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Convergence properties of the maximum partial sums for moving average process under ρ^--mixing assumption

Mingzhu Song[1] and Quanxin Zhu[2,3]*

*Correspondence: zqx22@126.com
[2]Key Laboratory of HPC-SIP (MOE), College of Mathematics and Statistics, Hunan Normal University, Changsha, China
[3]School of Mathematical Sciences, Qufu Normal University, Qufu, China
Full list of author information is available at the end of the article

Abstract

In this paper, we assume that the sequence $\{Y_n, -\infty < n < +\infty\}$ is a ρ^--mixing random variables which are stochastically dominated by a random variable Y. Moreover, a real number sequence $\{a_n, -\infty < n < +\infty\}$ is assumed to be absolute summable. Then, complete convergence and complete γ-order moment convergence of the maximum partial sums for the moving average process $\{X_n = \sum_{j=-\infty}^{+\infty} a_j Y_{n+j}, n \geq 1\}$ are obtained. The results in this paper extend and improve the corresponding ones under NA and ρ-mixing conditions in the literature.

MSC: 60F15; 62G05

Keywords: Moving average process; ρ^--mixing sequence; Complete convergence; Maximum partial sums; Complete γ-order moment convergence

1 Introduction

Let the $\{Y_n, -\infty < n < +\infty\}$ be a random variable sequence and assume that a real number sequence $\{a_n, -\infty < n < +\infty\}$ is absolute summable, i.e., $\sum_{j=-\infty}^{+\infty} |a_j| < \infty$. Then $\{X_n, n \geq 1\}$ is called a moving average process of the sequence $\{Y_n, -\infty < n < +\infty\}$, if

$$X_n = \sum_{j=-\infty}^{+\infty} a_j Y_{j+n}, \quad n \geq 1, \tag{1.1}$$

and $S_n = \sum_{i=1}^{n} X_i$, $n \geq 1$ are the partial sums of $\{X_n, n \geq 1\}$.

The complete convergence plays a very important role in the probability theory and mathematical statistics, which was introduced by Hsu and Robbins [1]. A sequence $\{X_n, n \geq 1\}$ of random variables is said to converge completely to a constant θ, if

$$\sum_{n=1}^{\infty} P\bigl(|X_n - \theta| > \varepsilon\bigr) < \infty, \quad \forall \varepsilon > 0.$$

In view of the Borel–Cantelli lemma, the complete convergence implies that $X_n \to \theta$ almost surely. Therefore, complete convergence is a very important tool in establishing almost sure convergence for partial sums of random variables, as well as weighted sums of random variables.

Let $\{X_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $\gamma > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|X_n| - \varepsilon\}_+^{\gamma} < \infty, \quad \forall \varepsilon > 0,$$

then the above result was called the complete γ-order moment convergence by Chow [2]. We observe that the complete γ-order moment convergence is more general than complete convergence.

After the moving average process was defined, many conclusions about its convergence properties have been obtained. When the sequence $\{Y_n, -\infty < n < +\infty\}$ is identically distributed, many results about the moving average process $\{X_n, n \geq 1\}$ are known. For example, Ibragimov [3] obtained several limit theorems, Burton and Dehling [4] established a large deviation result, Li et al. [5] got the complete convergence property, Chen and Wang [6] obtained convergence rates of moderate deviations, and so on. Recently, several results have been obtained under the assumption that the sequence $\{Y_n, -\infty < n < +\infty\}$ is dependent. For example, Zhang [7] and Chen et al. [8] discussed the limiting behavior of moving average processes under φ-mixing assumptions; Baek et al. [9] and Yu and Wang [10] established complete convergence under NA assumptions; Li and Zhang [11], Kim and Ko [12–14] and Zhou [15, 16] discussed the complete moment convergence under NA, φ-mixing and ρ-mixing random variables, and Qiu and Chen [17] obtained the complete moment convergence under END.

In this paper, we aim to study some convergence properties of the maximum partial sums for a moving average process under ρ^--mixing sequence assumption.

Let T and S be two nonempty disjoint sets of integers, and then define $\mathrm{dist}(T,S) =: \min\{|j-k|; j \in T, k \in S\}$, and let the σ-field $\sigma(T)$ ($\sigma(S)$) be generated by $\{Y_k, k \in T\}$ ($\{Y_k, k \in S\}$).

Definition 1.1 A sequence $\{Y_n, -\infty < n < +\infty\}$ is said to be ρ^--mixing, if

$$\rho^-(s) = \sup\{\rho^-(T,S); T,S \subset Z, \mathrm{dist}(T,S) \geq s\} \to 0, \quad \text{as } s \to \infty,$$
$$\rho^-(T,S) = 0 \vee \sup\{\mathrm{corr}(f(Y_i, i \in T), g(Y_j, j \in S))\},$$

where the supremum is taken over all increasing real functions f on R^T and g on R^S.

Definition 1.2 A sequence $\{Y_n, -\infty < n < +\infty\}$ is said to be ρ-mixing, if

$$\rho(s) = \sup\{\rho(T,S); T,S \subset Z, \mathrm{dist}(T,S) \geq s\} \to 0, \quad \text{as } s \to \infty,$$

where

$$\rho(T,S) = \sup\{|\mathrm{corr}(f,g)|; f \in L_2(\sigma(T)), f \in L_2(\sigma(S))\}.$$

Definition 1.3 A sequence $\{Y_n, -\infty < n < +\infty\}$ is said to be negatively associated (NA), if

$$\mathrm{Cov}\{f(Y_i, i \in S), g(Y_j, j \in T)\} \leq 0,$$

holds for every pair of disjoint subsets $T, S \subset Z$, and coordinatewise increasing functions f on R^S and g on R^T.

From the above definitions, we see that ρ^--mixing includes both ρ-mixing and the NA sequence. Recently many scholars have focused on the convergence properties of moving average processes under ρ-mixing and NA condition. For example, Wang and Lu [18] obtained weak convergence and Rosenthal-type moment inequality; Budsaba et al. [19, 20] and Zhang [21] got complete convergence for the moving average process under ρ-mixing and ρ^--mixing conditions; Tan et al. [22] gained the central limit theorem of partial sums for ρ-mixing sequences. However, to the best of our knowledge, there have been few results about the complete γ-order moment convergence for the moving average process under ρ^--mixing condition. The main aim of this paper is to present some results about complete convergence and complete γ-order moment convergence for the maximum partial sums of $\{X_n, n \geq 1\}$ under ρ^--mixing condition.

Definition 1.4 The sequence $\{Y_n, -\infty < n < +\infty\}$ is called be stochastically dominated by a random variable Y, if for all $x > 0$

$$P\big(|Y_n| > x\big) \leq DP\big(|Y| > x\big), \quad -\infty < n < +\infty,$$

where the constant $D > 0$; we denote $\{Y_n, -\infty < n < +\infty\} \prec Y$.

Throughout this paper, $I(A)$ denotes the indicator function of an event A, the symbol C represents a positive constant, which can take different values in different places, even in the same formula.

