{2q}} e^{-\frac{\gamma}{2}}}{1 + \alpha - \theta + \frac{\varepsilon}{q}} M\left(\frac{\alpha - \theta}{2} + \frac{\varepsilon}{2q}, \frac{\alpha - \theta}{2} + \frac{1}{2} + \frac{\varepsilon}{2q}, \gamma \right)$$

$$\quad + \gamma^{\theta - \beta - 1 - \frac{\varepsilon}{q}} \Gamma\left(1 + \beta - \theta + \frac{\varepsilon}{q}, \gamma \right)$$

$$\quad - \varepsilon \int_1^\infty x^{-1} \, dx \int_x^\infty t^{\beta - \theta + \frac{\varepsilon}{q}} \, dt$$

$$= \left[\frac{\gamma^{\frac{\theta - \alpha}{2} - 1} e^{-\frac{\gamma}{2}}}{1 + \alpha - \theta} M\left(\frac{\alpha - \theta}{2}, \frac{\alpha - \theta}{2} + \frac{1}{2}, \gamma \right) + \gamma^{\theta - \beta - 1} \Gamma(1 + \beta - \theta, \gamma) \right] (1 - o_2(1)).$$

To sum up, we have

$$\tilde{h} \cdot \varepsilon = \varepsilon \int_0^\infty \int_0^\infty \frac{(\min\{1, xy\})^\alpha (\max\{1, xy\})^\beta \tilde{f}(x) \tilde{g}(y)}{e^{\gamma xy}} \, dx \, dy$$

$$> C(\alpha, \beta, \gamma, \theta) \left(1 - o(1) \right) \quad \left(\varepsilon \to 0^+ \right). \qquad \square$$

3 Main results

Theorem 1 *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\theta \in \mathbb{R}$, $f(x), g(y) \geq 0$, satisfying $0 < \int_0^\infty x^{p\theta - 1} f^p(x) \, dx < \infty$, $0 < \int_0^\infty y^{q\theta - 1} g^q(y) \, dy < \infty$. $\gamma \geq 0$, and parameters α, β, γ, θ meet the following requirements: when $\gamma = 0$, $\alpha > \theta$, $\beta < \theta - 1$, and when $\gamma > 0$, $\alpha, \beta > \theta - 1$. Then the following inequality*

holds:

$$\int_0^\infty \int_0^\infty \frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta f(x)g(y)}{e^{\gamma xy}}\,dx\,dy$$

$$< C(\alpha,\beta,\gamma,\theta)\left\{\int_0^\infty x^{p\theta-1}f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{q\theta-1}g^q(y)\,dy\right\}^{\frac{1}{q}}. \tag{15}$$

The constant $C(\alpha,\beta,\gamma,\theta)$ appearing on its right-hand side is the best possible, where $C(\alpha,\beta,\gamma,\theta)$ has the same expression as (12).

Proof By weighted Hölder's inequality (see [19]) and Lemma 1, we have

$$\int_0^\infty \int_0^\infty \frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta f(x)g(y)}{e^{\gamma xy}}\,dx\,dy$$

$$= \int_0^\infty \int_0^\infty \frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta f(x)g(y)}{e^{\gamma xy}}\left[\frac{y^{-\frac{\theta}{p}}}{x^{-\frac{\theta}{q}}}\right]\left[\frac{x^{-\frac{\theta}{q}}}{y^{-\frac{\theta}{p}}}\right]dx\,dy$$

$$\leq \left\{\int_0^\infty \int_0^\infty \frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta f^p(x)}{e^{\gamma xy}}\frac{y^{-\theta}}{x^{-\frac{p\theta}{q}}}\,dx\,dy\right\}^{\frac{1}{p}}$$

$$\times \left\{\int_0^\infty \int_0^\infty \frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta g^q(y)}{e^{\gamma xy}}\frac{x^{-\theta}}{y^{-\frac{q\theta}{p}}}\,dx\,dy\right\}^{\frac{1}{q}}$$

$$= \left\{\int_0^\infty \omega(\alpha,\beta,\gamma,\theta,x)x^{p\theta-1}f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty \omega(\alpha,\beta,\gamma,\theta,y)y^{q\theta-1}g^q(y)\,dy\right\}^{\frac{1}{q}}$$

$$= C(\alpha,\beta,\gamma,\theta)\left\{\int_0^\infty x^{p\theta-1}f^p(x)\,dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty y^{q\theta-1}g^q(y)\,dy\right\}^{\frac{1}{q}}. \tag{16}$$

