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Hyperbolic curve flows in the plane

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Abstract

In this paper, we investigate the evolution of a strictly convex closed planar curve driven by a *hyperbolic* normal flow. The asymptotical behavior of the evolving curves has also been shown if the velocity of the initial curve is nonnegative.

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1 Introduction and our main conclusions

In this paper, we would like to consider a family of closed planar curves $F:\mathbb{S}^1\times[0,T)\to\mathbb{R}^2$, which satisfies the following initial value problem (IVP for short):

$$\begin{cases}\frac{\partial^2}{\partial t^2}F(u,t)=k^\alpha(u,t)\vec{N}(u,t)-\nabla\rho(u,t), & \forall u\in\mathbb{S}^1, t\in[0,T),\\ \frac{\partial F}{\partial t}(\cdot,0)=f(u)\vec{N}_0,\\ F(\cdot,0)=F_0,\end{cases}\tag{1.1}$$

where $F_0:\mathbb{S}^1\to\mathbb{R}^2$ is a smooth *strictly convex* closed curve in the plane $\mathbb{R}^2$, $\vec{N}_0$ is the unit inner normal vector of the initial curve F_0, $\alpha>0$ is a positive constant, $k(\cdot,t)$ is the curvature function of the evolving curve $F(\cdot,t)$, $\vec{N}(\cdot,t)$ is the unit inner normal vector of $F(\cdot,t)$, $f(u)\in C^\infty(\mathbb{S}^1)$, and $f(u)\vec{N}_0$ is the initial normal velocity. Besides, $\nabla\rho$ is defined by

$$\nabla\rho=\left\langle\frac{\partial^2 F}{\partial s\partial t},\frac{\partial F}{\partial t}\right\rangle\vec{T},\tag{1.2}$$

where $\langle\cdot,\cdot\rangle$ is the standard Euclidean inner product in $\mathbb{R}^2$, and $\vec{T}$, s are the unit tangent vector of $F(\cdot,t)$ and the arc-length parameter, respectively. For this flow, first, we can get the following.

Theorem 1.1 (Local existence and uniqueness) *Suppose that F_0 is a smooth strictly convex closed curve in $\mathbb{R}^2$. Then there exist a positive constant $T>0$ and a family of smooth strictly convex closed planar curves $F(u,t)$ satisfying* (1.1).

If furthermore the normal velocity of the initial curve $F(\cdot,0)$ is nonnegative, we can also describe the asymptotical behavior for the hyperbolic flow (1.1).

Theorem 1.2 *Suppose that F_0 is a smooth strictly convex closed curve in $\mathbb{R}^2$. Then there exist a finite time interval $[0, T_{\max})$ and a family of strictly convex closed planar curves $F(\cdot, t)$ such that $F(\cdot, t)$ satisfies (1.1) only on $[0, T_{\max})$, provided that $\alpha \geq 1$ and $f(u)$ is a smooth nonnegative function on $\mathbb{S}^1$. Moreover, as $t \to T_{\max}$, one of the following should be true*:

(i) *the solution $F(\cdot, t)$ converges to a point, that is to say, the curvature of the limit curve becomes unbounded*;

(ii) *the curvature k of the evolving curve is discontinuous so that the solution $F(\cdot, t)$ converges to a piecewise smooth curve, which implies that shocks and propagating discontinuities may be generated within the hyperbolic flow* (1.1).

Remark 1.3 If $\alpha = 1$, then the hyperbolic flow (1.1) degenerates into the one considered in [6], and correspondingly, our Theorem 1.1 would become [6, Theorem 1.2]. Therefore, our paper here is an interesting extension of [6]. Besides, as in [7], one can also add a term $c(t)F(u, t)$ to the RHS of the evolution equation in (1.1), which is actually a forcing term in the direction of the position vector, and then using the methods in [7] and the paper here, the evolution and the asymptotical behavior of the *new* hyperbolic planar flow can be expected without any big difficulty. The research of curve flows and related topics is important and has many interesting applications in other scientific branches (see, e.g., [1–4, 8, 9, 11, 12]).

2 Proof of Theorem 1.1

In this section, we will reparameterize the evolving curves so that the hyperbolic partial differential equation (PDE for short) can be derived for the support function defined by (2.5) below, which leads to the short-time existence and the uniqueness of the solution to the flow (1.1).

Definition 2.1 A curve $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ evolves normally if and only if its tangential velocity vanishes.

It is easy to know that the flow (1.1) is a normal flow.

Lemma 2.2 *The curve flow* (1.1) *is a normal flow.*

Proof Since

$$\begin{aligned}
\frac{d}{dt}\left\langle \frac{\partial F}{\partial t}, \frac{\partial F}{\partial s} \right\rangle &= \left\langle \frac{\partial^2 F}{\partial t^2}, \frac{\partial F}{\partial s} \right\rangle + \left\langle \frac{\partial F}{\partial t}, \frac{\partial^2 F}{\partial t \partial s} \right\rangle \\
&= \left\langle -\nabla \rho, \frac{\partial F}{\partial s} \right\rangle + \left\langle \frac{\partial F}{\partial t}, \frac{\partial^2 F}{\partial t \partial s} \right\rangle \\
&= -\left\langle \frac{\partial^2 F}{\partial s \partial t}, \frac{\partial F}{\partial t} \right\rangle + \left\langle \frac{\partial F}{\partial t}, \frac{\partial^2 F}{\partial t \partial s} \right\rangle \\
&= 0
\end{aligned}$$

and the initial velocity of the flow (1.1) is in the normal direction (i.e., the initial tangential velocity vanishes), then the tangential velocity of the evolving curve $F(\cdot, t)$ vanishes for all $t \geq 0$. □

By Lemma 2.2 and (1.1), it is easy to know that there exists a function $\sigma(u,t) =: \langle \frac{\partial F}{\partial t}, \vec{N}\rangle(u,t)$ satisfying

$$\begin{cases} \frac{\partial F}{\partial t}(u,t) = \sigma(u,t)\vec{N} = k^{\alpha}(u,t)\vec{N}, \\ F(u,0) = F_0(u), \end{cases} \tag{2.1}$$

where $\sigma(u,t) = f(u) + \int_0^t k^{\alpha}(u,\xi)\,d\xi$.