2 Several lemmas and main results

Lemma 2.1 (Wang and Lu [18]) *Let $p \geq 2$ and consider a ρ^--mixing random variables sequence of $\{Y_n, -\infty < n < +\infty\}$. For every $j \geq 1$, assume $EY_j = 0$ and $E|Y_j|^p < \infty$. Then, there exists a positive constant $C = C(q, \rho^-(\cdot))$ such that*

$$E\left(\max_{1\leq k\leq n}\left|\sum_{j=1}^{k} Y_j\right|^p\right) \leq C\left\{\sum_{j=1}^{n} E|Y_j|^p + \left(\sum_{j=1}^{n} EY_j^2\right)^{p/2}\right\}. \tag{2.1}$$

Lemma 2.2 (Wang and Lu [18]) *Let the sequence $\{Y_n, -\infty < n < +\infty\}$ be ρ^--mixing. If the functions of the sequence $\{f_n, -\infty < n < +\infty\}$ are all non-decreasing (non-increasing), then the sequence $\{f_n(Y_n), -\infty < n < +\infty\}$ is still ρ^--mixing.*

Lemma 2.3 (Wu [23]) *Let $\{Y_n, -\infty < n < +\infty\} \prec Y$. Then for all constants $s, t > 0$, there exist positive constants C_1, C_2 such that the following inequalities are true:*

$$E|Y_n|^s I\big(|Y_n| \leq t\big) \leq C_1\big(E|Y|^s I\big(|Y| \leq t\big) + t^s P\big(|Y| > t\big)\big),$$
$$E|Y_n|^s I\big(|Y_n| > t\big) \leq C_2 E|Y|^s I\big(|Y| > t\big).$$

We now present our main results, their proofs will be given in the next section.

Theorem 2.1 *Let $1 \leq p < 2$, $\alpha > 1$. Assume that $\{X_n, n \geq 1\}$ is a moving average process of a ρ^--mixing random variable sequence $\{Y_n, -\infty < n < +\infty\}$, and $\{Y_n, -\infty < n < +\infty\} \prec Y$. If $EY_j = 0$ for every j and $E|Y|^{ap} < \infty$, then*

$$\sum_{n=1}^{\infty} n^{\alpha-2} P\Big\{\max_{1\leq k\leq n} |S_k| \geq \varepsilon n^{1/p}\Big\} < \infty \quad \forall \varepsilon > 0. \tag{2.2}$$

If $\alpha = 1$, we have the following result.

Theorem 2.2 *Let $1 \leq p < 2$. Suppose the $\sum_{i=-\infty}^{+\infty} |a_i|^{\theta} < \infty$, where $\theta \in (0,1)$ if $p = 1$, and $\theta = 1$ if $1 < p < 2$. Let $\{X_n, n \geq 1\}$ be a moving average process of a ρ^--mixing random variable sequence $\{Y_n, -\infty < n < +\infty\}$, and $\{Y_n, -\infty < n < +\infty\} \prec Y$. If $EY_j = 0$ for every j and $E|Y|^p < \infty$, then*

$$\sum_{n=1}^{\infty} n^{-1} P\Big\{\max_{1\leq k\leq n} |S_k| \geq \varepsilon n^{1/p}\Big\} < \infty \quad \forall \varepsilon > 0. \tag{2.3}$$

Theorem 2.3 *Let $\gamma > 0$, $1 \leq p < 2$, $\alpha > 1$. Assume that $\{X_n, n \geq 1\}$ is a moving average process of a ρ^--mixing random variable sequence $\{Y_n, -\infty < n < +\infty\}$, and $\{Y_n, -\infty < n < +\infty\} \prec Y$. If $EY_j = 0$ for every j, and*

$$\begin{cases} E|Y|^{\alpha p} < \infty, & \text{if } \gamma < \alpha p, \\ E|Y|^{\alpha p} \log(1 + |Y|) < \infty, & \text{if } \gamma = \alpha p, \\ E|Y|^{\gamma} < \infty, & \text{if } \gamma > \alpha p, \end{cases}$$

then

$$\sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p} E\Big\{\max_{1\leq k\leq n} |S_k| - \varepsilon n^{1/p}\Big\}_+^{\gamma} < \infty \quad \forall \varepsilon > 0 \tag{2.4}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha-2} E\Big\{\sup_{k\geq n} \big|k^{-1/p} S_k\big| - \varepsilon\Big\}_+^{\gamma} < \infty \quad \forall \varepsilon > 0. \tag{2.5}$$

If $\alpha = 1$, we have the following result.

Theorem 2.4 *Let $\gamma > 0$, $1 \leq p < 2$, Suppose $\sum_{i=-\infty}^{+\infty} |a_i|^{\theta} < \infty$, as well as $\theta \in (0,1)$ if $p = 1$, and $\theta = 1$ if $1 < p < 2$. Let $\{X_n, n \geq 1\}$ be a moving average process of a ρ^--mixing random variable sequence $\{Y_n, -\infty < n < +\infty\}$, and $\{Y_n, -\infty < n < +\infty\} \prec Y$. If $EY_j = 0$ for every j, and*

$$\begin{cases} E|Y|^{p} < \infty, & \text{if } \gamma < p, \\ E|Y|^{p} \log(1 + |Y|) < \infty, & \text{if } \gamma = p, \\ E|Y|^{\gamma} < \infty, & \text{if } \gamma > p, \end{cases}$$

then

$$\sum_{n=1}^{\infty} n^{-1-\gamma/p} E\Bigl\{\max_{1\le k\le n} |S_k| - \varepsilon n^{1/p}\Bigr\}_+^{\gamma} < \infty \quad \forall \varepsilon > 0. \tag{2.6}$$

Remark 2.1 Theorems 2.1 and 2.2 give complete convergence of the moving average processes $\{X_n, n \ge 1\}$ for ρ^--mixing random variable sequences. It is worth pointing out that slowly varying functions are not needed in this paper, so our Theorems 2.1 and 2.2 extend and improve those in Chen [8], Budsaba [19, 20] and Zhang [21].

Remark 2.2 Theorems 2.3 and 2.4 give complete γ-order moment convergence of the moving average processes $\{X_n, n \ge 1\}$ for ρ^--mixing random variable sequences. Noting that the ρ^--mixing condition generalizes both NA and ρ-mixing conditions, our Theorems 2.3 and 2.4 also extend and improve the corresponding results of Li and Zhang [11], Budsaba [19, 20] and Tan [22].

Remark 2.3 It should be mentioned that our condition $\{Y_n, -\infty < n < +\infty\} \prec Y$ is weaker than the assumption of identical distribution for $\{Y_n, -\infty < n < +\infty\}$. Thus, our results still hold for identically distributed random variables.

Remark 2.4 Many dependent random variables satisfy condition (2.1). For example, one can consider NA, ρ-mixing and φ-mixing random variable sequences. Therefore, our results extend and improve the corresponding results of Zhou [15, 16] and Qiu [17].

