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The modified split generalized equilibrium problem for quasi-nonexpansive mappings and applications

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Abstract

In this paper, we introduce a new problem, the modified split generalized equilibrium problem, which extends the generalized equilibrium problem, the split equilibrium problem and the split variational inequality problem. We introduce a new method of an iterative scheme $\{x_n\}$ for finding a common element of the set of solutions of variational inequality problems and the set of common fixed points of a finite family of quasi-nonexpansive mappings and the set of solutions of the modified split generalized equilibrium problem without assuming a demicloseness condition and $T_\omega := (1-\omega)I + \omega T$, where T is a quasi-nonexpansive mapping and $\omega \in (0, \frac{1}{2})$; a difficult proof in the framework of Hilbert space. In addition, we give a numerical example to support our main result.

MSC: 47B40; 47H10; 47J20

Keywords: The modified split generalized equilibrium problem; Quasi-nonexpansive mapping; Variational Inequality problem; Fixed Point problem

1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. The set of fixed points of T is denoted by $F(T)$. The mapping $T : C \to C$ is said to be *quasi-nonexpansive* if

$$\|Tx - p\| \leq \|x - p\|,$$

for all $x \in C$ and $p \in F(T)$.

Definition 1.1 ([1]) Let $T : H \to H$. Then the following are equivalent:

1. T is firmly nonexpansive,
2. $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle$, $\forall x, y \in H$,
3. $\langle Tx - Ty, (I - T)x - (I - T)y\rangle \geq 0$, $\forall x, y \in H$.

Let $A : C \to H$ be a mapping. The *variational inequality* is to find a point $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \tag{1.1}$$

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for all $v \in C$. The set of solutions of (1.1) is denoted by $VI(C,A)$. A mapping $A : C \to H$ is called *α-inverse strongly monotone* if there exists a positive real number $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha \|Ax - Ay\|^2,$$

for all $x, y \in C$. They have been investigated in the literature; see, for example, [2, 3]. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The *equilibrium problem* for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $EP(F)$. Equilibrium problems were introduced by [4] in 1994 and included many well-known problems such as variational inequality, optimization problem, nonexpansive mapping and fixed point problem; see, for example, [5–8].

Let F be a function of $C \times C$ into $\mathbb{R}$ and let $f : H \to H$ be a mapping. The *generalized equilibrium problem* is to find $x \in C$ such that

$$F(x,y) + \langle f(x), y - x\rangle \geq 0, \tag{1.3}$$

for all $y \in C$. The set of solutions of (1.3) is denoted by $EP(F,f)$. When $f \equiv 0$, $EP(F,f)$ is denoted by $EP(F)$ and $F \equiv 0$, $EP(F,f)$ is denoted by $VI(C,f)$.

Throughout this section, let H_1, H_2 be real Hilbert spaces and let C, Q be nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator.

In 1994, Censor and Elfving [9] introduced the *split feasibility problem* (in short, SFP) which is to find a point $x \in C$ such that $Ax \in Q$. The set of all solutions of split feasibility problem is denoted by $\varphi = \{x \in C : Ax \in Q\}$.

To solve the SFP, Byrne [10] introduced CQ algorithm whose sequence $\{x_n\}$ is generated by

$$x_{n+1} = P_{C_1}\bigl(x_n - \gamma A^*(I - P_{C_2})Ax_n\bigr),$$

where the initial $x_0 \in H_1$ and $\gamma \in (0, 2/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Then the CQ algorithm converges to a solution of the SFP, whenever solutions exist. If there are no solutions of the SFP, the CQ algorithm converges to a minimizer of the function

$$\frac{1}{2}\bigl\|(I - P_{C_2})Ax\bigr\|^2,$$

whenever such minimizers exist.

Let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be two nonlinear operators. The *split common fixed points problem* (SCFPP) [11, 12] is to find a point x^* such that

$$x^* \in F(U) \quad \text{and} \quad Ax^* \in F(T).$$

The solution set of SCFPP is denoted by $\Phi = \{p^* \in F(U) : Ap^* \in F(T)\}$. The split common fixed point problem is a generalization of the split feasibility problem.

In 2017, Wang [13] introduced a new method for solving SCFPP as follows:

$$x_{n+1} = x_n - \rho_n\big((I-U)x_n + A^*(I-T)Ax_n\big),$$

where $\rho_n \subset (0,\infty)$ is chosen such that

$$\rho_n = \frac{\|(I-U)x_n\|^2 + \|(I-T)Ax_n\|^2}{\|(I-U)x_n + A^*(I-T)Ax_n\|^2} \tag{1.4}$$

and U and T are firmly quasi-nonexpansive mappings. Then the sequence $\{x_n\}$ converges weakly to z, where $z = \lim_{n\to\infty} P_\Phi x_n$.

Censor et al. [11, 14] introduced the prototypical *split inverse problem* (SIP) which is a generalization of the split common fixed points problem. In this, there are given two vector spaces X and Y and a linear operator $A : X \to Y$. In addition, two inverse problems are involved. The first one, denoted IP_1, is formulated in the space X and the second one, denoted IP_2, is formulated in the space Y. Given these data, the split inverse problem is formulated as follows:

$$\text{find a point } x^* \in X \text{ that solves } \mathrm{IP}_1, \tag{1.5}$$

and such that

$$\text{find a point } y^* \in Y \text{ that solves } \mathrm{IP}_2. \tag{1.6}$$

This problem is used in many modeling arising in sensor networks, radiation therapy treatment planning, color imaging, etc.

The *split equilibrium problem* (SEP) [12] is to find $\widehat{x} \in C$ such that

$$F_1(\widehat{x},x) \geq 0, \quad \forall x \in C, \tag{1.7}$$

and such that

$$\widehat{y} = A\widehat{x} \in Q \quad \text{solves } F_2(\widehat{y},y) \geq 0, \quad \forall y \in Q, \tag{1.8}$$

where $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be nonlinear bifunctions. If we consider only problem (1.7), it is the equilibrium problem and we denoted its solution set by $EP(F_1)$. The solution set of SEP is denoted by $\Gamma = \{\widehat{p} \in EP(F_1) : A\widehat{p} \in EP(F_2)\}$. SEP is reduced to $EP(F)$, where $H_1 \equiv H_2$, $F_1 \equiv F_2$ and $A \equiv I$. $EP(F)$ is an unifying model for several problems arising in physics, engineering, science, optimization, economics, etc.

The *split variational inequality problems* (in short, SVIP) were introduced and studied by Cencor et al. [11]: find $\overline{x} \in C$ such that

$$\big\langle f_1(\overline{x}), x - \overline{x}\big\rangle \geq 0, \quad \forall x \in C, \tag{1.9}$$

and such that

$$\overline{y} = A\overline{x} \in Q \quad \text{solves } \langle f_2(\overline{y}), y - \overline{y} \rangle \geq 0, \quad \forall y \in Q, \tag{1.10}$$

where $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ are nonlinear mappings. The solution set of SVIP is denoted by $\Psi = \{\overline{p} \in VI(C, f_1) : A\overline{p} \in VI(Q, f_2)\}$. The split variational inequality problems have already been studied and used in practice as a model in intensity-modulated radiation therapy (IMRT) treatment planning; see, for example, [15] and the modeling of many inverse problems arising for phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression; see, for example, [16, 17].

By investigating SEP and SVIP, we introduce the *modified split generalized equilibrium problem* (MSGEP) which is to find $x^* \in C$ such that

$$F_1(x^*, x) + \langle f_1(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{1.11}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves } F_2(y^*, y) + \langle f_2(y^*), y - y^* \rangle \geq 0, \quad \forall y \in Q, \tag{1.12}$$

where $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are nonlinear bifunctions and $f_1 : C \to H_1$ and $f_2 : Q \to H_2$ are nonlinear mappings. The solution set of MSGEP is denoted by $\Omega = \{p^* \in EP(F_1, f_1) : Ap^* \in EP(F_2, f_2)\}$.