Now, suppose that "$\leq$" in (16) takes the form of equality, then by the conclusion of Hölder's inequality, there exist constants A and B, which are not all zero, such that

$$A\frac{y^{-\theta}}{x^{-\frac{p\theta}{q}}}f^p(x) = B\frac{x^{-\theta}}{y^{-\frac{q\theta}{p}}}g^q(y) \quad \text{a.e. in } (0,\infty)\times(0,\infty),$$

so there is a constant $C \neq 0$, the expression

$$Ax^{p\theta}f^p(x) = By^{q\theta}g^q(y) = C \quad \text{a.e. in } (0,\infty)\times(0,\infty)$$

is valid. Assuming that $A \neq 0$, we have $x^{p\theta-1}f^p(x) = \frac{C}{Ax}$ a.e. in $(0,\infty)$. The integral as $\int_0^\infty \frac{C}{Ax}\,dx$ is divergent, which contradicts the fact that $0 < \int_0^\infty x^{p\theta-1}f^p(x)\,dx < \infty$. Hence expression (16) only takes the form of strict inequality.

We will prove by counter-proof that the constant factor $C(\alpha,\beta,\gamma,\theta)$ in (15) is the best possible. If the constant factor $C(\alpha,\beta,\gamma,\theta)$ in (15) is not the best possible, then there exists a positive number $K < C(\alpha,\beta,\gamma,\theta)$ such that inequality (15) is still valid when replacing $C(\alpha,\beta,\gamma,\theta)$ by K. But employing expressions (13) and (14), we get $C(\alpha,\beta,\gamma,\theta)(1 - o(1)) < K$. Letting $\varepsilon \to 0^+$, it follows that $K \geq C(\alpha,\beta,\gamma,\theta)$, which contradicts the previous hypothesis that $K < C(\alpha,\beta,\gamma,\theta)$, so the constant factor $C(\alpha,\beta,\gamma,\theta)$ in (15) is the best possible. □

Theorem 2 *Under the same conditions as Theorem* 1, *the inequality*

$$\int_0^\infty y^{\frac{p}{q}(1-q\theta)} \left\{ \int_0^\infty \frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta f(x)}{e^{\gamma xy}}\, dx \right\}^p dy$$

$$< C^p(\alpha,\beta,\gamma,\theta) \int_0^\infty x^{p\theta-1} f^p(x)\, dx \tag{17}$$

holds and the constant factor $C^p(\alpha,\beta,\gamma,\theta)$ appearing on its right-hand side is the best possible. In addition, inequality (17) *is equivalent to inequality* (15).

Proof First, we will derive (17) from (15).

Define a real function as $[f(x)]_n := \min\{n, f(x)\}$. For $0 < \int_0^\infty x^{p\theta-1} f^p(x)\, dx < \infty$, there exists $n_0 \in \mathbb{N}$ such that $0 < \int_{\frac{1}{n}}^n x^{p\theta-1} f^p(x)\, dx < \infty$ $(n \geq n_0)$. Setting a real function as

$$g_n(y) := y^{\frac{p}{q}(1-q\theta)} \left[\int_{\frac{1}{n}}^n \frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta [f(x)]_n}{e^{\gamma xy}}\, dx \right]^{\frac{p}{q}} \quad \left(\frac{1}{n} < y < n, n \geq n_0 \right)$$

when $n \geq n_0$, making use of (15), we find

$$0 < \int_{\frac{1}{n}}^n y^{q\theta-1} g_n^q(y)\, dy$$

$$= \int_{\frac{1}{n}}^n y^{q\theta-1} g_n^{q-1}(y) g_n(y)\, dy$$

$$= \int_{\frac{1}{n}}^n \int_{\frac{1}{n}}^n \frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta [f(x)]_n g_n(y)}{e^{\gamma xy}}\, dx\, dy$$

$$< C(\alpha,\beta,\gamma,\theta) \left\{ \int_{\frac{1}{n}}^n x^{p\theta-1} [f(x)]_n^p\, dx \right\}^{\frac{1}{p}} \left\{ \int_{\frac{1}{n}}^n y^{q\theta-1} g_n^q(y)\, dy \right\}^{\frac{1}{q}}. \tag{18}$$

Moreover, making use of (18), we find

$$0 < \int_{\frac{1}{n}}^n y^{q\theta-1} g_n^q(y)\, dy = \int_{\frac{1}{n}}^n y^{\frac{p}{q}(1-q\theta)} \left\{ \int_{\frac{1}{n}}^n \frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta [f(x)]_n}{e^{\gamma xy}}\, dx \right\}^p dy$$

$$< C^p(\alpha,\beta,\gamma,\theta) \int_{\frac{1}{n}}^n x^{p\theta-1} [f(x)]_n^p\, dx < \infty. \tag{19}$$

For $n \to \infty$, it follows that $0 < \int_0^\infty y^{q\theta-1} g_\infty^q(y)\, dy < \infty$ and $0 < \int_0^\infty x^{p\theta-1} f^p(x)\, dx < \infty$. By (15), we know that expressions (18) and (19) still keep the form of strict inequalities. Hence, inequality (17) holds.