3 Proof of theorems

Proof of Theorem 2.1 Write

$$Y_j' = -n^{1/p} I\bigl(Y_j < -n^{1/p}\bigr) + Y_j I\bigl(|Y_j| \le n^{1/p}\bigr) + n^{1/p} I\bigl(Y_j > n^{1/p}\bigr),$$
$$Y_j'' = Y_j - Y_j' = \bigl(Y_j + n^{1/p}\bigr) I\bigl(Y_j < -n^{1/p}\bigr) + \bigl(Y_j - n^{1/p}\bigr) I\bigl(Y_j > n^{1/p}\bigr).$$

Note that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \sum_{j=-\infty}^{+\infty} a_j Y_{k+j} = \sum_{j=-\infty}^{+\infty} a_j \sum_{i=j+1}^{j+n} Y_i.$$

Since $\sum_{j=-\infty}^{+\infty} |a_j| < \infty$, $EY_i = 0$, it follows from Lemma 2.3 that

$$\begin{aligned} & n^{-1/p} \left| E \max_{1\le k\le n} \sum_{j=-\infty}^{+\infty} a_j \sum_{i=j+1}^{j+k} Y_i' \right| \quad (\text{since } EY_i = 0) \\ & \le n^{-1/p} \sum_{j=-\infty}^{+\infty} |a_j| \sum_{i=j+1}^{j+n} \bigl(n^{1/p} P\bigl(|Y_i| > n^{1/p}\bigr) + E|Y_i| I\bigl(|Y_i| > n^{1/p}\bigr)\bigr) \\ & \le C n^{1-1/p} E|Y| I\bigl(|Y| > n^{1/p}\bigr) \\ & \le C E|Y|^p I\bigl(|Y| > n^{1/p}\bigr) \to 0, \quad n \to \infty. \end{aligned}$$

Therefore, for large enough n, we obtain

$$n^{-1/p}\left|E\max_{1\le k\le n}\sum_{j=-\infty}^{+\infty}a_j\sum_{i=j+1}^{j+k}Y_i'\right|<\varepsilon/4. \tag{3.1}$$

By (3.1), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}n^{\alpha-2}P\Big\{\max_{1\le k\le n}|S_k|\ge\varepsilon n^{1/p}\Big\}\\
&=\sum_{n=1}^{\infty}n^{\alpha-2}P\left\{\max_{1\le k\le n}\left|\sum_{j=-\infty}^{+\infty}a_j\sum_{i=j+1}^{j+k}Y_i\right|\ge\varepsilon n^{1/p}\right\}\\
&\le\sum_{n=1}^{\infty}n^{\alpha-2}P\left\{\max_{1\le k\le n}\left|\sum_{j=-\infty}^{+\infty}a_j\sum_{i=j+1}^{j+k}Y_i''\right|\ge\varepsilon n^{1/p}/2\right\}\\
&\quad+\sum_{n=1}^{\infty}n^{\alpha-2}P\left\{\max_{1\le k\le n}\left|\sum_{j=-\infty}^{+\infty}a_j\sum_{i=j+1}^{j+k}\big(Y_i'-EY_i'\big)\right|\ge\varepsilon n^{1/p}/4\right\}\\
&=:I_1+I_2.
\end{aligned}$$

For I_1, noting $\alpha p>1$, it follows from Lemma 2.3 and the Markov inequality that

$$\begin{aligned}
I_1&\le C\sum_{n=1}^{\infty}n^{\alpha-2-1/p}E\max_{1\le k\le n}\left|\sum_{j=-\infty}^{\infty}a_j\sum_{i=j+1}^{j+k}Y_i''\right|\\
&\le C\sum_{n=1}^{\infty}n^{\alpha-2-1/p}\sum_{j=-\infty}^{+\infty}|a_j|\sum_{i=j+1}^{j+n}E|Y_i|I\big(|Y_i|>n^{1/p}\big)\\
&\le C\sum_{n=1}^{\infty}n^{\alpha-1-1/p}E|Y|I\big(|Y|>n^{1/p}\big)\\
&=C\sum_{n=1}^{\infty}n^{\alpha-1-1/p}\sum_{l=n}^{\infty}E|Y|I\big(l^{1/p}<|Y|\le(l+1)^{1/p}\big)\\
&\le C\sum_{l=1}^{\infty}E|Y|I\big(l^{1/p}<|Y|\le(l+1)^{1/p}\big)\sum_{n=1}^{l}n^{\alpha-1-1/p}\\
&\le C\sum_{l=1}^{\infty}l^{\alpha-1/p}E|Y|I\big(l^{1/p}<|Y|\le(l+1)^{1/p}\big)\le E|Y|^{\alpha p}<\infty.
\end{aligned}\tag{3.2}$$

For I_2, from Lemma 2.2, we see that $\{Y_j'-EY_j'\}$ is still a ρ^--mixing random variable sequence. By Lemma 2.1, Markov and Hölder inequalities, we have that for any $\nu\ge2$,

$$\begin{aligned}
I_2&\le C\sum_{n=1}^{\infty}n^{\alpha-2-\nu/p}E\left\{\max_{1\le k\le n}\left|\sum_{j=-\infty}^{+\infty}a_j\left(\sum_{i=j+1}^{j+k}\big(Y_i'-EY_i'\big)\right)\right|\right\}^{\nu}\\
&\le C\sum_{n=1}^{\infty}n^{\alpha-2-\nu/p}E\left\{\sum_{j=-\infty}^{+\infty}|a_j|\max_{1\le k\le n}\left|\sum_{i=j+1}^{j+k}\big(Y_i'-EY_i'\big)\right|\right\}^{\nu}
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{n=1}^{\infty} n^{\alpha-2-\nu/p} E\left\{\sum_{j=-\infty}^{+\infty} |a_j|^{1-1/\nu}\left(|a_j|^{1/\nu}\max_{1\le k\le n}\left|\sum_{i=j+1}^{j+k}\left(Y_i'-EY_i'\right)\right|\right)\right\}^{\nu} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha-2-\nu/p}\left(\sum_{j=-\infty}^{+\infty}|a_j|\right)^{\nu-1}\left(\sum_{j=-\infty}^{+\infty}|a_j| E\left(\max_{1\le k\le n}\left|\sum_{i=j+1}^{j+k}\left(Y_i'-EY_i'\right)\right|^{\nu}\right)\right) \\
&\le C\sum_{n=1}^{\infty} n^{\alpha-2-\nu/p}\sum_{j=-\infty}^{+\infty}|a_j|\left\{\sum_{i=j+1}^{j+n} E\left|Y_i'\right|^{\nu}+\left(\sum_{i=j+1}^{j+n} E\left|Y_i'\right|^2\right)^{\nu/2}\right\} \\
&= C\sum_{n=1}^{\infty} n^{\alpha-2-\nu/p}\sum_{j=-\infty}^{+\infty}|a_j|\left\{\sum_{i=j+1}^{j+n}\left\{E|Y_i|^{\nu} I\left(|Y_i|\le n^{1/p}\right)+n^{\nu/p}P\left(|Y_i|>n^{1/p}\right)\right\}\right\} \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha-2-\nu/p}\sum_{j=-\infty}^{+\infty}|a_j|\left\{\sum_{i=j+1}^{j+n}\left\{E|Y_i|^{2} I\left(|Y_i|\le n^{1/p}\right)+n^{2/p}P\left(|Y_i|>n^{1/p}\right)\right\}^{\nu/2}\right\} \\
&\le C\sum_{n=1}^{\infty} n^{\alpha-1-\nu/p}\left\{E|Y|^{\nu} I\left(|Y|\le n^{1/p}\right)+n^{\nu/p}P\left(|Y|>n^{1/p}\right)\right\} \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha-2-\nu/p+\nu/2}\left\{E|Y|^{2} I\left(|Y|\le n^{1/p}\right)+n^{2/p}P\left(|Y|>n^{1/p}\right)\right\}^{\nu/2} \\
&=: I_{21}+I_{22}.
\end{aligned}
$$