Denote by $s = s(\cdot,t)$ the arc-length parameter of curve $F(\cdot,t) : \mathbb{S}^1 \to \mathbb{R}^2$. By the arc-length formula, we have

$$\frac{\partial}{\partial s} = \frac{1}{\sqrt{(\frac{\partial x}{\partial u})^2 + (\frac{\partial y}{\partial u})^2}}\frac{\partial}{\partial u} = \frac{1}{|\frac{\partial F}{\partial u}|}\frac{\partial}{\partial u} := \frac{1}{\upsilon}\frac{\partial}{\partial u},$$

where (x,y) is the Cartesian coordinates of $\mathbb{R}^2$. By the Frenet formula, for the orthogonal frame field $\{\vec{T},\vec{N}\}$ of $\mathbb{R}^2$, we have

$$\frac{\partial \vec{T}}{\partial s} = k\vec{N}, \qquad \frac{\partial \vec{N}}{\partial s} = -k\vec{T}.$$

Let θ be the unit outward normal angle of a closed convex curve $F : \mathbb{S}^1 \to \mathbb{R}^2$ w.r.t. the Cartesian coordinates of $\mathbb{R}^2$. Then we have

$$\vec{N} = (-\cos\theta, -\sin\theta),$$
$$\vec{T} = (-\sin\theta, \cos\theta).$$

Correspondingly, we have $\frac{\partial \theta}{\partial s} = k$ and, by the chain rule, it follows that

$$\frac{\partial \vec{N}}{\partial t} = \frac{\partial \vec{N}}{\partial \theta}\frac{\partial \theta}{\partial t} = -\frac{\partial \theta}{\partial t}\vec{T}, \qquad \frac{\partial \vec{T}}{\partial t} = \frac{\partial \vec{T}}{\partial \theta}\frac{\partial \theta}{\partial t} = \frac{\partial \theta}{\partial t}\vec{N}. \tag{2.2}$$

Clearly, by (2.1) and (2.2), we have

$$\begin{aligned} &\frac{\partial \sigma}{\partial t} = \frac{\partial}{\partial t}\left\langle \frac{\partial F}{\partial t}, \vec{N}\right\rangle = \left\langle \frac{\partial^2 F}{\partial t^2}, \vec{N}\right\rangle = k^{\alpha}, \\ &\sigma\frac{\partial \sigma}{\partial s} = \left\langle \frac{\partial F}{\partial t}, \vec{N}\right\rangle \cdot \left\langle \frac{\partial^2 F}{\partial s \partial t}, \vec{N}\right\rangle = \left\langle \frac{\partial F}{\partial t}, \frac{\partial^2 F}{\partial s\partial t}\right\rangle. \end{aligned} \tag{2.3}$$

By the definition of υ, (2.1), (2.2), and (2.3), we can obtain the following.

Lemma 2.3 *The derivative of υ with respect to t is $\frac{\partial \upsilon}{\partial t} = -k\sigma\upsilon$.*

Proof By a direct computation, we have

$$\begin{aligned} \frac{\partial}{\partial t}(\upsilon^2) &= 2\left\langle \frac{\partial F}{\partial u}, \frac{\partial^2 F}{\partial t \partial u}\right\rangle = 2\left\langle \left|\frac{\partial F}{\partial u}\right|\vec{T}, \frac{\partial}{\partial u}(\sigma\vec{N})\right\rangle \\ &= 2\left\langle \upsilon\vec{T}, \sigma\frac{\partial \vec{N}}{\partial u}\right\rangle = 2\left\langle \upsilon\vec{T}, \sigma\frac{\partial \vec{N}}{\partial s}\frac{\partial s}{\partial u}\right\rangle \end{aligned}$$

$$= 2\langle \upsilon \vec{T}, \sigma(-k\vec{T})\upsilon \rangle$$
$$= -2\upsilon^2 k\sigma,$$

which implies the lemma. □

By Lemma 2.3, we can obtain

$$\frac{\partial^2}{\partial t \partial s} = \frac{\partial}{\partial t}\left(\frac{1}{\upsilon}\frac{\partial}{\partial u}\right) = -\frac{1}{\upsilon^2}\frac{\partial \upsilon}{\partial t}\frac{\partial}{\partial u} + \frac{1}{\upsilon}\frac{\partial}{\partial u}\frac{\partial}{\partial t} = k\sigma\frac{\partial}{\partial s} + \frac{\partial^2}{\partial s \partial t}.$$

Noting that $\vec{T} = \frac{\partial F}{\partial s}$, and together with the above equality, we can deduce

$$\begin{aligned}
\frac{\partial \vec{T}}{\partial t} &= \frac{\partial^2 F}{\partial t \partial s} \\
&= k\sigma\frac{\partial F}{\partial s} + \frac{\partial^2 F}{\partial s \partial t} \\
&= k\sigma\vec{T} + \frac{\partial}{\partial s}(\sigma\vec{N}) \\
&= k\sigma\vec{T} + \frac{\partial \sigma}{\partial s}\vec{N} + \sigma(-k\vec{T}) \\
&= \frac{\partial \sigma}{\partial s}\vec{N},
\end{aligned}$$

which, combining with (2.2), implies

$$\frac{\partial \sigma}{\partial s} = \frac{\partial \theta}{\partial t}, \qquad \frac{\partial \vec{N}}{\partial t} = -\frac{\partial \sigma}{\partial s}\vec{T}.$$

Assume that $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ is a family of curves satisfying the flow (1.1). As in [7, 12], one can use the normal angle θ to reparameterize each evolving curve $F(u, t)$ as follows:

$$\widetilde{F}(\theta, \tau) = F\big(u(\theta, \tau), t(\theta, \tau)\big), \tag{2.4}$$

where $t(\theta, \tau) = \tau$. By the chain rule, we have

$$0 = \frac{\partial \theta}{\partial \tau} = \frac{\partial \theta}{\partial u}\frac{\partial u}{\partial \tau} + \frac{\partial \theta}{\partial t},$$

and

$$\frac{\partial \theta}{\partial t} = -\frac{\partial \theta}{\partial u}\frac{\partial u}{\partial \tau} = -\frac{\partial \theta}{\partial s}\frac{\partial s}{\partial u}\frac{\partial u}{\partial \tau} = -k\upsilon\frac{\partial u}{\partial \tau}.$$

Hence, by a direct calculation, one can obtain

$$\frac{\partial \vec{T}}{\partial \tau} = \frac{\partial \vec{T}}{\partial s}\frac{\partial s}{\partial u}\frac{\partial u}{\partial \tau} + \frac{\partial \theta}{\partial t}\vec{N} = \left(k\upsilon\frac{\partial u}{\partial \tau} + \frac{\partial \theta}{\partial t}\right)\vec{N} = 0$$

and

$$\frac{\partial \vec{N}}{\partial \tau} = \frac{\partial \vec{N}}{\partial s}\frac{\partial s}{\partial u}\frac{\partial u}{\partial \tau} - \frac{\partial \theta}{\partial t}\vec{T} = -\left(k\upsilon\frac{\partial u}{\partial \tau} + \frac{\partial \theta}{\partial t}\right)\vec{T} = 0,$$

which implies that $\vec{N}$ and $\vec{T}$ are independent of the parameter τ.