Next, we will derive (15) from (17). If inequality (17) holds, by Hölder's inequality, we have

$$\int_0^\infty \int_0^\infty \frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta f(x) g(y)}{e^{\gamma xy}}\, dx\, dy$$

$$= \int_0^\infty \left[y^{\frac{1-q\theta}{q}} \int_0^\infty \frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta f(x)}{e^{\gamma xy}}\, dx \right] \left[y^{\frac{q\theta-1}{q}} g(y) \right] dy$$

$$\leq \left\{ \int_0^\infty y^{\frac{p(1-q\theta)}{q}} \left[\int_0^\infty \frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta f(x)}{e^{\gamma xy}}\, dx \right]^p dy \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_0^\infty y^{q\theta-1} g^q(y)\, dy \right\}^{\frac{1}{q}}$$

$$< C(\alpha,\beta,\gamma,\theta) \left\{ \int_0^\infty x^{p\theta-1} f^p(x)\, dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty y^{q\theta-1} g^q(y)\, dy \right\}^{\frac{1}{q}}.$$

It is shown above that inequality (15) holds, so inequality (17) is equivalent to inequality (15).

In addition, suppose that the constant factor $C^p(\alpha,\beta,\gamma,\theta)$ is not the best possible. Then by (17), the constant factor $C(\alpha,\beta,\gamma,\theta)$ we acquired in (15) is not the best possible too, which contradicts the conclusion of Theorem 1. Thus the constant factor $C^p(\alpha,\beta,\gamma,\theta)$ in (17) is the best possible. □

4 Operator expression with norm

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\gamma \geq 0$, $\theta \in \mathbb{R}$, $f(x),g(y) \geq 0$, when $\gamma = 0$, $\alpha > \theta$, $\beta < \theta - 1$, and when $\gamma > 0$, $\alpha, \beta > \theta - 1$. $\varphi(x) = x^{p\theta-1}$, $\psi(y) = y^{q\theta-1}$ $(x, y > 0)$, apparently, $\psi^{1-p}(y) = y^{\frac{p}{q}(1-q\theta)}$. Now, define normed linear spaces as

$$L_\varphi^p(0,\infty) := \left\{ f : \|f\|_{p,\varphi} = \left[\int_0^\infty \varphi(x) |f(x)|^p\, dx \right]^{\frac{1}{p}} < \infty \right\},$$

$$L_\psi^q(0,\infty) := \left\{ g : \|g\|_{q,\psi} = \left[\int_0^\infty \psi(y) |g(y)|^q\, dy \right]^{\frac{1}{q}} < \infty \right\},$$

$$L_{\psi^{1-p}}^p(0,\infty) := \left\{ h : \|h\|_{p,\psi^{1-p}} = \left[\int_0^\infty \psi^{1-p}(y) |h(y)|^p\, dy \right]^{\frac{1}{p}} < \infty \right\}.$$

If $f \in L_\varphi^p(0,\infty)$, a singular integral operator is defined as $T : L_\varphi^p(0,\infty) \to L_{\psi^{1-p}}^p(0,\infty)$,

$$T(f)(y) := \int_0^\infty \frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta}{e^{\gamma xy}} f(x)\, dx, \quad y \in (0,\infty).$$

For $f \in L_\varphi^p(0,\infty)$, $g \in L_\psi^q(0,\infty)$, the formal inner product of Tf and g is defined as

$$(Tf,g) := \int_0^\infty \int_0^\infty \frac{(\min\{1,xy\})^\alpha (\max\{1,xy\})^\beta f(x)g(y)}{e^{\gamma xy}}\, dx\, dy.$$

With regard to (17), we have

$$\left\| T(f) \right\|_{p,\psi^{1-p}}^p = \int_0^\infty \psi^{1-p}(y) \left| T(f) \right|^p dy < C^p(\alpha,\beta,\gamma,\theta) \|f\|_{p,\varphi}^p < \infty. \tag{20}$$

According to the expression (20), the operator T is bounded, that is,

$$\|T\| := \sup_{f(\neq 0) \in L_{\varphi^{1-p}}^p(0,\infty)} \frac{\|T(f)\|_{p,\varphi^{1-p}}}{\|f\|_{p,\varphi}} \leq C(\alpha,\beta,\gamma,\theta).$$

Because the constant factor $C(\alpha,\beta,\gamma,\theta)$ is optimal, therefore $\|T\| = C(\alpha,\beta,\gamma,\theta)$.