Remark 1.1

1. If we put $f_1 \equiv f_2 \equiv 0$ in MSGEP then the MSGEP is reduced to SEP.
2. If we put $F_1 \equiv F_2 \equiv 0$ in MSGEP then the MSGEP is reduced to SVIP.
3. In the case of bifunctions F_1 and F_2 are according to (A1)–(A4). From (1.11), (1.12) and Lemma 2.2, we have $x^* \in F(T_r^{F_1}(I - rf_1))$ and $Ax^* \in F(T_s^{F_2}(I - sf_2))$, for all $r, s > 0$. So, MSGEP can be viewed as SCFPP.

MSGEP is a generalization of the generalized equilibrium problem, the split equilibrium problem and the split variational inequality problem. So, this problem can be used in sensor networks, data compression, practice as a model in intensity-modulated radiation therapy (IMRT) treatment planning, robustness to marginal changes and equilibrium stability etc.

Example 1.2 Let $H_1 = [0, 6]$, $H_2 = [0, 18]$, $C = [2, 5]$ and $Q = [6, 10]$. Let $A : H_1 \to H_2$ be defined by $Ax = 3x$ for all $x \in H_1$. Let the mapping $F_1 : C \times C \to \mathbb{R}$ be defined by

$$F_1(x^*, x) = -(x^* - 2)^2 + (x - 2)^2, \quad \forall x, y \in C,$$

and $F_2 : Q \times Q \to \mathbb{R}$ be defined by

$$F_2(y^*, y) = -(y^* - 6)^2 + (y - 6)^2, \quad \forall x, y \in Q.$$

Let the mapping $f_1 : C \to H_1$ be defined by $f_1 x = \frac{x-2}{9}$, $\forall x \in C$ and the mapping $f_2 : Q \to H_2$ be defined by $f_2 x = \frac{x-6}{7}$, $\forall x \in Q$.

Then $2 \in \Omega$. Therefore 2 is a solution of MSGEP.

In 2012, Tain and Jin [18] introduced iterative algorithms involving a quasi-nonexpansive mapping. They generated the iterative as follows:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) T_\omega x_n,$$

where A is a bounded linear operator on H, T is a quasi-nonexpansive mapping on H, f is a contraction with coefficient a under suitable conditions of the parameters α_n, γ and ω. By assuming $\omega \in (0, \frac{1}{2})$, $T_\omega := (1 - \omega)I + \omega T$ and T is demiclosed on H.

Motivated by SFP and SVIP, we introduced a new problem, the modified split generalized equilibrium problem, which extends the generalized equilibrium problem, the split equilibrium problem and the split variational inequality problem. Many authors proved strong convergence theorem involving a quasi-nonexpansive mapping T by assuming $T_\omega := (1 - \omega)I + \omega T$ and T is demiclosed on H; a difficult proof. Motivated by [19], we introduced Remark 2.5 and [11, 12] and [18], we introduce a new method of iterative scheme $\{x_n\}$ for finding a common element of the set of solutions of variational inequality problems and the set of common fixed points of a finite family of quasi-nonexpansive mappings and the set of solutions of the modified split generalized equilibrium problem without the condition above in the framework of a Hilbert space.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Throughout this paper, we use the notations of weak and strong convergence by "$\rightharpoonup$" and "$\to$", respectively. Recall that H satisfies *Opial's condition* [20], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\|$, holds for every $y \in H$ with $y \neq x$.

For solving the equilibrium problem, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfy the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$,

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \le 0$ for all $x, y \in C$,

(A3) for each $x, y, z \in C$, $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \le F(x, y)$,

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.1 ([4]) *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)–(A4). *Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \ge 0, \quad \forall y \in C.$$

Lemma 2.2 ([21]) *Assume that $F : C \times C \to \mathbb{R}$ satisfies* (A1)–(A4). *For $r > 0$, define a mapping $T_r : H \to C$ as follows*:

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \ge 0, \forall y \in C \right\}$$

for all $x \in H$. Then the following hold:

(1) T_r *is single-valued*,

(2) T_r *is firmly nonexpansive*, *i.e.*, *for any* $x, y \in H$,

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle,$$

(3) $F(T_r) = EP(F)$,

(4) $EP(F)$ *is closed and convex*.

Lemma 2.3 ([22]) *Let* H *be a real Hilbert space*, *let* C *be a nonempty closed convex subset of* H *and let* A *be a mapping of* C *into* H. *Let* $u \in C$. *Then*, *for* $\lambda > 0$,

$$u = P_C(I - \lambda A)u \quad \Leftrightarrow \quad u \in VI(C, A),$$

where P_C *is the metric projection of* H *onto* C.

Lemma 2.4 *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\{T_i\}_{i=1}^N$ *be a finite family of quasi-nonexpansive mappings of* C *into* H *with* $\bigcap_{i=1}^N F(T_i) \neq \emptyset$ *and let* $0 < a_i < 1$ *with* $\sum_{i=1}^N a_i = 1$. *Then*

$$\bigcap_{i=1}^N F(T_i) = VI\left(C, \sum_{i=1}^N a_i(I - T_i)\right).$$

Proof In this lemma, we show that $\bigcap_{i=1}^N F(T_i) = \bigcap_{i=1}^N VI(C, I - T_i)$ and $\bigcap_{i=1}^N VI(C, I - T_i) = VI(C, \sum_{i=1}^N a_i(I - T_i))$. Lastly, we have

$$\bigcap_{i=1}^N F(T_i) = VI\left(C, \sum_{i=1}^N a_i(I - T_i)\right).$$

To start with, it is easy to see that $\bigcap_{i=1}^N F(T_i) \subseteq \bigcap_{i=1}^N VI(C, I - T_i)$. Next, we show that $\bigcap_{i=1}^N VI(C, I - T_i) \subseteq \bigcap_{i=1}^N F(T_i)$. Let $u \in \bigcap_{i=1}^N VI(C, I - T_i)$ and $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. So, we get $u \in VI(C, I - T_i)$, $\forall i = 1, 2, \ldots, N$. We may write

$$\langle u - v, (I - T_i)u\rangle \leq 0, \quad \forall v \in C. \tag{2.1}$$

There exists $v^* \in C$ such that $v^* = T_i v^*$, $\forall i = 1, 2, \ldots, N$. Since T_i is a quasi-nonexpansive mapping, $\forall i = 1, 2, \ldots, N$, it follows that

$$\begin{aligned} \|T_i u - v^*\|^2 &= \|(u - v^*) - (I - T_i)u\|^2 \\ &= \|u - v^*\|^2 - 2\langle u - v^*, (I - T_i)u\rangle + \|(I - T_i)u\|^2 \\ &\leq \|u - v^*\|^2. \end{aligned} \tag{2.2}$$

By using (2.1) and (2.2), we conclude that

$$\|(I - T_i)u\|^2 \leq 2\langle u - v^*, (I - T_i)u\rangle \leq 0.$$

It implies that $u \in \bigcap_{i=1}^N F(T_i)$. Therefore $\bigcap_{i=1}^N VI(C, I - T_i) \subseteq \bigcap_{i=1}^N F(T_i)$. Hence

$$\bigcap_{i=1}^N F(T_i) = \bigcap_{i=1}^N VI(C, I - T_i).$$

After that, we show $\bigcap_{i=1}^N VI(C, I - T_i) = VI(C, \sum_{i=1}^N a_i(I - T_i))$ where $0 < a_i < 1$ and $\sum_{i=1}^N a_i = 1$. Observe that

$$\begin{aligned} & u \in \bigcap_{i=1}^N VI(C, I - T_i) \\ & \quad \Leftrightarrow \quad u \in VI(C, I - T_i), \quad \forall i = 1, 2, \dots, N \\ & \quad \Leftrightarrow \quad \langle (I - T_i)u, v - u \rangle \geq 0, \quad \forall v \in C \text{ and } \forall i = 1, 2, \dots, N \\ & \quad \Leftrightarrow \quad \sum_{i=1}^N a_i \langle (I - T_i)u, v - u \rangle \geq 0, \quad \forall v \in C \\ & \quad \Leftrightarrow \quad \left\langle \sum_{i=1}^N a_i (I - T_i)u, v - u \right\rangle \geq 0, \quad \forall v \in C \\ & \quad \Leftrightarrow \quad u \in VI\left(C, \sum_{i=1}^N a_i (I - T_i) \right). \end{aligned}$$

Therefore $\bigcap_{i=1}^N VI(C, I - T_i) = VI(C, \sum_{i=1}^N a_i(I - T_i))$. Hence $\bigcap_{i=1}^N F(T_i) = VI(C, \sum_{i=1}^N a_i(I - T_i))$. □

Remark 2.5 From Lemma 2.3 and Lemma 2.4, we have

$$\bigcap_{i=1}^N F(T_i) = VI\left(C, \sum_{i=1}^N a_i (I - T_i) \right) = F\left(P_C\left(I - \lambda\left(\sum_{i=1}^N a_i (I - T_i) \right) \right) \right),$$

for all $\lambda > 0$ and $0 < a_i < 1$ with $\sum_{i=1}^N a_i = 1$.