Define the support function of the evolving curve $\widetilde{F}(\theta,\tau) = (x(\theta,\tau), y(\theta,\tau))$ as follows:

$$S(\theta,\tau) = \langle \widetilde{F}(\theta,\tau), -\vec{N}\rangle = x(\theta,\tau)\cos\theta + y(\theta,\tau)\sin\theta. \tag{2.5}$$

Then we have

$$S_\theta(\theta,\tau) = -x(\theta,\tau)\sin\theta + y(\theta,\tau)\cos\theta = \langle \widetilde{F}(\theta,\tau), \vec{T}\rangle.$$

Solving the above two equations yields

$$\begin{cases} x(\theta,\tau) = S\cos\theta - S_\theta\sin\theta, \\ y(\theta,\tau) = S\sin\theta + S_\theta\cos\theta. \end{cases}$$

Furthermore, we have

$$\begin{aligned} S_{\theta\theta} + S &= \langle \widetilde{F}_\theta(\theta,\tau), \vec{T}\rangle + \langle \widetilde{F}(\theta,\tau), \vec{N}\rangle + \langle \widetilde{F}(\theta,\tau), -\vec{N}\rangle \\ &= \langle \widetilde{F}_\theta(\theta,\tau), \vec{T}\rangle = \left\langle \frac{\partial F}{\partial u}\frac{\partial u}{\partial s}\frac{\partial s}{\partial \theta}, \vec{T}\right\rangle \\ &= \frac{1}{k}, \end{aligned}$$

which implies

$$k = \frac{1}{S_{\theta\theta} + S}. \tag{2.6}$$

The equation $S_{\theta\theta} + S = \frac{1}{k}$ makes sense, since the evolving curve $\widetilde{F}(\theta,\tau) = F(u(\theta,\tau), t(\theta,\tau))$ is strictly convex under the flow (1.1), see Proposition 4.3 for details.

On the other hand, since $\vec{N}$ and $\vec{T}$ are independent of the parameter τ, together with (2.1) and (2.4), we can get

$$S_\tau = \left\langle \frac{\partial \widetilde{F}}{\partial \tau}, -\vec{N}\right\rangle = \left\langle \frac{\partial F}{\partial u}\frac{\partial u}{\partial \tau} + \frac{\partial F}{\partial t}, -\vec{N}\right\rangle = \left\langle \frac{\partial F}{\partial t}, -\vec{N}\right\rangle = -\widetilde{\sigma}(\theta,\tau),$$

and moreover,

$$\begin{aligned} S_{\tau\tau} &= \left\langle \frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial \tau^2} + \frac{\partial^2 F}{\partial u^2}\left(\frac{\partial u}{\partial \tau}\right)^2 + 2\frac{\partial^2 F}{\partial u \partial t}\frac{\partial u}{\partial \tau} + \frac{\partial^2 F}{\partial t^2}, -\vec{N}\right\rangle \\ &= \left\langle \frac{\partial^2 F}{\partial u^2}\left(\frac{\partial u}{\partial \tau}\right)^2 + \frac{\partial^2 F}{\partial u \partial t}\frac{\partial u}{\partial \tau}, -\vec{N}\right\rangle + \left\langle \frac{\partial^2 F}{\partial u \partial t}\frac{\partial u}{\partial \tau} + \frac{\partial^2 F}{\partial t^2}, -\vec{N}\right\rangle \\ &= \frac{\partial u}{\partial \tau}\cdot\left\langle \left(\frac{\partial F}{\partial u}\right)_\tau, -\vec{N}\right\rangle + \left\langle \frac{\partial^2 F}{\partial u \partial t}\frac{\partial u}{\partial \tau} + \frac{\partial^2 F}{\partial t^2}, -\vec{N}\right\rangle \end{aligned}$$

$$\begin{aligned}
&= \left\langle \frac{\partial^2 F}{\partial u \partial t}\frac{\partial u}{\partial \tau} + \frac{\partial^2 F}{\partial t^2}, -\vec{N} \right\rangle \\
&= \left\langle \frac{\partial^2 F}{\partial u \partial t}\frac{\partial u}{\partial \tau}, -\vec{N} \right\rangle - k^\alpha .
\end{aligned}$$

Since, by Lemma 2.2, we know that $F : \mathbb{S}^1 \times [0, T) \to \mathbb{R}^2$ is a normal flow,

$$\left\langle \frac{\partial F}{\partial t}, \vec{T} \right\rangle \equiv 0$$

holds for all $t \in [0, T)$. Then by a straightforward computation, we can get

$$\begin{aligned}
S_{\tau\theta} &= \frac{\partial}{\partial \tau}\langle \widetilde{F}, \vec{T} \rangle = \left\langle \frac{\partial \widetilde{F}}{\partial \tau}, \vec{T} \right\rangle \\
&= \left\langle \frac{\partial F}{\partial u}\frac{\partial u}{\partial \tau} + \frac{\partial F}{\partial t}, \vec{T} \right\rangle \\
&= \upsilon \frac{\partial u}{\partial \tau}
\end{aligned}$$

and

$$\begin{aligned}
S_{\theta\tau} &= \frac{\partial}{\partial \theta}\left\langle \frac{\partial F}{\partial t}, -\vec{N} \right\rangle = \left\langle \frac{\partial^2 F}{\partial u \partial t}\frac{\partial u}{\partial \theta}, -\vec{N} \right\rangle \\
&= \left\langle \frac{\partial^2 F}{\partial u \partial t}\frac{\partial u}{\partial s}\frac{\partial s}{\partial \theta}, -\vec{N} \right\rangle \\
&= \frac{1}{k\upsilon}\left\langle \frac{\partial^2 F}{\partial u \partial t}, -\vec{N} \right\rangle .
\end{aligned}$$

Hence, the support function $S(\theta, \tau)$ satisfies

$$\begin{aligned}
S_{\tau\tau} &= \left\langle \frac{\partial^2 F}{\partial u \partial t}\frac{\partial u}{\partial \tau}, -\vec{N} \right\rangle - k^\alpha \\
&= k\upsilon S_{\theta\tau}\frac{\partial u}{\partial \tau} - k^\alpha \\
&= kS_{\theta\tau}^2 - k^\alpha ,
\end{aligned}$$

which is equivalent to

$$S_{\tau\tau} = \frac{S_{\theta\tau}^2}{S_{\theta\theta} + S} - \left(\frac{1}{S_{\theta\theta} + S}\right)^\alpha , \quad \forall (\theta, \tau) \in \mathbb{S}^1 \times [0, T). \tag{2.7}$$

Therefore, we have

$$\begin{cases}
SS_{\tau\tau} + S_{\tau\tau}S_{\theta\theta} - S_{\theta\tau}^2 + (S_{\theta\theta} + S)^{1-\alpha} = 0, \\
S(\theta, 0) = h(\theta) = (F_0, \vec{N}), \\
S_\tau(\theta, 0) = -\widetilde{f}(\theta) = -f(u(\theta, 0)),
\end{cases} \tag{2.8}$$

where h is the support function of $F_0(u(\theta))$, and $\widetilde{f}$ is the initial velocity of the initial curve F_0.