For I_{21}, taking $\nu \ge \alpha p$, we have

$$
\begin{aligned}
I_{21} &= C\sum_{n=1}^{\infty} n^{\alpha-1-\nu/p} E|Y|^{\nu} I\left(|Y|\le n^{1/p}\right)+C\sum_{n=1}^{\infty} n^{\alpha-1}P\left(|Y|>n^{1/p}\right) \\
&= C\sum_{n=1}^{\infty} n^{\alpha-1-\nu/p}\sum_{l=1}^{n} E|Y|^{\nu} I\left((l-1)^{1/p}<|Y|\le l^{1/p}\right) \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha-1}\sum_{l=n}^{\infty} P\left(l^{1/p}<|Y|\le (l+1)^{1/p}\right) \\
&= C\sum_{l=1}^{\infty} E|Y|^{\nu} I\left((l-1)^{1/p}<|Y|\le l^{1/p}\right)\sum_{n=l}^{\infty} n^{\alpha-1-\nu/p} \\
&\quad + C\sum_{l=1}^{\infty} l^{\alpha} P\left(l^{1/p}<|Y|\le (l+1)^{1/p}\right) \\
&\le C\sum_{l=1}^{\infty} l^{\alpha-\nu/p} E|Y|^{\nu} I\left((l-1)^{1/p}<|Y|\le l^{1/p}\right)+CE|Y|^{\alpha p} \\
&\le CE|Y|^{\alpha p}<\infty.
\end{aligned}
\tag{3.3}
$$

For I_{22}, we will prove the claim based on two cases.

If $\alpha p < 2$, setting $\nu = 2$, similar to the proof of I_{21}, we obtain

$$
I_{22} = C\sum_{n=1}^{\infty} n^{\alpha-1-2/p}\left\{E|Y|^{2} I\left(|Y|\le n^{1/p}\right)+n^{2/p}P\left(|Y|>n^{1/p}\right)\right\}<\infty. \tag{3.4}
$$

If $\alpha p \geq 2$, noting that $E|Y|^2 < \infty$, choosing $\nu > \max\{2, (\alpha - 1)2p/(2 - p)\}$, we have that $\alpha - \nu/p + \nu/2 < 1$, and therefore

$$I_{22} \leq C \sum_{n=1}^{\infty} n^{\alpha-2-\nu/p+\nu/2} \bigl\{E|Y|^2\bigr\}^{\nu/2} < \infty. \tag{3.5}$$

Thus, from (3.2)–(3.5), we see that (2.2) is satisfied. □

Next, we prove Theorem 2.2.

Proof of Theorem 2.2 From the proof of Theorem 2.1, we only need to prove $I_1 < \infty$, $I_2 < \infty$.

For I_1, noting $\theta \in (0,1)$ if $p = 1$, and $\theta = 1$ if $1 < p < 2$, by the Markov and C_r-inequalities, we have

$$\begin{aligned} I_1 &\leq C \sum_{n=1}^{\infty} n^{-1-\theta/p} E \max_{1\leq k\leq n} \left| \sum_{j=-\infty}^{+\infty} a_j \sum_{i=j+1}^{j+k} Y_i'' \right|^{\theta} \\ &\leq C \sum_{n=1}^{\infty} n^{-1-\theta/p} \sum_{j=-\infty}^{+\infty} |a_j|^{\theta} \sum_{i=j+1}^{j+n} E|Y_i|^{\theta} I\bigl(|Y_i| > n^{1/p}\bigr) \\ &\leq C \sum_{n=1}^{\infty} n^{-\theta/p} E|Y|^{\theta} I\bigl(|Y| > n^{1/p}\bigr) \\ &= C \sum_{n=1}^{\infty} n^{-\theta/p} \sum_{l=n}^{\infty} E|Y|^{\theta} I\bigl(l^{1/p} < |Y| \leq (l+1)^{1/p}\bigr) \\ &= C \sum_{l=1}^{\infty} E|Y|^{\theta} I\bigl(l^{1/p} < |Y| \leq (l+1)^{1/p}\bigr) \sum_{n=1}^{l} n^{-\theta/p} \\ &\leq C \sum_{l=1}^{\infty} l^{1-\theta/p} E|Y|^{\theta} I\bigl(l^{1/p} < |Y| \leq (l+1)^{1/p}\bigr) < CE|Y|^p < \infty. \end{aligned} \tag{3.6}$$

For I_2, by Markov and Hölder inequalities, as well as Lemmas 2.1–2.3, we obtain

$$\begin{aligned} I_2 &= \sum_{n=1}^{\infty} n^{-1-2/p} E\left\{ \max_{1\leq k\leq n} \left| \sum_{j=-\infty}^{+\infty} a_j \left(\sum_{i=j+1}^{j+k} (Y_i' - EY_i') \right) \right|^2 \right\} \\ &= \sum_{n=1}^{\infty} n^{-1-2/p} E\left\{ \sum_{j=-\infty}^{+\infty} |a_j|^{1/2} \left(|a_j|^{1/2} \max_{1\leq k\leq n} \left| \sum_{i=j+1}^{j+k} (Y_i' - EY_i') \right| \right) \right\}^2 \\ &\leq \sum_{n=1}^{\infty} n^{-1-2/p} \left(\sum_{j=-\infty}^{+\infty} |a_j| \right) \left(\sum_{j=1}^{\infty} |a_j| E\left(\max_{1\leq k\leq n} \left| \sum_{i=j+1}^{j+k} (Y_i' - EY_i') \right|^2 \right) \right) \\ &\leq C \sum_{n=1}^{\infty} n^{-1-2/p} \sum_{j=-\infty}^{+\infty} |a_j| \sum_{i=j+1}^{j+n} E\bigl|Y_i'\bigr|^2 \\ &\leq C \sum_{n=1}^{\infty} n^{-1-2/p} \sum_{j=-\infty}^{+\infty} |a_j| \left\{ \sum_{i=j+1}^{j+n} \bigl\{ E|Y_i|^2 I\bigl(|Y_i| \leq n^{1/p}\bigr) + n^{2/p} P\bigl(|Y_i| > n^{1/p}\bigr) \bigr\} \right\} \end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty} n^{-2/p}\left\{E|Y|^2 I\left(|Y|\leq n^{1/p}\right)+n^{2/p}P\left(|Y|>n^{1/p}\right)\right\} \\
&\leq C\sum_{n=1}^{\infty} n^{-2/p}\sum_{l=1}^{n} E|Y|^2 I\left((l-1)^{1/p}<|Y|\leq l^{1/p}\right)+CE|Y|^p \\
&\leq C\sum_{l=1}^{\infty} l^{1-2/p}E|Y|^2 I\left((l-1)^{1/p}<|Y|<l^{1/p}\right)+CE|Y|^p \\
&\leq CE|Y|^p<\infty.
\end{aligned} \tag{3.7}
$$