Lemma 2.6 ([23]) *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

$$\text{(1)} \quad \sum_{n=1}^\infty \alpha_n = \infty, \qquad \text{(2)} \quad \limsup_{n \to \infty} \frac{\delta_n}{\alpha_n} \leq 0 \quad \textit{or} \quad \sum_{n=1}^\infty |\delta_n| < \infty.$$

Then $\lim_{n \to \infty} s_n = 0$.

3 Main results

Lemma 3.1 *Let C and Q be nonempty closed convex subsets of a real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $F_1 : C \times C \to \mathbb{R}$ and*

$F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying (A1)–(A4). *Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Then*

1. $T_r^{F_1}(I - rf_1)$ *and* $T_s^{F_2}(I - sf_2)$ *are nonexpansive mapping,*
2. $$\begin{aligned}&\bigl\| T_r^{F_1}(I - rf_1)\bigl(p + \gamma A^*\bigl(T_s^{F_2}(I - sf_2) - I\bigr)Ap\bigr) \\ &\quad - T_r^{F_1}(I - rf_1)\bigl(q + \gamma A^*\bigl(T_s^{F_2}(I - sf_2) - I\bigr)Aq\bigr)\bigr\|^2 \\ &\quad \le \|p - q\|^2 + \gamma(\gamma L - 1)\bigl\|\bigl(T_s^{F_2}(I - sf_2) - I\bigr)Ap - \bigl(T_s^{F_2}(I - sf_2) - I\bigr)Aq\bigr\|^2,\end{aligned}$$

for all $p, q \in C$, where $r \in (0, 2\rho)$, $s \in (0, 1)$, $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A, $T_r^{F_1} : H_1 \to C$ defined by

$$T_r^{F_1}(x) = \left\{ z \in C : F_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C \right\},$$

for all $x \in H_1$ and $T_s^{F_2} : H_2 \to Q$ defined by

$$T_s^{F_2}(\overline{x}) = \left\{ \overline{z} \in Q : F_2(\overline{z}, y) + \frac{1}{s}\langle y - \overline{z}, \overline{z} - \overline{x}\rangle \ge 0, \forall y \in Q \right\},$$

for all $\overline{x} \in H_2$.

Proof Let $p, q \in C$. First, we show 1 is true. Since f_1 is a ρ-inverse strongly monotone mapping and $r \in (0, 2\rho)$, we obtain

$$\begin{aligned}\bigl\| T_r^{F_1}(I - rf_1)p - T_r^{F_1}(I - rf_1)q\bigr\|^2 &\le \|p - q\|^2 - 2r\langle p - q, f_1p - f_1q\rangle + r^2\|f_1p - f_1q\|^2 \\ &\le \|p - q\|^2 + r(r - 2\rho)\|f_1p - f_1q\|^2 \\ &\le \|p - q\|^2.\end{aligned}$$

Thus $T_r^{F_1}(I - rf_1)$ is a nonexpansive mapping. Since f_2 is a firmly nonexpansive mapping and $s \in (0, 1)$, we get

$$\begin{aligned}\bigl\| T_s^{F_2}(I - sf_2)\overline{p} - T_s^{F_2}(I - sf_2)\overline{q}\bigr\|^2 &\le \|\overline{p} - \overline{q}\|^2 - 2s\langle \overline{p} - \overline{q}, f_2\overline{p} - f_2\overline{q}\rangle + s^2\|f_2\overline{p} - f_2\overline{q}\|^2 \\ &\le \|\overline{p} - \overline{q}\|^2 - s(2 - s)\|f_2\overline{p} - f_2\overline{q}\|^2 \\ &\le \|\overline{p} - \overline{q}\|^2,\end{aligned}$$

for all $\overline{p}, \overline{q} \in Q$. Therefore $T_s^{F_2}(I - sf_2)$ is a nonexpansive mapping.

Next, we show 2 is true. From Lemma 3.1(1), we have

$$\begin{aligned}&\bigl\| T_r^{F_1}(I - rf_1)\bigl(p + \gamma A^*\bigl(T_s^{F_2}(I - sf_2) - I\bigr)Ap\bigr) \\ &\qquad - T_r^{F_1}(I - rf_1)\bigl(q + \gamma A^*\bigl(T_s^{F_2}(I - sf_2) - I\bigr)Aq\bigr)\bigr\|^2 \\ &\quad \le \bigl\|(p - q) + \gamma\bigl(A^*\bigl(T_s^{F_2}(I - sf_2) - I\bigr)Ap - A^*\bigl(T_s^{F_2}(I - sf_2) - I\bigr)Aq\bigr)\bigr\|^2 \\ &\quad \le \|p - q\|^2 + 2\gamma\bigl\langle Ap - Aq, \bigl(T_s^{F_2}(I - sf_2) - I\bigr)Ap - \bigl(T_s^{F_2}(I - sf_2) - I\bigr)Aq\bigr\rangle \\ &\qquad + \gamma^2L\bigl\|\bigl(T_s^{F_2}(I - sf_2) - I\bigr)Ap - \bigl(T_s^{F_2}(I - sf_2) - I\bigr)Aq\bigr\|^2.\end{aligned} \tag{3.1}$$

From the property of $T_s^{F_2}$, we get

$$\begin{aligned}
&\bigl\|(I-sf_2)Ap-(I-sf_2)Aq\bigr\|^2\\
&\quad\geq \bigl\|T_s^{F_2}(I-sf_2)Ap-T_s^{F_2}(I-sf_2)Aq-(Ap-Aq)+(Ap-Aq)\bigr\|^2\\
&\quad= \bigl\|\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Ap-\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Aq\bigr\|^2\\
&\qquad+2\bigl\langle\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Ap-\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Aq,Ap-Aq\bigr\rangle\\
&\qquad+\|Ap-Aq\|^2.
\end{aligned}\tag{3.2}$$

We have

$$\begin{aligned}
&\bigl\|(I-sf_2)Ap-(I-sf_2)Aq\bigr\|^2\\
&\quad=\|Ap-Aq\|^2-2s\langle Ap-Aq,f_2Ap-f_2Aq\rangle\\
&\qquad+s^2\|f_2Ap-f_2Aq\|^2.
\end{aligned}\tag{3.3}$$

From (3.2), (3.3) and the property of firmly nonexpansive mapping, we get

$$\begin{aligned}
&2\bigl\langle\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Ap-\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Aq,Ap-Aq\bigr\rangle\\
&\quad\leq -\bigl\|\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Ap-\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Aq\bigr\|^2\\
&\qquad-2s\langle Ap-Aq,f_2Ap-f_2Aq\rangle+s^2\|f_2Ap-f_2Aq\|^2\\
&\quad\leq -\bigl\|\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Ap-\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Aq\bigr\|^2.
\end{aligned}$$

That is,

$$\begin{aligned}
&2\gamma\bigl\langle\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Ap-\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Aq,Ap-Aq\bigr\rangle\\
&\quad\leq -\gamma\bigl\|\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Ap-\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Aq\bigr\|^2.
\end{aligned}\tag{3.4}$$