Here we would like to get the short-time existence of the flow (2.8) by the linearization method. First, we have the following conclusion.

Lemma 2.4 *Suppose that F_0 is a smooth strictly convex closed curve and $k_0 > 0$ is the curvature of the curve F_0. Then the wave equation*

$$\begin{cases} S_{\tau\tau} = S_{\theta\theta} + k_0^{\alpha}, \\ S(\theta,0) = h(\theta), \\ S_\tau(\theta,0) = -\widetilde{f}(\theta), \end{cases} \tag{2.9}$$

has a unique solution $S_0 \in C^\infty(\mathbb{S}^1 \times [0,T_1))$ with some $T_1 > 0$, where $\alpha > 0$, $S_{\tau\tau}$, $S_{\theta\theta}$, $h(\theta)$, and $\widetilde{f}(\theta)$ have the same meaning as those in (2.8).

Next, we want to consider the linearization of (2.8) around S_0.

Lemma 2.5 *Let $S_0 \in C^\infty(\mathbb{S}^1 \times [0,T_1))$ be the solution of the wave equation (2.9) and $\xi \in C^\infty(\mathbb{S}^1 \times [0,T_1))$. Then there exists some $T > 0$ such that the linearization of (2.8) around S_0 given by*

$$\begin{cases} L_{S_0}S := S_{\tau\tau} - [aS_{\theta\theta} + bS_{\theta\tau} + cS_\theta + dS_\tau + eS] = \xi, \\ S(\theta,0) = h(\theta), \\ S_\tau(\theta,0) = -\widetilde{f}(\theta), \end{cases} \tag{2.10}$$

has a unique solution $S \in C^\infty(\mathbb{S}^1 \times [0,T))$.

Proof For equation (2.7), set

$$\frac{S_{\theta\tau}^2}{S_{\theta\theta} + S} - \left(\frac{1}{S_{\theta\theta} + S}\right)^{\alpha} = \phi(x, S_{\theta\theta}, S_{\theta\tau}, S_\theta, S_\tau, S).$$

Let $S_\varepsilon := S_0 + \varepsilon S$. We obtain the linearized operator L_{S_0} of $\frac{\partial^2}{\partial t^2} - \phi$ around S_0 as follows:

$$\begin{aligned} L_{S_0}S &:= \frac{d}{d\varepsilon}|_{\varepsilon=0}\left(\frac{\partial^2 S_\varepsilon}{\partial\tau^2} - \phi\big(x, (S_\varepsilon)_{\theta\theta}, (S_\varepsilon)_{\theta\tau}, (S_\varepsilon)_\theta, (S_\varepsilon)_\tau, S_\varepsilon\big)\right) \\ &= S_{\tau\tau} - \left[\frac{\partial\phi}{\partial(S_\varepsilon)_{\theta\theta}}\frac{d(S_\varepsilon)_{\theta\theta}}{d\varepsilon} + \frac{\partial\phi}{\partial(S_\varepsilon)_{\theta\tau}}\frac{d(S_\varepsilon)_{\theta\tau}}{d\varepsilon}\right. \\ &\quad \left. + \frac{\partial\phi}{\partial(S_\varepsilon)_\theta}\frac{d(S_\varepsilon)_\theta}{d\varepsilon} + \frac{\partial\phi}{\partial(S_\varepsilon)_\tau}\frac{d(S_\varepsilon)_\tau}{d\varepsilon} + \frac{\partial\phi}{\partial(S_\varepsilon)}\frac{d(S_\varepsilon)}{d\varepsilon}\right]|_{\varepsilon=0} \\ &= S_{\tau\tau} - \left[\frac{\partial\phi}{\partial(S_0)_{\theta\theta}}S_{\theta\theta} + \frac{\partial\phi}{\partial(S_0)_{\theta\tau}}S_{\theta\tau} + \frac{\partial\phi}{\partial(S_0)_\theta}S_\theta + \frac{\partial\phi}{\partial(S_0)_\tau}S_\tau + \frac{\partial\phi}{\partial(S_0)}S\right], \end{aligned}$$

which implies in (2.10)

$$a = \frac{-S_{\theta\tau}^2}{(S_{\theta\theta} + S)^2} + \alpha\left(\frac{1}{S_{\theta\theta} + S}\right)^{\alpha+1}, \qquad b = \frac{2S_{\theta\tau}}{S_{\theta\theta} + S}.$$

Consider the principal matrix

$$\begin{pmatrix} -1 & \frac{S_{\theta\tau}}{S_{\theta\theta}+S} \\ \frac{S_{\theta\tau}}{S_{\theta\theta}+S} & \frac{-S_{\theta\tau}^2}{(S_{\theta\theta}+S)^2} + \alpha(\frac{1}{S_{\theta\theta}+S})^{\alpha+1} \end{pmatrix}$$

which, by a proper linear transformation, can be transformed into

$$\begin{pmatrix} -1 & 0 \\ 0 & \alpha k^{\alpha+1} \end{pmatrix}.$$

At $t = 0$, since k_0 is strictly positive, thus L_{S_0} is uniformly hyperbolic in some time interval $[0, T)$. Therefore, the theory of second-order linear hyperbolic PDEs yields the result. □

By Lemma 2.5, we have the following.

Lemma 2.6 *Suppose that F_0 is a smooth strictly convex closed curve and $k_0 > 0$ is the curvature of the curve F_0. Then there exist some $T > 0$ and a family of strictly convex closed curves $F(\cdot, t)$ such that (2.8) has a unique solution $S \in C^\infty(\mathbb{S}^1 \times [0, T))$ with S the support function of $F(\cdot, t)$.*

Proof We want to translate the solvability of (2.8) to the invertibility of some operator A defined as

$$S \to AS := S_{\tau\tau} - \phi(x, S_{\theta\theta}, S_{\theta\tau}, S_\theta, S_\tau, S).$$

The inverse function theorem states that if $\mathrm{DA}(S_0)$ is a linear homeomorphism from S to AS, then there exists a neighborhood U_{S_0} such that $A : U_{S_0} \longrightarrow A(U_{S_0})$ is a homeomorphism.

Let S_0 be the solution of (2.8), then $\mathrm{DA}(S_0)$ is given by

$$\mathrm{DA}(S_0) : S \to \mathrm{DA}(S_0)(S) = L_{S_0} S.$$

By Lemma 2.5, we know that there exists a unique solution S to (2.9), which shows that $\mathrm{DA}(S_0)$ is invertible. Since $\mathrm{DA}(S_0)$ is a linear homeomorphism, A is invertible in a neighborhood U_{S_0} of S_0, which implies the conclusion of Lemma 2.6. □

Then Theorem 1.1 follows by applying Lemma 2.6 directly.