Combing (3.6) and (3.7), we see that Theorem 2.2 holds. □

Proof of Theorem 2.3 Let $x > n^{\gamma/p}$ and consider

$$
\begin{aligned}
&Y_j' = -x^{1/\gamma} I\left(Y_j < -x^{1/\gamma}\right) + Y_j I\left(|Y_j| \leq x^{1/\gamma}\right) + x^{1/\gamma} I\left(Y_j > x^{1/\gamma}\right), \\
&Y_j'' = Y_j - Y_j' = \left(Y_j + x^{1/\gamma}\right) I\left(Y_j < -x^{1/\gamma}\right) + \left(Y_j - x^{1/\gamma}\right) I\left(Y_j > x^{1/\gamma}\right).
\end{aligned}
$$

Similarly to the proof of (3.1), for large enough x, we get

$$
x^{-1/\gamma}\left|E\max_{1\leq k\leq n}\sum_{j=-\infty}^{+\infty} a_j \sum_{i=j+1}^{j+k} Y_i'\right| < \varepsilon/4. \tag{3.8}
$$

Then, we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p} E\left\{\max_{1\leq k\leq n}|S_k| - \varepsilon n^{1/p}\right\}_+^{\gamma} \\
&\quad= \sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p}\int_0^{\infty} P\left(\max_{1\leq k\leq n}|S_k| > \varepsilon n^{1/p} + x^{1/\gamma}\right)dx \\
&\quad\leq \sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p}\int_0^{n^{\gamma/p}} P\left(\max_{1\leq k\leq n}|S_k| > \varepsilon n^{1/p}\right)dx \\
&\qquad+ \sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p}\int_{n^{\gamma/p}}^{\infty} P\left(\max_{1\leq k\leq n}|S_k| > x^{1/\gamma}\right)dx \\
&\quad\leq \sum_{n=1}^{\infty} n^{\alpha-2} P\left(\max_{1\leq k\leq n}|S_k| > \varepsilon n^{1/p}\right) \\
&\qquad+ \sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p}\int_{n^{\gamma/p}}^{\infty} P\left(\max_{1\leq k\leq n}|S_k| > x^{1/\gamma}\right)dx.
\end{aligned}
$$

It follows from Theorem 2.1 that

$$
\sum_{n=1}^{\infty} n^{\alpha-2} P\left(\max_{1\leq k\leq n}|S_k| > \varepsilon n^{1/p}\right) < \infty. \tag{3.9}
$$

To prove (2.4) of Theorem 2.3, we only need to prove

$$I_3 =: \sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p} \int_{n^{\gamma/p}}^{\infty} P\Big(\max_{1\le k\le n} |S_k| > x^{1/\gamma}\Big)\,dx < \infty.$$

By (3.8), we get

$$\begin{aligned} I_3 &\le C\sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p} \int_{n^{\gamma/p}}^{\infty} P\left\{\max_{1\le k\le n}\left|\sum_{j=-\infty}^{+\infty} a_j \sum_{i=j+1}^{j+k} Y_i''\right| \ge x^{1/\gamma}/2\right\} dx \\ &\quad + C\sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p} \int_{n^{\gamma/p}}^{\infty} P\left\{\max_{1\le k\le n}\left|\sum_{j=-\infty}^{+\infty} a_j \sum_{i=j+1}^{j+k} (Y_i' - EY_i')\right| \ge x^{1/\gamma}/4\right\} dx \\ &=: I_{31} + I_{32}. \end{aligned}$$

For I_{31}, by Markov inequality and Lemma 2.3, we get

$$\begin{aligned} I_{31} &\le C\sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p} \int_{n^{\gamma/p}}^{\infty} x^{-1/\gamma} E\max_{1\le k\le n}\left|\sum_{j=-\infty}^{+\infty} a_j \sum_{i=j+1}^{j+k} Y_i''\right| dx \\ &\le C\sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p} \int_{n^{\gamma/p}}^{\infty} x^{-1/\gamma} \sum_{j=-\infty}^{+\infty} |a_j| \sum_{i=j+1}^{j+n} E\big|Y_i''\big|\, dx \\ &\le C\sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p} \int_{n^{\gamma/p}}^{\infty} x^{-1/\gamma} \sum_{j=-\infty}^{+\infty} |a_j| \sum_{i=j+1}^{j+n} E|Y_i| I\big(|Y_i| > x^{1/\gamma}\big)\, dx \\ &\le C\sum_{n=1}^{\infty} n^{\alpha-1-\gamma/p} \int_{n^{\gamma/p}}^{\infty} x^{-1/\gamma} E|Y| I\big(|Y| > x^{1/\gamma}\big)\, dx \\ &= C\sum_{n=1}^{\infty} n^{\alpha-1-\gamma/p} \sum_{l=n}^{\infty} \int_{l^{\gamma/p}}^{(l+1)^{\gamma/p}} x^{-1/\gamma} E|Y| I\big(|Y| > x^{1/\gamma}\big)\, dx \\ &\le C\sum_{n=1}^{\infty} n^{\alpha-1-\gamma/p} \sum_{l=n}^{\infty} l^{\gamma/p(1-1/\gamma)-1} E|Y| I\big(|Y| > l^{1/p}\big) \\ &= C\sum_{l=1}^{\infty} l^{\gamma/p-1/p-1} E|Y| I\big(|Y| > l^{1/p}\big) \sum_{n=1}^{l} n^{\alpha-1-\gamma/p} \\ &\le \begin{cases} C\sum_{l=1}^{\infty} l^{\gamma/p-1/p-1} E|Y| I(|Y| > l^{1/p}) l^{\alpha-\gamma/p}, & \text{if } \gamma < \alpha p, \\ C\sum_{l=1}^{\infty} l^{\gamma/p-1/p-1} E|Y| I(|Y| > l^{1/p}) \log(l+1), & \text{if } \gamma = \alpha p, \\ C\sum_{l=1}^{\infty} l^{\gamma/p-1/p-1} E|Y| I(|Y| > l^{1/p}), & \text{if } \gamma > \alpha p \end{cases} \\ &\le \begin{cases} C\sum_{l=1}^{\infty} l^{\alpha-1/p} E|Y| I(|Y| > l^{1/p}), & \text{if } \gamma < \alpha p, \\ C\sum_{l=1}^{\infty} l^{\alpha-1/p} \log(l+1) E|Y| I(|Y| > l^{1/p}), & \text{if } \gamma = \alpha p, \\ C\sum_{l=1}^{\infty} l^{\gamma/p-1/p} E|Y| I(|Y| > l^{1/p}), & \text{if } \gamma > \alpha p \end{cases} \\ &\le \begin{cases} CE|Y|^{\alpha p} < \infty, & \text{if } \gamma < \alpha p, \\ CE|Y|^{\alpha p}\log(|Y|+1) < \infty, & \text{if } \gamma = \alpha p, \\ CE|Y|^{\gamma} < \infty, & \text{if } \gamma > \alpha p. \end{cases} \end{aligned} \tag{3.10}$$