Substituting (3.4) in (3.1), we obtain

$$\begin{aligned}
&\bigl\|T_r^{F_1}(I-rf_1)\bigl(p+\gamma A^*\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Ap\bigr)\\
&\qquad-T_r^{F_1}(I-rf_1)\bigl(q+\gamma A^*\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Aq\bigr)\bigr\|^2\\
&\quad\leq\|p-q\|^2-\gamma\bigl\|\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Ap-\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Aq\bigr\|^2\\
&\qquad+\gamma^2L\bigl\|\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Ap-\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Aq\bigr\|^2\\
&\quad=\|p-q\|^2+\gamma(\gamma L-1)\bigl\|\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Ap-\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Aq\bigr\|^2.
\end{aligned}$$ □

Lemma 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T:C\to C$ be a quasi-nonexpansive mapping with $F(T)\neq\emptyset$. Then*

$$\bigl\|(I-T)x\bigr\|^2\leq 2\bigl\langle x-z,(I-T)x\bigr\rangle,\quad\forall x\in C.$$

Proof Let $x \in C$ and $z \in F(T)$. Since T is a quasi-nonexpansive mapping, we get

$$\begin{aligned}\|Tx - z\|^2 &= \big\|(x - z) - (I - T)x\big\|^2 \\ &= \|x - z\|^2 - 2\langle x - z, (I - T)x\rangle + \big\|(I - T)x\big\|^2 \\ &\le \|x - z\|^2.\end{aligned}$$

We can conclude that

$$\big\|(I - T)x\big\|^2 \le 2\langle x - z, (I - T)x\rangle. \qquad \square$$

Lemma 3.3 *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Then*

$$\left\|P_C\left(I - \overline{\lambda}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)x - z\right\|^2 \le \|x - z\|^2,$$

for all $x \in C$, where $0 < k_i < 1$ with $\sum_{i=1}^N k_i = 1$ and $0 < \overline{\lambda} < 1$.

Proof Let $x \in C$ and $z \in \bigcap_{i=1}^N F(T_i)$. From Remark 2.5 and $z \in \bigcap_{i=1}^N F(T_i)$, we have $z \in F(P_C(I - \overline{\lambda}(\sum_{i=1}^N k_i(I - T_i))))$ and $z = T_i z$, $\forall i = 1, 2, \dots, N$. Since P_C is nonexpansive mapping, $0 < \overline{\lambda} < 1$ and Lemma 3.2, we have

$$\begin{aligned}&\left\|P_C\left(I - \overline{\lambda}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)x - z\right\|^2 \\ &\quad = \left\|P_C\left(I - \overline{\lambda}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)x - P_C\left(I - \overline{\lambda}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)z\right\|^2 \\ &\quad \le \|x - z\|^2 - 2\overline{\lambda}\sum_{i=1}^N k_i\langle x - z, (I - T_i)x\rangle + \overline{\lambda}^2\sum_{i=1}^N k_i\big\|(I - T_i)x\big\|^2 \\ &\quad \le \|x - z\|^2 - \overline{\lambda}\sum_{i=1}^N k_i\big\|(I - T_i)x\big\|^2 + \overline{\lambda}^2\sum_{i=1}^N k_i\big\|(I - T_i)x\big\|^2 \\ &\quad \le \|x - z\|^2. \end{aligned} \tag{3.5}$$

Next, we prove a strong convergence theorem for solving the modified split generalized equilibrium problem (MSGEP).

Theorem 3.4 *Let C and Q be nonempty closed convex subsets of a real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying* (A1)–(A4). *Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Assume $\mathcal{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Omega \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}$, $\{u_n\}$ and*

$\{y_n\}$ *be sequences generated by*

$$\begin{cases} u_n = T_r^{F_1}(I - rf_1)(x_n + \gamma A^*(T_s^{F_2}(I - sf_2) - I)Ax_n), \\ y_n = P_C(I - d_1D_1)(au_n + (1-a)P_C(I - d_2D_2)u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(\sum_{i=1}^N k_i(I - T_i)))y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.6}$$

where $d_1 \in (0, 2\alpha)$, $d_2 \in (0, 2\beta)$, $r \in (0, 2\rho)$, $s \in (0,1)$, $a \in [0,1]$, $0 < k_i < 1$ *with* $\sum_{i=1}^N k_i = 1$, $\gamma \in (0, 1/L)$, *L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *are sequences in* $[0,1]$ *with* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$. *Suppose the following conditions hold*:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $0 < c \le \beta_n, \gamma_n \le d < 1$ *for some* $c, d > 0$ *for all* $n \ge 1$,
(iii) $\sum_{n=1}^\infty \lambda_n < \infty$ *and* $0 < \lambda_n < 1$,
(iv) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < \infty$.

Then $\{x_n\}$, $\{u_n\}$ *and* $\{y_n\}$ *converge strongly to* $z = P_{\mathcal{F}}u$.

Proof Let $x, y \in C$ and $z \in \mathcal{F}$. First, we show that $(I - d_1D_1)$ is a nonexpansive mapping. Since D_1 is an α-inverse strongly monotone mapping, we obtain

$$\begin{aligned} \|(I - d_1D_1)x - (I - d_1D_1)y\|^2 &= \|x - y\|^2 - 2d_1\langle x - y, D_1x - D_1y\rangle + d_1^2\|D_1x - D_1y\|^2 \\ &\le \|x - y\|^2 + d_1(d_1 - 2\alpha)\|D_1x - D_1y\|^2 \le \|x - y\|^2. \end{aligned}$$

Thus $(I - d_1D_1)$ is a nonexpansive mapping. By using the same method as above, we see that $(I - d_2D_2)$ is a nonexpansive mapping. Since f_1 is a ρ-inverse strongly monotone mapping and f_2 is a firmly nonexpansive mapping. From Lemma 3.1(1), we have $(T_r^{F_1}(I - rf_1))$ and $(T_s^{F_2}(I - sf_2))$ are nonexpansive mappings. Since $z \in \bigcap_{i=1}^N F(T_i)$ and Lemma 3.3, we have

$$\left\| P_C\left(I - \lambda_n\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)y_n - z\right\|^2 \le \|y_n - z\|^2. \tag{3.7}$$

Since $z \in VI(C, D_1)$ and $z \in VI(C, D_2)$ and using the property of $(I - d_1D_1)$ and $(I - d_2D_2)$, we get

$$\begin{aligned} \|y_n - z\|^2 &= \|P_C(I - d_1D_1)(au_n + (1-a)P_C(I - d_2D_2)u_n) - P_C(I - d_1D_1)z\|^2 \\ &\le a\|u_n - z\|^2 + (1-a)\|P_C(I - d_2D_2)u_n - z\|^2 &(3.8) \\ &\le \|u_n - z\|^2. &(3.9) \end{aligned}$$

Since $z \in \Omega$, we have $z = T_r^{F_1}(I - rf_1)z$ and $Az = T_s^{F_2}(I - sf_2)Az$. From Lemma 3.1(2) and $\gamma \in (0, 1/L)$, we obtain

$$\begin{aligned} \|u_n - z\|^2 &= \|T_r^{F_1}(I - rf_1)(x_n + \gamma A^*(T_s^{F_2}(I - sf_2) - I)Ax_n) - T_r^{F_1}(I - rf_1)z\|^2 \\ &\le \|x_n - z\|^2 + \gamma(L\gamma - 1)\|(T_s^{F_2}(I - sf_2) - I)Ax_n\|^2 &(3.10) \\ &\le \|x_n - z\|^2. &(3.11) \end{aligned}$$

Using the definition of x_n, (3.7), (3.9) and (3.11), we get

$$\begin{aligned}\|x_{n+1}-z\| &= \left\|\alpha_n(u-z)+\beta_n(x_n-z)\right.\\ &\quad \left.+\gamma_n\left(P_C\left(I-\lambda_n\left(\sum_{i=1}^N k_i(I-T_i)\right)\right)y_n-z\right)\right\|\\ &\le \alpha_n\|u-z\|+\beta_n\|x_n-z\|+\gamma_n\|y_n-z\|\\ &\le \alpha_n\|u-z\|+\beta_n\|x_n-z\|+\gamma_n\|u_n-z\|\\ &\le \alpha_n\|u-z\|+(1-\alpha_n)\|x_n-z\|.\end{aligned}$$