3 An interesting example

Example 3.1 Let $F(u, t)$ be a family of round circles, with the radius $r(t)$ centered at the origin, given by

$$F(u, t) = r(t)(\cos\theta, \sin\theta)$$

with $F(u, 0) = r_0(\cos\theta, \sin\theta)$, which implies $r(0) = r_0 \geq 0$. Then the support function and the curvature of circles are given by

$$S(\theta, t) = r(t)$$

and

$$k(\theta,t)=\frac{1}{r(t)}.$$

In this setting the hyperbolic flow (1.1) becomes

$$\begin{cases} r_{tt}=-\frac{1}{r^{\alpha}}, \\ r(0)=r_0>0, \quad r_t(0)=r_1, \end{cases} \tag{3.1}$$

where $\alpha>0$. For IVP (3.1), we divide the discussion into two cases as follows:

Case (I). Assume that $r_1\le 0$. Since $r_{tt}=-\frac{1}{r^{\alpha}}<0$, which implies the acceleration and the initial velocity are in the same direction, it is easy to know that $r(t)$ decreases and then there must exist a finite time $t_0>0$ such that $r(t_0)=0$. That is to say, the initial circle $F(u,0)$ contracts to a single point as $t\to t_0$. Especially, by [5, Lemma 3.1], if $\alpha=1$ and $r_1=0$, then $t_0=\sqrt{\frac{\pi}{2}}r_0$.

Case (II). Assume that $r_1>0$. By [5, Lemma 3.1], if $\alpha=1$, we know that the solution r to IVP (3.1) increases first and then deceases and attains its zero point at some finite time t_0. Assume in addition that $\alpha\neq 1$. Multiplying both sides of the first equation in (3.1) by r_t, and then integrating from 0 to $t>0$, we have

$$r_t^2=\frac{2}{\alpha-1}\bigl[r^{1-\alpha}(t)-r_0^{1-\alpha}\bigr]+r_1^2.$$

So, it follows that

$$r\le\left(r_0^{1-\alpha}+\frac{1-\alpha}{2}r_1^2\right)^{1/(1-\alpha)}:=\Upsilon(r_0,r_1).$$

Therefore, if r increases for all time, i.e., $r_t>0$ for $t>0$, we have $r_0\le r(t)\le\Upsilon(r_0,r_1)$ and $-\frac{1}{r_0^{\alpha}}\le r_{tt}\le-\frac{1}{\Upsilon^{\alpha}(r_0,r_1)}$, which implies that the curve r_t is bounded by two straight lines $r_t=-\frac{1}{r_0^{\alpha}}t+r_1$ and $r_t=-\frac{1}{\Upsilon^{\alpha}(r_0,r_1)}t+r_1$. On the other hand, since $(r_t)_{tt}=\alpha r^{-(\alpha+1)}r_t>0$ for $t>0$, r_t is a convex function. Hence r_t would vanish at some finite time and change sign after that time, which contradicts the assumption that $r_t>0$ for $t>0$. Thus, for $\alpha>0$ and $r_1>0$, the solution $r(t)$ to IVP (3.1) increases first and then deceases and attains its zero point at some finite time.

4 Some propositions of the hyperbolic flow

Consider the general second-order operator L defined by

$$L[\omega]:=a\omega_{\theta\theta}+2b\omega_{\theta t}+c\omega_{tt}+d\omega_{\theta}+e\omega_t, \tag{4.1}$$

where a, b, c are twice continuously differentiable and d, e are continuously differentiable w.r.t. θ and t. If at a point (θ,t) the inequality

$$b^2-ac>0$$

holds, then the operator L is said to be hyperbolic at point (θ,t). It is hyperbolic in a domain D if it is hyperbolic at each point of D, and uniformly hyperbolic in a domain D if there exists a constant μ such that $b^2-ac\geq\mu>0$ in D.

Assume that ω and the conormal derivative

$$\frac{\partial\omega}{\partial\vec{\nu}}:=-b\frac{\partial\omega}{\partial\theta}-c\frac{\partial\omega}{\partial t}$$

are given at $t=0$. The adjoint operator L^* of L can be defined as follows:

$$\begin{aligned}L^*[\omega]&:=(a\omega)_{\theta\theta}+2(b\omega)_{\theta t}+(c\omega)_{tt}-(d\omega)_\theta-(e\omega)_t\\&=a\omega_{\theta\theta}+2b\omega_{\theta t}+c\omega_{tt}+(2a_\theta+2b_t-d)\omega_\theta+(2b_\theta+2c_t-e)\omega_t\\&\quad+(a_{\theta\theta}+2b_{\theta t}+c_{tt}-d_\theta-e_t)\omega.\end{aligned}$$

As shown in [6, pp. 502–503], for any hyperbolic operator L, there exists a function l satisfying the following condition:

$$\begin{cases}2\sqrt{b^2-ac}[l_t-\frac{1}{c}(\sqrt{b^2-ac}-b)l_\theta]+lK_+\geq0,\\2\sqrt{b^2-ac}[l_t+\frac{1}{c}(\sqrt{b^2-ac}-b)l_\theta]+lK_-\geq0,\\(L^*+g)[\omega]\geq0,\end{cases}\tag{4.2}$$

in a sufficiently small strip $0\leq t\leq t_0$, where

$$\begin{aligned}K_+:={}&\left(\sqrt{b^2-ac}\right)_\theta+\frac{b}{c}\left(\sqrt{b^2-ac}\right)_\theta+\frac{1}{c}(b_\theta+c_t-e)\sqrt{b^2-ac}\\&+\left[-\frac{1}{2c}\left(b^2-ac\right)_\theta+a_\theta+b_t-d-\frac{b}{c}(b_\theta+c_t-e)\right]\end{aligned}$$

and

$$\begin{aligned}K_-:={}&\left(\sqrt{b^2-ac}\right)_\theta+\frac{b}{c}\left(\sqrt{b^2-ac}\right)_\theta+\frac{1}{c}(b_\theta+c_t-e)\sqrt{b^2-ac}\\&-\left[-\frac{1}{2c}\left(b^2-ac\right)_\theta+a_\theta+b_t-d-\frac{b}{c}(b_\theta+c_t-e)\right].\end{aligned}$$

Choose the function l to be

$$l(\theta,t):=1+\eta t-\beta t^2,$$

and then it is easy to check that the coefficients restriction (4.2) becomes

$$\begin{cases}2\sqrt{b^2-ac}(\eta-2\beta t)+(1+\eta t-\beta t^2)K_+\geq0,\\2\sqrt{b^2-ac}(\eta-2\beta t)+(1+\eta t-\beta t^2)K_-\geq0,\\(L^*+g)[\omega]\geq0,\end{cases}\tag{4.3}$$

for $0\leq t\leq t_0$. However, (4.3) can be assured by suitably choosing values for η and β; for this fact, see [6, p. 503]. Once the function l is determined, the condition on the conormal derivative becomes

$$\frac{\partial \omega}{\partial \vec{\nu}} + (b_\theta + c_t - e + c\eta)\omega \le 0$$

at $t = 0$, and if we choose a constant M large enough, the following inequality

$$M \ge -(b_\theta + c_t - e + c\eta) \tag{4.4}$$

holds on Γ_0, where Γ_0 is the boundary of the initial domain.