For I_{32}, from Lemmas 2.1–2.3, Markov and Hölder inequalities, we have that for any $\nu \geq 2$,

$$\begin{aligned}
I_{32} &\leq C\sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p}\int_{n^{\gamma/p}}^{\infty} x^{-\nu/\gamma} E\left\{\max_{1\leq k\leq n}\left|\sum_{j=-\infty}^{+\infty} a_j \sum_{i=j+1}^{j+k}\left(Y_i' - EY_i'\right)\right|\right\}^{\nu} dx \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p}\int_{n^{\gamma/p}}^{\infty} x^{-\nu/\gamma} E\left\{\sum_{j=-\infty}^{+\infty} |a_j| \max_{1\leq k\leq n}\left|\sum_{i=j+1}^{j+k}\left(Y_i' - EY_i'\right)\right|\right\}^{\nu} dx \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p}\int_{n^{\gamma/p}}^{\infty} x^{-\nu/\gamma} E\left\{\sum_{j=-\infty}^{+\infty} |a_j|^{1-1/\nu}\left(|a_j|^{1/\nu} \max_{1\leq k\leq n}\left|\sum_{i=j+1}^{j+k}\left(Y_i' - EY_i'\right)\right|\right)\right\}^{\nu} dx \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p}\int_{n^{\gamma/p}}^{\infty} x^{-\nu/\gamma} \left(\sum_{j=-\infty}^{+\infty} |a_j|\right)^{\nu-1} \\
&\quad\times \left(\sum_{j=-\infty}^{+\infty} |a_j| E\left(\max_{1\leq k\leq n}\left|\sum_{i=j+1}^{j+k}\left(Y_i' - EY_i'\right)\right|^{\nu}\right)\right) dx \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p}\int_{n^{\gamma/p}}^{\infty} x^{-\nu/\gamma} \sum_{j=-\infty}^{+\infty} |a_j| \left\{\sum_{i=j+1}^{j+n} E\left|Y_i'\right|^{\nu} + \left(\sum_{i=j+1}^{j+n} E\left|Y_j'\right|^{2}\right)^{\nu/2}\right\} dx \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p}\int_{n^{\gamma/p}}^{\infty} x^{-\nu/\gamma} \sum_{j=-\infty}^{+\infty} |a_j| \\
&\quad\times \left\{\sum_{i=j+1}^{j+n}\left\{E|Y_i|^{\nu} I\left(|Y_i| \leq x^{1/\gamma}\right) + x^{\nu/\gamma} P\left(|Y_i| > x^{1/\gamma}\right)\right\}\right\} dx \\
&\quad + C\sum_{n=1}^{+\infty} n^{\alpha-2-\gamma/p}\int_{n^{\gamma/p}}^{\infty} x^{-\nu/\gamma} \sum_{j=-\infty}^{+\infty} |a_j| \\
&\quad\times \left\{\sum_{i=j+1}^{j+n}\left\{E|Y_i|^{2} I\left(|Y_i| \leq x^{1/\gamma}\right) + x^{2/\gamma} P\left(|Y_i| > x^{1/\gamma}\right)\right\}\right\}^{\nu/2} dx \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha-1-\gamma/p}\int_{n^{\gamma/p}}^{\infty} x^{-\nu/\gamma} \left\{E|Y|^{\nu} I\left(|Y| \leq x^{1/\gamma}\right) + x^{\nu/\gamma} P\left(|Y| > x^{1/\gamma}\right)\right\} dx \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha-2-\gamma/p+\nu/2}\int_{n^{\gamma/p}}^{\infty} x^{-\nu/\gamma} \left\{E|Y|^{2} I\left(|Y| \leq x^{1/\gamma}\right) + x^{2/\gamma} P\left(|Y| > x^{1/\gamma}\right)\right\}^{\nu/2} dx \\
&=: I_{321} + I_{322}.
\end{aligned}$$

For I_{321}, taking $\nu > \alpha p$, we have

$$\begin{aligned}
I_{321} &= C\sum_{n=1}^{\infty} n^{\alpha-1-\gamma/p}\sum_{l=n}^{\infty}\int_{l^{\gamma/p}}^{(l+1)^{\gamma/p}} \left[x^{-\nu/\gamma} E|Y|^{\nu} I\left(|Y| \leq x^{1/\gamma}\right) + P\left(|Y| > x^{1/\gamma}\right)\right] dx \\
&\leq C\sum_{n=1}^{\infty} n^{\alpha-1-\gamma/p}\sum_{l=n}^{\infty}\left[l^{(\gamma/p(1-\nu/\gamma)-1)} E|Y|^{\nu} I\left(|Y| \leq (l+1)^{1/p}\right) + l^{\gamma/p-1} P\left(|Y| > l^{1/p}\right)\right] \\
&\leq C\sum_{l=1}^{\infty}\left[l^{(\gamma/p-\nu/p-1)} E|Y|^{\nu} I\left(|Y| \leq (l+1)^{1/p}\right) + l^{\gamma/p-1} P\left(|Y| > l^{1/p}\right)\right]\sum_{n=1}^{l} n^{\alpha-1-\gamma/p}
\end{aligned}$$

$$\leq \begin{cases} C\sum_{l=1}^{\infty}[l^{\alpha-v/p-1}E|Y|^{v}I(|Y|\leq(l+1)^{1/p})+l^{\alpha-1}P(|Y|>l^{1/p})], \\ \quad \text{if } \gamma<\alpha p, \\ C\sum_{l=1}^{\infty}[l^{\alpha-v/p-1}E|Y|^{v}I(|Y|\leq(l+1)^{1/p})+l^{\alpha-1}P(|Y|>l^{1/p})]\log(l+1), \\ \quad \text{if } \gamma=\alpha p, \\ C\sum_{l=1}^{\infty}[l^{\gamma/p-v/p-1}E|Y|^{v}I(|Y|\leq(l+1)^{1/p})+l^{\gamma/p-1}P(|Y|>l^{1/p})], \\ \quad \text{if } \gamma>\alpha p \end{cases}$$