Using induction, we can conclude that

$$\|x_n-z\|\le \max\bigl\{\|u-z\|,\|x_1-z\|\bigr\}$$

for all $n\ge 1$. This implies that the sequence $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{u_n\}$. From Lemma 3.1 (2) and $\gamma\in(0,1/L)$, we obtain

$$\begin{aligned}&\|u_n-u_{n-1}\|^2\\ &\quad=\bigl\|T_r^{F_1}(I-rf_1)\bigl(x_n+\gamma A^*\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Ax_n\bigr)\\ &\qquad -T_r^{F_1}(I-rf_1)\bigl(x_{n-1}+\gamma A^*\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Ax_{n-1}\bigr)\bigr\|^2\\ &\quad\le \|x_n-x_{n-1}\|^2+\gamma(\gamma L-1)\bigl\|\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Ax_n-\bigl(T_s^{F_2}(I-sf_2)-I\bigr)Ax_{n-1}\bigr\|^2\\ &\quad\le \|x_n-x_{n-1}\|^2. \end{aligned}\tag{3.12}$$

Next, we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. According to Eq. (3.12), we have

$$\begin{aligned}&\|x_{n+1}-x_n\|\\ &\quad=\left\|\left(\alpha_n u+\beta_n x_n+\gamma_n P_C\left(I-\lambda_n\left(\sum_{i=1}^N k_i(I-T_i)\right)\right)y_n\right)\right.\\ &\qquad\left.-\left(\alpha_{n-1}u+\beta_{n-1}x_{n-1}+\gamma_{n-1}P_C\left(I-\lambda_{n-1}\left(\sum_{i=1}^N k_i(I-T_i)\right)\right)y_{n-1}\right)\right\|\\ &\quad\le|\alpha_n-\alpha_{n-1}|\|u\|+\beta_n\|x_n-x_{n-1}\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+\gamma_n\|y_n-y_{n-1}\|\\ &\qquad+\lambda_n\left\|\left(\sum_{i=1}^N k_i(I-T_i)\right)y_n-\left(\sum_{i=1}^N k_i(I-T_i)\right)y_{n-1}\right\|\\ &\qquad+|\lambda_n-\lambda_{n-1}|\left\|\left(\sum_{i=1}^N k_i(I-T_i)\right)y_{n-1}\right\|\\ &\qquad+|\gamma_n-\gamma_{n-1}|\left\|P_C\left(I-\lambda_{n-1}\left(\sum_{i=1}^N k_i(I-T_i)\right)\right)y_{n-1}\right\|\\ &\quad\le(1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|u\|+|\beta_n-\beta_{n-1}|\|x_{n-1}\|\end{aligned}$$

$$
\begin{aligned}
&+ \lambda_n \left\| \left(\sum_{i=1}^{N} k_i (I - T_i) \right) y_n - \left(\sum_{i=1}^{N} k_i (I - T_i) \right) y_{n-1} \right\| \\
&+ |\lambda_n - \lambda_{n-1}| \left\| \left(\sum_{i=1}^{N} k_i (I - T_i) \right) y_{n-1} \right\| \\
&+ |\gamma_n - \gamma_{n-1}| \left\| P_C \left(I - \lambda_{n-1} \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_{n-1} \right\| \\
\leq{}& (1 - \alpha_n) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| M + |\beta_n - \beta_{n-1}| M + \lambda_n M \\
&+ |\lambda_n - \lambda_{n-1}| M + |\gamma_n - \gamma_{n-1}| M,
\end{aligned}
$$

where

$$
\begin{aligned}
M := \max_{n \in \mathbb{N}} \Bigg\{ &\|u\|, \|x_n\|, \left\| \left(\sum_{i=1}^{N} k_i (I - T_i) \right) y_{n+1} - \left(\sum_{i=1}^{N} k_i (I - T_i) \right) y_n \right\|, \\
&\left\| \left(\sum_{i=1}^{N} k_i (I - T_i) \right) y_n \right\|, \left\| P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_n \right\| \Bigg\}.
\end{aligned}
$$

From condition (i), (iii), (iv) and Lemma 2.6, we have

$$
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.13}
$$

According to Eqs. (3.7), (3.9) and (3.10), we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 \leq{}& \alpha_n \|u - z\|^2 + \gamma_n \left\| P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_n - z \right\|^2 \\
&+ \beta_n \|x_n - z\|^2 - \beta_n \gamma_n \left\| x_n - P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_n \right\|^2 \\
\leq{}& \alpha_n \|u - z\|^2 + \beta_n \|x_n - z\|^2 + \gamma_n \|y_n - z\|^2 \\
&- \beta_n \gamma_n \left\| x_n - P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_n \right\|^2 \qquad (3.14) \\
\leq{}& \alpha_n \|u - z\|^2 + \beta_n \|x_n - z\|^2 + \gamma_n \|u_n - z\|^2 \\
&- \beta_n \gamma_n \left\| x_n - P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_n \right\|^2 \qquad (3.15) \\
\leq{}& \alpha_n \|u - z\|^2 + (1 - \alpha_n) \|x_n - z\|^2 + \gamma_n \gamma (L\gamma - 1) \left\| \left(T_s^{F_2} (I - sf_2) - I \right) A x_n \right\|^2 \\
&- \beta_n \gamma_n \left\| x_n - P_C \left(I - \lambda_n \left(\sum_{i=1}^{N} k_i (I - T_i) \right) \right) y_n \right\|^2.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\gamma_n \gamma (1 - L\gamma) \left\| \left(T_s^{F_2} (I - sf_2) - I \right) A x_n \right\|^2 \\
&\quad \leq \alpha_n \|u - z\|^2 + \|x_n - x_{n+1}\| \left(\|x_n - z\| + \|x_{n+1} - z\| \right).
\end{aligned}
$$

By using condition (i) and (3.13), we have

$$\lim_{n\to\infty}\left\|\left(T_s^{F_2}(I-sf_2)-I\right)Ax_n\right\|=0. \tag{3.16}$$

By using the same method as (3.16), we have

$$\lim_{n\to\infty}\left\|x_n-P_C\left(I-\lambda_n\left(\sum_{i=1}^{N}k_i(I-T_i)\right)\right)y_n\right\|=0. \tag{3.17}$$

Let $M_n=x_n+\gamma A^*(T_s^{F_2}(I-sf_2)-I)Ax_n$. Applying the inequality (3.11), we have

$$\|M_n-z\|\leq\|x_n-z\|. \tag{3.18}$$

Using the property of inverse strongly monotone operators and (3.18), we have

$$\begin{aligned}\|u_n-z\|^2&=\left\|T_r^{F_1}(I-rf_1)M_n-T_r^{F_1}(I-rf_1)z\right\|^2\\&\leq\left\|(I-rf_1)M_n-(I-rf_1)z\right\|^2\\&=\|M_n-z\|^2-2r\langle M_n-z,f_1M_n-f_1z\rangle+r^2\|f_1M_n-f_1z\|^2\\&\leq\|x_n-z\|^2+r(r-2\rho)\|f_1M_n-f_1z\|^2.\end{aligned} \tag{3.19}$$

Substituting (3.19) in (3.15), we have

$$\begin{aligned}\|x_{n+1}-z\|^2&\leq\alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2\\&\quad+\gamma_n\left(\|x_n-z\|^2+r(r-2\rho)\|f_1M_n-f_1z\|^2\right)\\&\leq\alpha_n\|u-z\|^2+(1-\alpha_n)\|x_n-z\|^2+\gamma_nr(r-2\rho)\|f_1M_n-f_1z\|^2.\end{aligned}$$

That is,

$$\gamma_nr(2\rho-r)\|f_1M_n-f_1z\|^2\leq\alpha_n\|u-z\|^2+\|x_n-x_{n+1}\|\left(\|x_n-z\|+\|x_{n+1}-z\|\right).$$