Using the above facts, one can easily get the following maximum principle for the trip adjacent to the θ-axis (see, e.g., [6, 10]).

Lemma 4.1 *Suppose that the coefficients of the operator L given by (4.1) are bounded and have bounded first and second derivatives. Let D be an admissible domain. If t_0 and M are selected in accordance with (4.3) and (4.4), then any function ω which satisfies*

$$\begin{cases} (L+g)[\omega] \ge 0 & \textit{in } D, \\ \frac{\partial \omega}{\partial \vec{\nu}} - M\omega \le 0 & \textit{on } \Gamma_0, \\ \omega \le 0 & \textit{on } \Gamma_0, \end{cases}$$

also satisfies $\omega \le 0$ in the part of D which lies in the strip $0 \le t \le t_0$. The constants t_0 and M depend only on lower bounds for $-c$ and $\sqrt{b^2 - ac}$ and on bounds for the coefficients of L and their derivatives.

Lemma 4.1 can be used to get the following principle.

Proposition 4.2 (Containment principle) *Let F_1 and $F_2 : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^2$ be two convex solutions of (2.8). Suppose that $F_2(u,0)$ lies in the domain enclosed by $F_1(u,0)$, and $f_2(u) \ge f_1(u) \ge 0$. Then $F_2(u,t)$ is contained in the domain enclosed by $F_1(u,t)$ for all $t \in [0,T)$.*

Proof Assume that $S_1(\theta,t)$ and $S_2(\theta,t)$ are the support functions of $F_1(u,t)$ and $F_2(u,t)$, respectively. Then, under the above assumptions, it is easy to know that $S_1(\theta,t)$ and $S_2(\theta,t)$ satisfy the first equation of (2.8) with $S_2(\theta,0) \le S_1(\theta,0)$ and $S_{2t}(\theta,0) \le S_{1t}(\theta,0)$. That is, we have

$$\begin{cases} S_1 S_{1tt} + S_{1tt} S_{1\theta\theta} - S_{1\theta t}^2 + k_1^{\alpha-1} = 0, \\ S_2 S_{2tt} + S_{2tt} S_{2\theta\theta} - S_{2\theta t}^2 + k_2^{\alpha-1} = 0, \end{cases}$$

where k_1 and k_2 are the curvatures of curves $F_1(u,t)$ and $F_2(u,t)$, respectively. Set

$$\omega(\theta,t) = S_2(\theta,t) - S_1(\theta,t).$$

By direct calculation, we have

$$\begin{aligned} \omega_{tt} = S_{2tt} - S_{1tt} &= \frac{S_{2\theta t}^2 - k_2^{\alpha-1}}{S_2 + S_{2\theta\theta}} - \frac{S_{1\theta t}^2 - k_1^{\alpha-1}}{S_1 + S_{1\theta\theta}} \\ &= g(k_1,k_2)\cdot \omega_{\theta\theta} + (k_1 S_{1\theta t} + k_2 S_{2\theta t})\omega_{\theta t} + g(k_1,k_2)\cdot \omega, \end{aligned}$$

where

$$g(k_1,k_2) := \frac{k_1k_2}{k_1-k_2}\left[k_2\left(S_{1\theta t}S_{2\theta t}-k_2^{\alpha-1}\right)-k_1\left(S_{1\theta t}S_{2\theta t}-k_1^{\alpha-1}\right)\right].$$

Therefore, we have

$$\begin{cases} \omega_{tt} = g(k_1,k_2)\cdot\omega_{\theta\theta} + (k_1S_{1\theta t}+k_2S_{2\theta t})\omega_{\theta t} + g(k_1,k_2)\cdot\omega, \\ \omega_t(\theta,0) = f_1(\theta)-f_2(\theta) := \omega_1(\theta), \\ \omega(\theta,0) = h_2(\theta)-h_1(\theta) := \omega_0(\theta). \end{cases} \tag{4.5}$$

Define the operator L as

$$L[\omega] := g(k_1,k_2)\omega_{\theta\theta} + (k_1S_{1\theta t}+k_2S_{2\theta t})\omega_{\theta t} - \omega_{tt}.$$

So, it is easy to check that

$$a = g(k_1,k_2), \qquad b = \frac{1}{2}(k_1S_{1\theta t}+k_2S_{2\theta t}), \qquad c = -1$$

are twice continuously differentiable functions w.r.t. θ and t. Since

$$\begin{aligned} b^2-ac &= \frac{1}{4}(k_1S_{1\theta t}+k_2S_{2\theta t})^2 - \frac{k_1k_2}{k_1-k_2}\left[k_2\left(S_{1\theta t}S_{2\theta t}-k_2^{\alpha-1}\right)-k_1\left(S_{1\theta t}S_{2\theta t}-k_1^{\alpha-1}\right)\right]\cdot(-1) \\ &= \frac{1}{4}(k_1S_{1\theta t}-k_2S_{2\theta t})^2 + \frac{k_1k_2}{k_1-k_2}\left(k_1^{\alpha}-k_2^{\alpha}\right) \\ &> 0, \end{aligned}$$

which implies that L is uniformly hyperbolic in $\mathbb{S}^1\times[0,T)$. The last inequality holds since for $\alpha>0$, $(k_1^{\alpha}-k_2^{\alpha})/(k_1-k_2)$ is strictly positive. By Lemma 4.1, we have $S_2(\theta,t)\le S_1(\theta,t)$ for all $t\in[0,T)$. □

Proposition 4.3 (Preserving convexity) *Let k_0 be the mean curvature of F_0 and let*

$$\delta = \min_{\theta\in[0,2\pi)}\left\{k_0(\theta)\right\} > 0.$$

If $\alpha\ge 1$, then for a solution S of (2.8), *we have*

$$k(\theta,t)\ge\delta$$

for all $t\in[0,T_{\max})$, where $[0,T_{\max})$ is the maximal time interval for solution $F(u,t)$ of (1.1).