$$\leq \begin{cases} C\sum_{k=1}^{\infty}k^{\alpha-v/p}E|Y|^{v}I((k-1)^{1/p}<|Y|\leq k^{1/p})+CE|Y|^{\alpha p}, \\ \quad \text{if } \gamma<\alpha p, \\ C\sum_{k=1}^{\infty}k^{\alpha-v/p}E|Y|^{v}I((k-1)^{1/p}<|Y|\leq k^{1/p})\log(k+1)+CE|Y|^{\alpha p}\log(|Y|+1), \\ \quad \text{if } \gamma=\alpha p, \\ C\sum_{k=1}^{\infty}k^{\gamma/p-v/p}E|Y|^{v}I((k-1)^{1/p}<|Y|\leq k^{1/p})+CE|Y|^{\gamma}, \\ \quad \text{if } \gamma>\alpha p \end{cases}$$

$$\leq \begin{cases} CE|Y|^{\alpha p}<\infty, & \text{if } \gamma<\alpha p, \\ CE|Y|^{\alpha p}\log(|Y|+1)<\infty, & \text{if } \gamma=\alpha p, \\ CE|Y|^{\gamma}<\infty, & \text{if } \gamma>\alpha p. \end{cases} \tag{3.11}$$

For I_{322}, we prove the claim by considering two cases.

If $\alpha p<2$, setting $v=2$, we have

$$I_{322}=C\sum_{n=1}^{\infty}n^{r-1-\gamma/p}\int_{n^{\gamma/p}}^{\infty}x^{-2/\gamma}\left\{E|Y|^{2}I\left(|Y|\leq x^{1/\gamma}\right)+x^{2/\gamma}P\left(|Y|>x^{1/\gamma}\right)\right\}dx.$$

Similarly to the proof of I_{321}, we have

$$I_{322}<\infty. \tag{3.12}$$

If $\alpha p\geq 2$, noting that $E|Y|^2<\infty$, choosing $v>\max\{2,\gamma,(\alpha-1)2p/(2-p)\}$, we obtain $\alpha-v/p+v/2<1$. Therefore, we have

$$\begin{aligned} I_{322} &\leq C\sum_{n=1}^{\infty}n^{\alpha-2-\gamma/p+v/2}\int_{n^{\gamma/p}}^{\infty}x^{-v/\gamma}\left\{E|Y|^{2}\right\}^{v/2}dx \\ &\leq C\sum_{n=1}^{\infty}n^{\alpha-2+v/2-v/p}\left\{E|Y|^{2}\right\}^{v/2}<\infty. \end{aligned} \tag{3.13}$$

Then, by (3.6)–(3.13), we see that (2.4) is true.

Next, we prove (2.5).

$$\begin{aligned} &\sum_{n=1}^{\infty}n^{\alpha-2}E\left\{\sup_{k\geq n}\left|k^{-1/p}S_k\right|-\varepsilon\right\}_{+}^{\gamma} \\ &\quad=\sum_{n=1}^{\infty}n^{\alpha-2}\int_{0}^{\infty}P\left(\sup_{k\geq n}\left|k^{-1/p}S_k\right|>\varepsilon+x^{1/\gamma}\right)dx \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{\infty} \sum_{n=2^{j-1}}^{2^j-1} n^{\alpha-2} \int_0^{\infty} P\Big(\sup_{k\geq n}\big|k^{-1/p}S_k\big| > \varepsilon + x^{1/\gamma}\Big)\,dx \\
&\leq C \sum_{j=1}^{\infty} \int_0^{\infty} P\Big(\sup_{k\geq 2^{j-1}}\big|k^{-1/p}S_k\big| > \varepsilon + x^{1/\gamma}\Big)\,dx \sum_{n=2^{j-1}}^{2^j-1} 2^{j(\alpha-2)} \\
&\leq C \sum_{j=1}^{\infty} 2^{j(\alpha-1)} \int_0^{\infty} P\Big(\sup_{k\geq 2^{j-1}}\big|k^{-1/p}S_k\big| > \varepsilon + x^{1/\gamma}\Big)\,dx \\
&\leq C \sum_{j=1}^{\infty} 2^{j(\alpha-1)} \sum_{l=j}^{\infty} \int_0^{\infty} P\Big(\max_{2^{l-1}\leq k\leq 2^l}\big|k^{-1/p}S_k\big| > \varepsilon + x^{1/\gamma}\Big)\,dx \\
&\leq C \sum_{l=1}^{\infty} \int_0^{\infty} P\Big(\max_{2^{l-1}\leq k\leq 2^l}\big|k^{-1/p}S_k\big| > \varepsilon + x^{1/\gamma}\Big)\,dx \sum_{j=1}^{l} 2^{j(\alpha-1)} \\
&\leq C \sum_{l=1}^{\infty} 2^{l(\alpha-1)} \int_0^{\infty} P\Big(\max_{2^{l-1}\leq k\leq 2^l}|S_k| > \big(\varepsilon + x^{1/\gamma}\big)2^{(l-1)/p)}\Big)\,dx \\
&\quad \big(\text{Let } y = 2^{(l-1)\gamma/p}x\big) \\
&\leq C \sum_{l=1}^{\infty} 2^{l(\alpha-1-\gamma/p)} \int_0^{\infty} P\Big(\max_{1\leq k\leq 2^l}|S_k| > \varepsilon 2^{(l-1)/p} + y^{1/\gamma}\Big)\,dy \\
&\leq C \sum_{n=1}^{\infty} n^{(\alpha-2-\gamma/p)} \int_0^{\infty} P\Big(\max_{1\leq k\leq n}|S_k| > \varepsilon n^{1/p}2^{-1/p} + y^{1/\gamma}\Big)\,dy \\
&\quad \big(\text{Let } \varepsilon_0 = 2^{-1/p}\varepsilon\big) \\
&\leq C \sum_{n=1}^{\infty} n^{(\alpha-2-\gamma/p)} E\Big\{\max_{1\leq k\leq n}|S_k| - \varepsilon_0 n^{1/p}\Big\}_+^{\gamma} < \infty.
\end{aligned}
$$

Thus, (2.5) holds. □

Next, we prove Theorem 2.4.

Proof of Theorem 2.4 Since the proof of Theorem 2.4 is similar to that of Theorem 2.3, we only give the proof outline as follows.