According to condition (i) and (3.13), we get

$$\lim_{n\to\infty}\|f_1M_n-f_1z\|=0. \tag{3.20}$$

By the property of firmly nonexpansive mappings, we have

$$\begin{aligned}\|u_n-z\|^2&=\left\|T_r^{F_1}(I-rf_1)M_n-T_r^{F_1}(I-rf_1)z\right\|^2\\&\leq\left\langle u_n-z,(I-rf_1)M_n-(I-rf_1)z\right\rangle\\&=\frac{1}{2}\left(\|u_n-z\|^2+\left\|(I-rf_1)M_n-(I-rf_1)z\right\|^2\right.\\&\quad\left.-\left\|(u_n-z)-\left((I-rf_1)M_n-(I-rf_1)z\right)\right\|^2\right).\end{aligned} \tag{3.21}$$

That is,

$$\begin{aligned}\|u_n - z\|^2 &\leq \big\|(I - rf_1)M_n - (I - rf_1)z\big\|^2 - \big\|(u_n - M_n) + r(f_1M_n - f_1z)\big\|^2 \\ &\leq \|M_n - z\|^2 - \big(\|u_n - M_n\|^2 + 2r\langle u_n - M_n, f_1M_n - f_1z\rangle \\ &\quad + r^2\|f_1M_n - f_1z\|^2\big) \\ &\leq \|M_n - z\|^2 - \|u_n - M_n\|^2 + 2r\|u_n - M_n\|\|f_1M_n - f_1z\| \\ &\quad - r^2\|f_1M_n - f_1z\|^2. \end{aligned} \tag{3.22}$$

Substituting (3.22) in (3.15), we get

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\big(\|M_n - z\|^2 - \|u_n - M_n\|^2 \\ &\quad + 2r\|u_n - M_n\|\|f_1M_n - f_1z\| - r^2\|f_1M_n - f_1z\|^2\big) \\ &\leq \alpha_n\|u - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 - \gamma_n\|u_n - M_n\|^2 \\ &\quad + 2r\gamma_n\|u_n - M_n\|\|f_1M_n - f_1z\|.\end{aligned}$$

It follows that

$$\begin{aligned}\gamma_n\|u_n - M_n\|^2 &\leq \alpha_n\|u - z\|^2 + \|x_n - x_{n+1}\|\big(\|x_n - z\| + \|x_{n+1} - z\|\big) \\ &\quad + 2r\gamma_n\|u_n - M_n\|\|f_1M_n - f_1z\|.\end{aligned}$$

From condition (i), (3.13) and (3.20), we ensure that

$$\lim_{n\to\infty}\|u_n - M_n\| = 0. \tag{3.23}$$

From (3.16) and (3.23), we also have

$$\begin{aligned}\|u_n - x_n\| &\leq \|u_n - M_n\| + \|M_n - x_n\| \\ &= \|u_n - M_n\| + \big\|x_n + \gamma A^*\big(T_s^{F_2}(I - sf_2) - I\big)Ax_n - x_n\big\| \\ &\leq \|u_n - M_n\| + \gamma\|A\|\big\|\big(T_s^{F_2}(I - sf_2) - I\big)Ax_n\big\|.\end{aligned}$$

Then we have

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.24}$$

By using the same method as (3.19), we have

$$\big\|P_C(I - d_2D_2)u_n - z\big\|^2 \leq \|x_n - z\|^2 + d_2(d_2 - 2\beta)\|D_2u_n - D_2z\|^2. \tag{3.25}$$

Substituting (3.8) and (3.25) in (3.14), we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq \alpha_n\|u - z\|^2 + \beta_n\|x_n - z\|^2 + \gamma_n\big(a\|u_n - z\|^2 \\ &\quad + (1 - a)\big\|P_C(I - d_2D_2)u_n - z\big\|^2\big)\end{aligned}$$

$$\begin{aligned}&\le \alpha_n\|u-z\|^2+(1-\alpha_n)\|x_n-z\|^2\\&\quad+\gamma_n(1-a)d_2(d_2-2\beta)\|D_2u_n-D_2z\|^2.\end{aligned}$$

We can conclude that

$$\begin{aligned}&\gamma_n(1-a)d_2(2\beta-d_2)\|D_2u_n-D_2z\|^2\\&\quad\le \alpha_n\|u-z\|^2+\|x_n-x_{n+1}\|\bigl(\|x_n-z\|+\|x_{n+1}-z\|\bigr).\end{aligned}$$

According to condition (i) and (3.13), we get

$$\lim_{n\to\infty}\|D_2u_n-D_2z\|=0. \tag{3.26}$$

Since P_C is a firmly nonexpansive mapping and using the same method as (3.21), we get

$$\begin{aligned}&\bigl\|P_C(I-d_2D_2)u_n-z\bigr\|^2\\&\quad\le\frac{1}{2}\bigl(\bigl\|P_C(I-d_2D_2)u_n-z\bigr\|^2+\bigl\|(I-d_2D_2)u_n-(I-d_2D_2)z\bigr\|^2\\&\qquad-\bigl\|P_C(I-d_2D_2)u_n-z-(I-d_2D_2)u_n+(I-d_2D_2)z\bigr\|^2\bigr).\end{aligned}$$

That is,

$$\begin{aligned}\bigl\|P_C(I-d_2D_2)u_n-z\bigr\|^2&\le\|u_n-z\|^2-\bigl\|\bigl(P_C(I-d_2D_2)u_n-u_n\bigr)+d_2(D_2u_n-D_2z)\bigr\|^2\\&\le\|x_n-z\|^2-\bigl\|P_C(I-d_2D_2)u_n-u_n\bigr\|^2\\&\quad+2d_2\bigl\|P_C(I-d_2D_2)u_n-u_n\bigr\|\|D_2u_n-D_2z\|\\&\quad-d_2^2\|D_2u_n-D_2z\|^2.\end{aligned} \tag{3.27}$$

Substituting (3.8) and (3.27) in (3.14), we have

$$\begin{aligned}&\|x_{n+1}-z\|^2\\&\quad\le\alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\gamma_n\bigl(a\|u_n-z\|^2+(1-a)\bigl\|P_C(I-d_2D_2)u_n-z\bigr\|^2\bigr)\\&\quad\le\alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\gamma_n\bigl(a\|x_n-z\|^2+(1-a)\bigl(\|x_n-z\|^2\\&\qquad-\bigl\|P_C(I-d_2D_2)u_n-u_n\bigr\|^2+2d_2\bigl\|P_C(I-d_2D_2)u_n-u_n\bigr\|\|D_2u_n-D_2z\|\\&\qquad-d_2^2\|D_2u_n-D_2z\|^2\bigr)\bigr)\\&\quad\le\alpha_n\|u-z\|^2+(1-\alpha_n)\|x_n-z\|^2-\gamma_n(1-a)\bigl\|P_C(I-d_2D_2)u_n-u_n\bigr\|^2\\&\qquad+2d_2\gamma_n(1-a)\bigl\|P_C(I-d_2D_2)u_n-u_n\bigr\|\|D_2u_n-D_2z\|.\end{aligned}$$

Therefore

$$\begin{aligned}&\gamma_n(1-a)\bigl\|P_C(I-d_2D_2)u_n-u_n\bigr\|^2\\&\quad\le\alpha_n\|u-z\|^2+\|x_n-x_{n+1}\|\bigl(\|x_n-z\|+\|x_{n+1}-z\|\bigr)\\&\qquad+2d_2\gamma_n(1-a)\bigl\|P_C(I-d_2D_2)u_n-u_n\bigr\|\|D_2u_n-D_2z\|.\end{aligned}$$