Proof By Theorem 1.1, it is easy to know that the evolving curve $F(\cdot,t)$ of the flow (1.1) remains strictly convex on some short time interval $[0,T)$ with $T\le T_{\max}$. Moreover, the support function S of $F(\cdot,t)$ satisfies

$$S_{tt} = kS_{\theta t}^2 - k^{\alpha}$$

for any $(\theta, t) \in \mathbb{S}^1 \times [0, T)$. By direct calculations, we have

$$\begin{aligned}
k_t &= \left(\frac{1}{S + S_{\theta\theta}}\right)_t = -k^2(S_t + S_{\theta\theta t}), \\
S_t + S_{\theta\theta t} &= -(S + S_{\theta\theta})^2 k_t = -\frac{1}{k^2}k_t, \\
S_{\theta t} + S_{\theta\theta\theta t} &= \left(-\frac{1}{k^2}k_t\right)_\theta = \frac{2}{k^3}k_t k_\theta - \frac{1}{k^2}k_{\theta t},
\end{aligned}$$

and

$$\begin{aligned}
k_{tt} &= \left[-\frac{1}{(S + S_{\theta\theta})^2}(S_t + S_{\theta\theta t})\right]_t \\
&= k^2\left[k^{\alpha-1} - S_{\theta t}^2 + (\alpha - 1)k^{\alpha-2}\right]k_{\theta\theta} + 2kS_{\theta t}k_{\theta t} + 4k^2 S_{\theta t}S_t k_\theta \\
&\quad - 4kS_t k_t + k^3\left[S_{\theta t}^2 - 2S_t^2 + k^{\alpha-1} + (\alpha - 1)k^\alpha(\alpha - 2 + 2k)(S_\theta + S_{\theta\theta\theta})^2\right].
\end{aligned}$$

Define the operator L as

$$L[k] := k^2\left[k^{\alpha-1} - S_{\theta t}^2 + (\alpha - 1)k^{\alpha-2}\right]k_{\theta\theta} + 2kS_{\theta t}k_{\theta t} - k_{tt} + 4k^2 S_{\theta t}S_t k_\theta - 4kS_t k_t.$$

By the definition of (4.1), we have

$$a = k^2\left[k^{\alpha-1} - S_{\theta t}^2 + (\alpha - 1)k^{\alpha-2}\right], \qquad b = kS_{\theta t}, \qquad c = -1,$$

which are twice continuously differentiable functions w.r.t. θ and t, and

$$d = 4k^2 S_{\theta t}S_t, \qquad e = -4kS_t,$$

which are continuously differentiable functions w.r.t. θ and t. Moreover, since

$$b^2 - ac = (kS_{\theta t})^2 - k^2\left[k^{\alpha-1} - S_{\theta t}^2 + (\alpha - 1)k^{\alpha-2}\right] \cdot (-1) = k^{\alpha+1} + (\alpha - 1)k^\alpha > 0$$

provided $\alpha \geq 1$, the operator L is hyperbolic in $\mathbb{S}^1 \times [0, T)$. Determine a function $k(\theta, t)$ by the following system:

$$\begin{cases}
(L + \widetilde{h})[k] & \\
\quad := L[k] + k^3[S_{\theta t}^2 - 2S_t^2 + k^{\alpha-1} + (\alpha - 1)k^\alpha(\alpha - 2 + 2k)(S_\theta + S_{\theta\theta\theta})^2] & \text{in } \mathbb{S}^1 \times [0, \mathrm{T}), \\
k(\theta, 0) = k_0(\theta) & \text{on } \Gamma_0, \\
0 \leq \frac{\partial k}{\partial \bar{\nu}} := -bk_\theta - ck_t = \beta(\theta) & \text{on } \Gamma_0,
\end{cases}$$

where Γ_0 is the boundary of the domain enclosed by the initial curve $F(\cdot, 0)$. It is easy to check that the function $\widetilde{k}(\theta, t) := \min_{\theta \in [0, 2\pi)}\{k_0(\theta)\} = \delta$ satisfies

$$\begin{cases}
(L + \widetilde{h})[\widetilde{k}] = 0 & \text{in } \mathbb{S}^1 \times [0, \mathrm{T}), \\
\widetilde{k}(\theta, 0) \leq k_0(\theta) & \text{on } \Gamma_0, \\
\frac{\partial \widetilde{k}}{\partial \bar{\nu}} - M\widetilde{k} \leq \beta(\theta) - Mk_0(\theta) & \text{on } \Gamma_0,
\end{cases}$$

where M is a constant given by (4.4). Applying Lemma 4.1 to the difference $\widetilde{k} - k$ yields

$$\widetilde{k} \leq k(\theta, t) \quad \text{in } \mathbb{S}^1 \times [0, t_0)$$

with $t_0 \leq T$. Therefore, $F(\cdot, t)$ remains convex on $[0, T_{\max})$ and the curvature of $F(\cdot, t)$ has a uniform lower bound δ on $\mathbb{S}^1 \times [0, T_{\max})$. This completes the proof. □

We also need the following properties of the evolving curves $F(\cdot, t)$ and the Blaschke selection theorem.

Lemma 4.4 *Under the hyperbolic flow* (1.1), *the arc-length* $\ell(t)$ *of the closed curve* $F(\cdot, t)$ *satisfies*

$$\frac{d\ell(t)}{dt} = -\int_0^{2\pi} \widetilde{\sigma}(\theta, t)\, d\theta$$

and

$$\frac{d^2\ell(t)}{dt^2} = \int_0^{2\pi} \left[k \left(\frac{\partial \widetilde{\sigma}}{\partial \theta} \right)^2 - k^\alpha \right] d\theta.$$

Proof By Lemma 2.3, the first- and second-order derivatives of the arc-length $\ell(t)$ are given by

$$\frac{d\ell(t)}{dt} = \frac{d}{dt} \int_0^{2\pi} \upsilon\, d\theta = \int_0^{2\pi} \frac{\partial \upsilon}{\partial t}\, d\theta = -\int_0^{2\pi} k\upsilon\widetilde{\sigma}\, d\theta = -\int_0^{2\pi} \widetilde{\sigma}\, d\theta$$

and

$$\begin{aligned} \frac{d^2\ell(t)}{dt^2} &= -\int_0^{2\pi} \frac{\partial}{\partial t} \big(\widetilde{\sigma}(\theta, t) \big)\, d\theta = \int_0^{2\pi} \frac{\partial}{\partial t} (S_t)\, d\theta = \int_0^{2\pi} S_{tt}\, d\theta \\ &= \int_0^{2\pi} \big(k S_{\theta t}^2 - k^\alpha \big)\, d\theta = \int_0^{2\pi} \left[k \left(\frac{\partial}{\partial \theta} S_t \right)^2 - k^\alpha \right] d\theta \\ &= \int_0^{2\pi} \left[k \left(\frac{\partial \widetilde{\sigma}}{\partial \theta} \right)^2 - k^\alpha \right] d\theta, \end{aligned}$$

which finishes the proof. □

Lemma 4.5 *Under the assumption of Proposition* 4.3, *the following inequality*

$$\left(\frac{\partial \widetilde{\sigma}}{\partial \theta} \right)^2 - k^{\alpha - 1} < 0$$

holds for all $t \in [0, T_{\max})$.