By (3.1), we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1-\gamma/p} E\Big\{\max_{1\leq k\leq n}|S_k| - \varepsilon n^{1/p}\Big\}_+^{\gamma} \\
&\quad\leq \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq k\leq n}|S_k| > \varepsilon n^{1/p}\Big) + \sum_{n=1}^{\infty} n^{-1-\gamma/p} \int_{n^{\gamma/p}}^{\infty} P\Big(\max_{1\leq k\leq n}|S_k| > x^{1/\gamma}\Big)\,dx \\
&\quad =: J_1 + J_2.
\end{aligned}
$$

For J_1, by Theorem 2.2, we have

$$
J_1 = \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq k\leq n}|S_k| > \varepsilon n^{1/p}\Big) < \infty. \tag{3.14}
$$

For J_2, similar to the proof of I_3, we have

$$\begin{aligned}J_2 &\le C\sum_{n=1}^{\infty} n^{-1-\gamma/p}\int_{n^{\gamma/p}}^{\infty} P\left\{\max_{1\le k\le n}\left|\sum_{j=-\infty}^{+\infty} a_j\sum_{i=j+1}^{j+k} Y_i''\right| \ge x^{1/\gamma}/2\right\} dx \\ &\quad + C\sum_{n=1}^{\infty} n^{-1-\gamma/p}\int_{n^{\gamma/p}}^{\infty} P\left\{\max_{1\le k\le n}\left|\sum_{j=-\infty}^{+\infty} a_j\sum_{i=j+1}^{j+k} \left(Y_i' - EY_i'\right)\right| \ge x^{1/\gamma}/4\right\} dx \\ &=: J_{21} + J_{22}.\end{aligned}$$

For J_{21}, by Lemma 2.3, Markov and C_r inequalities, similarly to the proof of I_{31}, we get

$$\begin{aligned}J_{21} &\le C\sum_{n=1}^{\infty} n^{-1-\gamma/p}\int_{n^{\gamma/p}}^{\infty} x^{-\theta/\gamma} E\max_{1\le k\le n}\left|\sum_{j=-\infty}^{+\infty} a_j\sum_{i=j+1}^{j+k} Y_i''\right|^{\theta} dx \\ &\le C\sum_{n=1}^{\infty} n^{-\gamma/p}\sum_{l=n}^{\infty}\int_{l^{\gamma/p}}^{(l+1)^{\gamma/p}} x^{-\theta/\gamma} E|Y|^{\theta} I\left(|Y| > x^{1/\gamma}\right) dx \\ &\le C\sum_{n=1}^{\infty} n^{-\gamma/p}\sum_{l=n}^{\infty} l^{\gamma/p(1-\theta/\gamma)-1} E|Y|^{\theta} I\left(|Y| > l^{/p}\right) \\ &\le \begin{cases} C\sum_{l=1}^{\infty} l^{-\theta/p} E|Y|^{\theta} I(|Y| > l^{1/p}), & \text{if } \gamma < p, \\ C\sum_{l=1}^{\infty} l^{-\theta/p} E|Y|^{\theta} I(|Y| > l^{1/p})\log(l+1), & \text{if } \gamma = p, \\ C\sum_{l=1}^{\infty} l^{\gamma/p-\theta/p-1} E|Y|^{\theta} I(|Y| > l^{1/p}), & \text{if } \gamma > p \end{cases} \\ &\le \begin{cases} CE|Y|^{p} < \infty, & \text{if } \gamma < p, \\ CE|Y|^{p}\log(|Y|+1) < \infty, & \text{if } \gamma = p, \\ CE|Y|^{\gamma} < \infty, & \text{if } \gamma > p. \end{cases}\end{aligned} \tag{3.15}$$

For J_{22}, similar to the proof of I_{32}, taking $\nu = 2$, we obtain

$$\begin{aligned}J_{22} &\le C\sum_{n=1}^{\infty} n^{-1-\gamma/p}\int_{n^{\gamma/p}}^{\infty} x^{-2/\gamma} E\left\{\max_{1\le k\le n}\left|\sum_{j=-\infty}^{+\infty} a_j\sum_{i=j+1}^{j+k}\left(Y_i' - EY_i'\right)\right|^2\right\} dx \\ &\le C\sum_{n=1}^{\infty} n^{-1-\gamma/p}\int_{n^{\gamma/p}}^{\infty} x^{-2/\gamma} E\left\{\sum_{j=-\infty}^{+\infty} |a_j|^{1/2}\left(|a_j|^{1/2}\max_{1\le k\le n}\left|\sum_{i=j+1}^{j+k}\left(Y_i' - EY_i'\right)\right|\right)\right\}^2 dx \\ &\le C\sum_{n=1}^{\infty} n^{-1-\gamma/p}\int_{n^{\gamma/p}}^{\infty} x^{-2/\gamma}\sum_{j=-\infty}^{+\infty} |a_j|\sum_{i=j+1}^{i+n} E\left|Y_i'\right|^2 dx \\ &\le C\sum_{n=1}^{\infty} n^{-\gamma/p}\int_{n^{\gamma/p}}^{\infty} x^{-2/\gamma}\left\{E|Y|^2 I\left(|Y| \le x^{1/\gamma}\right) + x^{2/\gamma} P\left(|Y| > x^{1/\gamma}\right)\right\} dx \\ &= C\sum_{n=1}^{\infty} n^{-\gamma/p}\sum_{l=n}^{\infty}\int_{l^{\gamma/p}}^{(l+1)^{\gamma/p}}\left[x^{-2/\gamma} E|Y|^2 I\left(|Y| \le x^{1/\gamma}\right) + P\left(|Y| > x^{1/\gamma}\right)\right] dx \\ &\le C\sum_{l=1}^{\infty}\left[l^{(\gamma/p-2/p-1)} E|Y|^2 I\left(|Y| \le (l+1)^{1/p}\right) + l^{\gamma/p-1} P\left(|Y| > l^{1/p}\right)\right]\sum_{n=1}^{l} n^{-\gamma/p}\end{aligned}$$

$$\leq \begin{cases} C\sum_{k=1}^{\infty}[k^{1-2/p}E|Y|^2I((k-1)^{1/p}<|Y|\leq k^{1/p})]+CE|Y|^p, \\ \quad \text{if } \gamma<p, \\ C\sum_{k=1}^{\infty}[k^{1-2/p}E|Y|^2I((k-1)^{1/p}<|Y|\leq k^{1/p})]\log(l+1)+CE|Y|^p\log(|Y|+1), \\ \quad \text{if } \gamma=p, \\ C\sum_{k=1}^{\infty}[k^{\gamma/p-2/p}E|Y|^2I((k-1)^{1/p}<|Y|\leq k^{1/p})]+CE|Y|^{\gamma}, \\ \quad \text{if } \gamma>p \end{cases}$$

$$\leq \begin{cases} CE|Y|^p<\infty, & \text{if } \gamma<p, \\ CE|Y|^p\log(|Y|+1)<\infty, & \text{if } \gamma=p, \\ CE|Y|^{\gamma}<\infty, & \text{if } \gamma>p. \end{cases} \tag{3.16}$$

Therefore, from (3.15)–(3.16), we see that (2.6) holds. □

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Authors' contributions
The authors contributed equally and significantly in writing this paper. Both authors read and approved the final manuscript.

Author details
[1]Department of Mathematics and Computer Science, Tongling University, Tongling, China. [2]Key Laboratory of HPC-SIP (MOE), College of Mathematics and Statistics, Hunan Normal University, Changsha, China. [3]School of Mathematical Sciences, Qufu Normal University, Qufu, China.

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