From condition (i), (3.13) and (3.26), we get

$$\lim_{n\to\infty}\left\|P_C(I-d_2D_2)u_n-u_n\right\|=0. \tag{3.28}$$

Let $k_n = au_n + (1-a)P_C(I-d_2D_2)u_n$. By using the same method as (3.19), we have

$$\|y_n-z\|^2\le\|x_n-z\|^2+d_1(d_1-2\alpha)\|D_1k_n-D_1z\|^2. \tag{3.29}$$

Substituting (3.29) in (3.14), we have

$$\begin{aligned}
&\|x_{n+1}-z\|^2\\
&\quad\le\alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\gamma_n\bigl(\|x_n-z\|^2+d_1(d_1-2\alpha)\|D_1k_n-D_1z\|^2\bigr)\\
&\quad\le\alpha_n\|u-z\|^2+(1-\alpha_n)\|x_n-z\|^2+d_1(d_1-2\alpha)\gamma_n\|D_1k_n-D_1z\|^2.
\end{aligned}$$

This implies that

$$d_1(2\alpha-d_1)\gamma_n\|D_1k_n-D_1z\|^2\le\alpha_n\|u-z\|^2+\|x_n-x_{n+1}\|\bigl(\|x_n-z\|+\|x_{n+1}-z\|\bigr).$$

According to condition (i) and (3.13), we have

$$\lim_{n\to\infty}\|D_1k_n-D_1z\|=0. \tag{3.30}$$

By using the same method as (3.21), we have

$$\begin{aligned}
\|y_n-z\|^2\le{}&\frac{1}{2}\bigl(\|y_n-z\|^2+\left\|(I-d_1D_1)k_n-(I-d_1D_1)z\right\|^2\\
&-\left\|(y_n-k_n)+d_1(D_1k_n-D_1z)\right\|^2\bigr).
\end{aligned}$$

That is,

$$\begin{aligned}
\|y_n-z\|^2&\le\|k_n-z\|^2-\bigl(\|y_n-k_n\|^2+2d_1\langle y_n-k_n,D_1k_n-D_1z\rangle\\
&\qquad+d_1^2\|D_1k_n-D_1z\|^2\bigr)\\
&\le\|x_n-z\|^2-\|y_n-k_n\|^2+2d_1\|y_n-k_n\|\|D_1k_n-D_1z\|\\
&\qquad-d_1^2\|D_1k_n-D_1z\|^2.
\end{aligned} \tag{3.31}$$

Substituting (3.31) in (3.14), we have

$$\begin{aligned}
\|x_{n+1}-z\|^2&\le\alpha_n\|u-z\|^2+\beta_n\|x_n-z\|^2+\gamma_n\bigl(\|x_n-z\|^2-\|y_n-k_n\|^2\\
&\qquad+2d_1\|y_n-k_n\|\|D_1k_n-d_1z\|-d_1^2\|D_1k_n-D_1z\|^2\bigr)\\
&\le\alpha_n\|u-z\|^2+(1-\alpha_n)\|x_n-z\|^2-\gamma_n\|y_n-k_n\|^2\\
&\qquad+2\gamma_nd_1\|y_n-k_n\|\|D_1k_n-D_1z\|.
\end{aligned} \tag{3.32}$$

This implies that

$$\gamma_n\|y_n - k_n\|^2 \leq \alpha_n\|u - z\|^2 + \|x_n - x_{n+1}\|\big(\|x_n - z\| + \|x_{n+1} - z\|\big) \\ + 2\gamma_n d_1\|y_n - k_n\|\|D_1k_n - D_1z\|.$$

According to condition (i), (3.13) and (3.30), we get

$$\lim_{n\to\infty}\|y_n - k_n\| = 0. \tag{3.33}$$

From (3.28) and (3.33)

$$\begin{aligned}\|y_n - u_n\| &\leq \|y_n - k_n\| + \|k_n - u_n\| \\ &\leq \|y_n - k_n\| + (1-a)\big\|P_C(I - d_2D_2)u_n - u_n\big\|,\end{aligned}$$

we conclude that

$$\lim_{n\to\infty}\|y_n - u_n\| = 0. \tag{3.34}$$

By (3.24) and (3.34), we also conclude that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.35}$$

Afterward, we show that $\limsup_{n\to\infty}\langle u - z, x_n - z\rangle \leq 0$, where $z = P_{\mathcal{F}}u$.
To show this, choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z, x_n - z\rangle = \lim_{j\to\infty}\langle u - z, x_{n_j} - z\rangle. \tag{3.36}$$

Without loss of generality, we may assume that $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$. From (3.35), we obtain $y_{n_j} \rightharpoonup \omega$ as $j \to \infty$. From Lemma 2.3, we have $VI(C, D_1) = F(P_C(I - d_1D_1))$. Assume that $\omega \notin VI(C, D_1)$, we have $\omega \neq P_C(I - d_1D_1)\omega$. Using Opial's condition, (3.33), we obtain

$$\begin{aligned}\liminf_{j\to\infty}\|y_{n_j} - \omega\| &< \liminf_{j\to\infty}\big\|y_{n_j} - P_C(I - d_1D_1)\omega\big\| \\ &\leq \liminf_{j\to\infty}\big(\big\|P_C(I - d_1D_1)k_{n_j} - P_C(I - d_1D_1)y_{n_j}\big\| \\ &\quad + \big\|P_C(I - d_1D_1)y_{n_j} - P_C(I - d_1D_1)\omega\big\|\big) \\ &\leq \liminf_{j\to\infty}\big(\|k_{n_j} - y_{n_j}\| + \|y_{n_j} - \omega\|\big) \\ &\leq \liminf_{j\to\infty}\|y_{n_j} - \omega\|.\end{aligned}$$

This is a contradiction, so we have

$$\omega \in VI(C, D_1). \tag{3.37}$$

From (3.24), we have $u_{n_j} \rightharpoonup \omega$ as $j \to \infty$. By (3.28) and using the same method as (3.37), we obtain

$$\omega \in VI(C, D_2). \tag{3.38}$$

Next, we show that $\omega \in \bigcap_{i=1}^N F(T_i)$. From Lemma 2.5, we have

$$\bigcap_{i=1}^N F(T_i) = F\left(P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)\right).$$

Assume that $\omega \notin \bigcap_{i=1}^N F(T_i)$, and that $\omega \neq P_C(I - \lambda_{n_j}(\sum_{i=1}^N k_i(I - T_i)))\omega$. Using Opial's condition, (3.17) and (3.35), we obtain

$$\begin{aligned}
&\liminf_{j\to\infty} \|x_{n_j} - \omega\| \\
&\quad < \liminf_{j\to\infty} \left\| x_{n_j} - P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)\omega \right\| \\
&\quad \le \liminf_{j\to\infty} \left(\left\| x_{n_j} - P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)y_{n_j} \right\| \right. \\
&\qquad + \left\| P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)y_{n_j} - P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)x_{n_j} \right\| \\
&\qquad \left. + \left\| P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)x_{n_j} - P_C\left(I - \lambda_{n_j}\left(\sum_{i=1}^N k_i(I - T_i)\right)\right)\omega \right\| \right) \\
&\quad \le \liminf_{j\to\infty} \left(\|y_{n_j} - x_{n_j}\| + \lambda_{n_j} \left\| \left(\sum_{i=1}^N k_i(I - T_i)\right)y_{n_j} - \left(\sum_{i=1}^N k_i(I - T_i)\right)x_{n_j} \right\| \right. \\
&\qquad \left. + \|x_{n_j} - \omega\| + \lambda_{n_j} \left\| \left(\sum_{i=1}^N k_i(I - T_i)\right)x_{n_j} - \left(\sum_{i=1}^N k_i(I - T_i)\right)\omega \right\| \right) \\
&\quad \le \liminf_{j\to\infty} \|x_{n_j} - \omega\|.
\end{aligned}$$

This is a contradiction, so we have

$$\omega \in \bigcap_{i=1}^N F(T_i). \tag{3.39}$$

After that, we show that $\omega \in \Omega$. Assume $\omega \notin EP(F_1, f_1)$. Since $EP(F_1, f_1) = F(T_r^{F_1}(I - rf_1))$, we obtain $\omega \neq T_r^{F_1}(I - rf_1)\omega$. Using Opial's condition and (3.23), we get

$$\begin{aligned}
\liminf_{j\to\infty} \|u_{n_j} - \omega\| &< \liminf_{j\to\infty} \left\| u_{n_j} - T_r^{F_1}(I - rf_1)\omega \right\| \\
&\le \liminf_{j\to\infty} \left(\left\| T_r^{F_1}(I - rf_1)M_{n_j} - T_r^{F_1}(I - rf_1)u_{n_j} \right\| \right. \\
&\quad \left. + \left\| T_r^{F_1}(I - rf_1)u_{n_j} - T_r^{F_1}(I - rf_1)\omega \right\| \right)
\end{aligned}$$

$$\begin{aligned} &\leq \liminf_{j\to\infty}\big(\|M_{n_j} - u_{n_j}\| + \|u_{n_j} - \omega\|\big) \\ &\leq \liminf_{j\to\infty}\|u_{n_j} - \omega\|. \end{aligned}$$