Proof Since

$$\frac{\partial \sigma}{\partial t} = k^\alpha > 0$$

for $t \in [0, T_{\max})$, we have

$$\sigma(u,t) > \sigma(u,0)$$

for $t \in [0, T_{\max})$, which is equivalent to say

$$\widetilde{\sigma}(\theta,t) = \sigma(u,t) > \sigma(u,0) = \widetilde{\sigma}(\theta,0)$$

for $t \in [0, T_{\max})$. This leads to the fact that

$$\frac{\partial \widetilde{\sigma}}{\partial t} > 0$$

for $t \in [0, T_{\max})$. On the other hand, by the chain rule, we can obtain

$$\frac{\partial \sigma}{\partial t} = \frac{\partial \widetilde{\sigma}}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial \widetilde{\sigma}}{\partial t} = \frac{\partial \widetilde{\sigma}}{\partial \theta}\frac{\partial \sigma}{\partial s} + \frac{\partial \widetilde{\sigma}}{\partial t} = \frac{\partial \widetilde{\sigma}}{\partial \theta}\frac{\partial \widetilde{\sigma}}{\partial \theta}\frac{\partial \theta}{\partial s} + \frac{\partial \widetilde{\sigma}}{\partial t},$$

which implies

$$\frac{\partial \widetilde{\sigma}}{\partial t} = \frac{\partial \sigma}{\partial t} - \left(\frac{\partial \widetilde{\sigma}}{\partial \theta}\right)^2 \frac{\partial \theta}{\partial s} = k^{\alpha} - k\left(\frac{\partial \widetilde{\sigma}}{\partial \theta}\right)^2.$$

Hence, together with the convexity of evolving curves on the time interval $[0, T_{\max})$, one has

$$\left(\frac{\partial \widetilde{\sigma}}{\partial \theta}\right)^2 - k^{\alpha-1} < 0.$$

This completes the proof. □

Theorem 4.6 (Blaschke selection theorem) *Let $\{K_j\}$ be a sequence of convex sets which are contained in a bounded set. Then there exist a subsequence K_{jk} and a convex set K such that $\{K_{jk}\}$ converges to K in the Hausdorff metric.*

5 Proof of Theorem 1.2

By reasonably using Example 3.1 and the containment principle, we can get the convergence of the hyperbolic flow (1.1).

Proof of Theorem 1.2 Let $[0, T_{\max})$ be the maximal time interval of the hyperbolic flow (1.1). We divide the proof into the following several steps.

Step 1. Preserving convexity

By Proposition 4.3, the evolving curves $F(\cdot,t)$ remain strictly convex on $[0, T_{\max})$ and their curvatures have a uniformly positive lower bound $\max_{\theta\in\mathbb{S}^1} k_0(\theta)$ on $\mathbb{S}^1 \times [0, T_{\max})$.

Step 2. Short-time existence

Enclose the initial curve F by a large enough round circle γ_0, and then let this circle evolve under the hyperbolic flow (1.1) with the initial velocity $\min_{u\in\mathbb{S}^1} f(u)$ to get a solution $\gamma(\cdot,t)$. By Example 3.1, we know that the solution $\gamma(\cdot,t)$ exists only at a finite time interval $[0, T^*)$, with $T^* < \infty$, and $\gamma(\cdot,t)$ shrinks into a point as $t \to T^*$. By Proposition 4.2, we know that $F(\cdot,t)$ is always enclosed by $\gamma(\cdot,t)$ for all $t \in [0,T^*)$. Therefore, $F(\cdot,t)$ must become singular at some time $T_{\max} \le T^*$.

Step 3. Hausdorff convergence

From Example 3.1, since $f(u)$ is nonnegative, the round circle $\gamma(\cdot,t)$ constructed in Step 2 is shrinking. Since, for any time $t \in [0,T_{\max})$, $F(\cdot,t)$ is enclosed by $\gamma(\cdot,t)$ and $\gamma(\cdot,t)$ is shrinking, every convex set $K_{F(\cdot,t)}$ enclosed by $F(\cdot,t)$ must be contained in an open bounded disk enclosed by $\gamma(\cdot,0) = \gamma_0$. By Theorem 4.6, we know that $F(\cdot,t)$ converges to a (maybe degenerate and non-smooth) weakly convex curve $F(\cdot,T_{\max})$ in the Hausdorff metric.

Step 4. Asymptotical behavior

By Lemma 4.5, we know that, for all $t \in [0,T_{\max})$,

$$\left(\frac{\partial\widetilde{\sigma}}{\partial\theta}\right)^2 - k^{\alpha-1} < 0$$

holds, then one can obtain from Lemma 4.4 that, for all $t \in [0,T_{\max})$,

$$\frac{d\ell(t)}{dt} < 0 \quad \text{and} \quad \frac{d^2\ell(t)}{dt^2} < 0.$$

Therefore, there exists a finite time T_0 such that $\ell(T_0) = 0$. There will be two situations (the rest is similar to Step 4 of the proof of [6, Theorem 1.2], however, for readers' convenience, we would like to write down all the details here):

Case I. $T_0 \le T_{\max}$. On the one hand, there exists a unique solution of the evolution equation (1.1) on the interval $[0,T_0)$. On the other hand, $\ell(t) \to 0$ as $t \to T_0$, which implies that the curvature k goes to infinity when $t \to T_0$, and then $F(\cdot,t)$ will blow up at time T_0. Hence, by the definition of $T_{\max}$, we have $T_0 = T_{\max}$. That is to say, $F(\cdot,t)$ converges to a point as $t \to T_{\max}$.

Case II. $T_0 > T_{\max}$. In this situation, we have $\ell(T_{\max}) > 0$, then the solution $F(\cdot,T_{\max})$ must be non-smooth. We divide the argument into the following three cases:

(1) $\|F(u,T_{\max})\| = \sup_{u\in\mathbb{S}^1}|F(u,T_{\max})| = \infty$. However, as shown in Step 2, we know that $F(\cdot,t)$ is contained by the initial curve F_0, and then $\|F(u,T_{\max})\|$ must be bounded, which is a contradiction. So, this case is impossible.

(2) $\|F_u(u,T_{\max})\| = \infty$, then the length of the limit curve satisfies

$$\begin{aligned}
\ell(T_{\max}) &= \lim_{t\to T_{\max}} \int_{F(u,t)} ds \\
&= \lim_{t\to T_{\max}} \int_{F(u,t)} \left|F_u(u,t)\right| du \\
&= \int_{F(u,t)} \lim_{t\to T_{\max}} \left|F_u(u,t)\right| du \\
&= \infty,
\end{aligned}$$

which is contradict with the fact $\ell(T_{\max}) < \ell(0) < \infty$. So, this case is also impossible.

(3) The curvature k is discontinuous. We cannot exclude this case, and then this phenomenon will occur if the above shocks are impossible.

This completes our proof. □

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Authors' contributions
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