Theorem 3 *By the preceding Theorems* 1 *and* 2, *inequalities* (15) *and* (17) *can be expressed as the following operator expressions with norm*:

$$(Tf, g) < \|T\| \|f\|_{p,\varphi} \|g\|_{q,\psi}, \tag{21}$$

$$\|T(f)\|_{p,\psi^{1-p}}^p < \|T\|^p \|f\|_{p,\varphi}^p. \tag{22}$$

5 Simple applications

We select the appropriate parameter values in (15) and (17) (first selecting the values of γ and θ, then determining the range of α and β) and calculate the value of constant factor $C(\alpha, \beta, \gamma, \theta)$ using Maple mathematical software. At the same time, combining with the representation methods of (21) and (22), some results in the references and some handsome Hilbert-type integral inequalities and their equivalent forms with single kernels or mixed kernels are obtained.

Example 1 Letting $\gamma = 1$, $\theta = \frac{1}{2}$, $\alpha = \beta = 0$, $p = q = 2$, we can get $C(0, 0, 1, \frac{1}{2}) = \sqrt{\pi}$ by calculating formula (12). If $f, g > 0$, $\varphi(x) = 1$, satisfying $0 < \|f\|_2$, $\|g\|_2 < \infty$, then we obtain (2) and its equivalent form

$$\int_0^\infty \left[\int_0^\infty e^{-xy} f(x) \, dx \right]^2 dy < \pi \|f\|_2^2, \tag{23}$$

where the constant factor π is the best possible.

Example 2 Letting $\gamma = 0$, $\theta = \frac{1}{2}$, $\alpha = 1$, $\beta = -1$, $p = q = 2$, we can get $C(1, -1, 0, \frac{1}{2}) = \frac{8}{3}$ by calculating formula (12). If $f, g > 0$, $\varphi(x) = 1$, satisfying $0 < \|f\|_2$, $\|g\|_2 < \infty$, then we have the equivalent inequalities

$$\int_0^\infty \int_0^\infty \frac{\min\{1, xy\}}{\max\{1, xy\}} f(x) g(y) \, dx \, dy < \frac{8}{3} \|f\|_2 \|g\|_2, \tag{24}$$

$$\int_0^\infty \left[\int_0^\infty \frac{\min\{1, xy\}}{\max\{1, xy\}} f(x) \, dx \right]^2 dy < \frac{64}{9} \|f\|_2^2, \tag{25}$$

where the constant factors $\frac{8}{3}$, $\frac{64}{9}$ are the best possible.

Example 3 Letting $\gamma = 0$, $\theta = -\frac{1}{2}$, $\alpha = 0$, $\beta = -2$, $p = q = 2$, we get $C(0, -2, 0, -\frac{1}{2}) = \frac{8}{3}$ by calculating formula (12). If $f, g > 0$, $\varphi(x) = \frac{1}{x^2}$, satisfying $0 < \|f\|_{2,\varphi}$, $\|g\|_{2,\varphi} < \infty$, then we have the equivalent inequalities

$$\int_0^\infty \int_0^\infty \frac{f(x) g(y)}{(\max\{1, xy\})^2} \, dx \, dy < \frac{8}{3} \|f\|_{2,\varphi} \|g\|_{2,\varphi}, \tag{26}$$

$$\int_0^\infty y^2 \left[\int_0^\infty \frac{f(x)}{(\max\{1, xy\})^2} \, dx \right]^2 dy < \frac{64}{9} \|f\|_{2,\varphi}^2, \tag{27}$$

where the constant factors $\frac{8}{3}$, $\frac{64}{9}$ are the best possible.

Example 4 Letting $\gamma = 0$, $\theta = \frac{3}{2}$, $\alpha = 2$, $\beta = 0$, $p = q = 2$, we get $C(2, 0, 0, \frac{3}{2}) = \frac{8}{3}$ by calculating formula (12). If $f, g > 0$, $\varphi(x) = x^2$, satisfying $0 < \|f\|_{2,\varphi}$, $\|g\|_{2,\varphi} < \infty$, then we have the

equivalent inequalities

$$\int_0^\infty \int_0^\infty \big(\min\{1, xy\}\big)^2 f(x)g(y)\,dx\,dy < \frac{8}{3}\|f\|_{2,\varphi}\|g\|_{2,\varphi}, \tag{28}$$

$$\int_0^\infty \frac{1}{y^2}\left[\int_0^\infty \big(\min\{1, xy\}\big)^2 f(x)\,dx\right]^2 dy < \frac{64}{9}\|f\|_{2,\varphi}^2, \tag{29}$$

where the constant factors $\frac{8}{3}$, $\frac{64}{9}$ are the best possible.

Example 5 Letting $\gamma = 1$, $\theta = 0$, $\alpha = 1$, $\beta = 0$, $p = q = 2$, we get $C(1, 0, 1, 0) = \frac{e-1}{e}$ by calculating formula (12). If $f, g > 0$, $\varphi(x) = \frac{1}{x}$, satisfying $0 < \|f\|_{2,\varphi}$, $\|g\|_{2,\varphi} < \infty$, then we have the equivalent inequalities

$$\int_0^\infty \int_0^\infty \frac{\min\{1, xy\}f(x)g(y)}{e^{xy}}\,dx\,dy < \frac{e-1}{e}\|f\|_{2,\varphi}\|g\|_{2,\varphi}, \tag{30}$$

$$\int_0^\infty y\left[\int_0^\infty \frac{\min\{1, xy\}f(x)}{e^{xy}}\,dx\right]^2 dy < \left(\frac{e-1}{e}\right)^2\|f\|_{2,\varphi}^2, \tag{31}$$

where the constant factors $\frac{e-1}{e}$, $(\frac{e-1}{e})^2$ are the best possible.

Example 6 Letting $\gamma = 1$, $\theta = 0$, $\alpha = 0$, $\beta = 1$, $p = q = 2$, we get $C(0, 1, 1, 0) = \frac{e+1}{e}$ by calculating formula (12). If $f, g > 0$, $\varphi(x) = \frac{1}{x}$, satisfying $0 < \|f\|_{2,\varphi}$, $\|g\|_{2,\varphi} < \infty$, then we have the equivalent inequalities

$$\int_0^\infty \int_0^\infty \frac{\max\{1, xy\}f(x)g(y)}{e^{xy}}\,dx\,dy < \frac{e+1}{e}\|f\|_{2,\varphi}\|g\|_{2,\varphi}, \tag{32}$$

$$\int_0^\infty y\left[\int_0^\infty \frac{\max\{1, xy\}f(x)}{e^{xy}}\,dx\right]^2 dy < \left(\frac{e+1}{e}\right)^2\|f\|_{2,\varphi}^2, \tag{33}$$

where the constant factors $\frac{e+1}{e}$, $(\frac{e+1}{e})^2$ are the best possible.

Example 7 Letting $\gamma = 1$, $\theta = \frac{1}{2}$, $\alpha = 1$, $\beta = -1$, $p = q = 2$, we get $C(1, -1, 1, \frac{1}{2}) = \frac{\sqrt{\pi}}{2}(5\,erf(1) - 4) + e^{-1} = 0.5570924045^+$ by calculating formula (12). If $f, g > 0$, $\varphi(x) = 1$, satisfying $0 < \|f\|_{2,\varphi}$, $\|g\|_{2,\varphi} < \infty$, then we have the equivalent inequalities

$$\int_0^\infty \int_0^\infty \frac{\min\{1, xy\}f(x)g(y)}{\max\{1, xy\}e^{xy}}\,dx\,dy < \left(\frac{\sqrt{\pi}}{2}\big(5\,\mathrm{erf}(1) - 4\big) + e^{-1}\right)\|f\|_{2,\varphi}\|g\|_{2,\varphi}, \tag{34}$$

$$\int_0^\infty \left[\int_0^\infty \frac{\min\{1, xy\}f(x)}{\max\{1, xy\}e^{xy}}\,dx\right]^2 dy < \left(\frac{\sqrt{\pi}}{2}\big(5\,\mathrm{erf}(1) - 4\big) + e^{-1}\right)^2\|f\|_{2,\varphi}^2, \tag{35}$$

where the constant factors $\frac{\sqrt{\pi}}{2}(5\,\mathrm{erf}(1) - 4) + e^{-1}$, $(\frac{\sqrt{\pi}}{2}(5\,\mathrm{erf}(1) - 4) + e^{-1})^2$ are the best possible, and $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$ is an error function.

Competing interests
The author declares that they have no competing interests.

Authors' contributions
The author read and approved the final version of the manuscript.

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