This is a contradiction, so we have

$$\omega \in EP(F_1,f_1). \tag{3.40}$$

Next, we show that $A\omega \in EP(F_2,f_2)$. Since A is bounded linear operator so that $Ax_{n_j} \rightharpoonup A\omega$ as $j \to \infty$. Assume $A\omega \notin EP(F_2,f_2)$. Since $EP(F_2,f_2) = F(T_s^{F_2}(I - sf_2))$, we obtain $A\omega \neq T_s^{F_2}(I - sf_s)A\omega$. Using Opial's condition and (3.16), we have

$$A\omega \in EP(F_2,f_2). \tag{3.41}$$

We can conclude that $\omega \in \Omega$. Therefore $\omega \in \mathcal{F}$. Since $x_{n_j} \rightharpoonup \omega$ as $j \to \infty$, we have

$$\begin{aligned} \limsup_{n\to\infty}\langle u - z, x_n - z\rangle &= \lim_{j\to\infty}\langle u - z, x_{n_j} - z\rangle \\ &= \langle u - z, \omega - z\rangle \leq 0. \end{aligned} \tag{3.42}$$

Finally, we show that the sequence $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}u$. By (3.7), (3.9) and (3.11), we get

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \left\|\alpha_n(u - z) + \beta_n(x_n - z) + \gamma_n\left(P_C\left(I - \lambda_n\left(\sum_{i=1}^{N} k_i(I - T_i)\right)\right)y_n - z\right)\right\|^2 \\ &\leq \left\|\beta_n(x_n - z) + \gamma_n\left(P_C\left(I - \lambda_n\left(\sum_{i=1}^{N} k_i(I - T_i)\right)\right)y_n - z\right)\right\|^2 \\ &\quad + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\ &\leq \big(\beta_n\|x_n - z\| + \gamma_n\|u_n - z\|\big)^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle \\ &\leq (1 - \alpha_n)\|x_n - z\|^2 + 2\alpha_n\langle u - z, x_{n+1} - z\rangle. \end{aligned}$$

According to condition (i), (3.42) and Lemma 2.6, we can conclude that $\{x_n\}$ converges strongly to $z = P_{\mathcal{F}}u$. By (3.24) and (3.35), we have $\{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathcal{F}}u$. This completes the proof. □

These results are directly proved from Theorem 3.4. Therefore, we omit the proof.

Corollary 3.5 *Let C and Q be nonempty closed convex subsets of a real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying* (A1)–(A4). *Let T be a quasi-nonexpansive mapping of C into itself. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Assume $\mathcal{F} = VI(C,D_1) \cap VI(C,D_2) \cap F(T) \cap \Omega \neq \emptyset$. For* *given $x_1, u \in C$, and let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} u_n = T_r^{F_1}(I - rf_1)(x_n + \gamma A^*(T_s^{F_2}(I - sf_2) - I)Ax_n), \\ y_n = P_C(I - d_1D_1)(au_n + (1 - a)P_C(I - d_2D_2)u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(I - T))y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha)$, $d_2 \in (0, 2\beta)$, $r \in (0, 2\rho)$, $s \in (0, 1)$, $a \in [0, 1]$, $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the conditions (i)–(iv) *of Theorem* 3.4 *hold. Then $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathcal{F}}u$.*

Corollary 3.6 *Let C be nonempty closed convex subset of a real Hilbert space H_1. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ be the bifunction satisfying* (A1)–(A4). *Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping. Assume $\mathcal{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap EP(F_1, f_1) \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}, \{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} u_n = T_r^{F_1}(I - rf_1)x_n, \\ y_n = P_C(I - d_1D_1)(au_n + (1 - a)P_C(I - d_2D_2)u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(\sum_{i=1}^N k_i(I - T_i)))y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha)$, $d_2 \in (0, 2\beta)$, $r \in (0, 2\rho)$, $a \in [0, 1]$, $0 < k_i < 1$ with $\sum_{i=1}^N k_i = 1$. Also $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the conditions (i)–(iv) *of Theorem* 3.4 *hold. Then $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathcal{F}}u$.*

Corollary 3.7 *Let C and Q be nonempty closed convex subsets of a real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying* (A1)–(A4). *Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive mappings of C into itself with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$. Assume $\mathcal{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Gamma \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} u_n = T_r^{F_1}(x_n + \gamma A^*(T_s^{F_2} - I)Ax_n), \\ y_n = P_C(I - d_1D_1)(au_n + (1 - a)P_C(I - d_2D_2)u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(\sum_{i=1}^N k_i(I - T_i)))y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha)$, $d_2 \in (0, 2\beta)$, $a \in [0, 1]$, $0 < k_i < 1$ with $\sum_{i=1}^N k_i = 1$, $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0, 1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the conditions (i)–(iv) *of Theorem* 3.4 *hold. Then $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathcal{F}}u$.*

Remark 3.8 If we take $N = 1$ in Theorem 3.4, we have a strong convergence for finding a common element of the set of solutions of variational inequality problems and the set of fixed points of a quasi-nonexpansive mapping and the set of solutions of the modified split generalized equilibrium problem. From previous result, we can apply by using the same method as Theorem 4.5 in [24]. We have a strong convergence for finding a common element of the set of solutions of variational inequality problems and the set of fixed points of a finite family of nonspreading mappings and the set of solutions of the modified split generalized equilibrium problem. By using our main result, Theorem 3.4 reduces to the Corollary 3.6, the solution of the generalized equilibrium problem and Corollary 3.7, the split equilibrium problem. All theorems are found as regards the solution of common fixed points of a finite family of quasi-nonexpansive mappings without assuming $T_\omega := (1-\omega)I + \omega T$ and T is demiclosed; a difficult proof in a framework of Hilbert space.

4 Application

The following knowledge is used to prove Theorem 4.4. A mapping $T : C \to C$ is called nonspreading if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C. \tag{4.1}$$

Such a mapping is defined by Kohsaka and Takahashi [25].

In 2009, Iemoto and Takahashi [26] proved that (4.1) is equivalent to

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C. \tag{4.2}$$

Remark 4.1 A nonspreading mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive mapping T.

Lemma 4.2 ([25]) *Let H be a Hilbert space, let C be a nonempty closed convex subset of H, and let S be a nonspreading mapping of C into itself. Then $F(S)$ is closed and convex.*

In 2009, Kangtunyakarn and Suantai[27] introduced the S-mapping generated by $T_1, T_2, T_3, \dots, T_N$ and $\lambda_1, \lambda_2, \dots, \lambda_N$ as follows.

Definition 4.1 Let C be a nonempty convex subset of a real Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of (nonexpansive) mappings of C into itself. For each $j = 1, 2, \dots, N$, let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, where $I \in [0,1]$ and $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$. Define the mapping $S : C \to C$ as follows:

$$\begin{aligned}
&U_0 = I, \\
&U_1 = \alpha_1^1 T_1 U_0 + \alpha_2^1 U_0 + \alpha_3^1 I, \\
&U_2 = \alpha_1^2 T_2 U_1 + \alpha_2^2 U_1 + \alpha_3^2 I, \\
&U_3 = \alpha_1^3 T_3 U_2 + \alpha_2^3 U_2 + \alpha_3^3 I, \\
&\vdots \\
&U_{N-1} = \alpha_1^{N-1} T_{N-1} U_{N-2} + \alpha_2^{N-1} U_{N-2} + \alpha_3^{N-1} I, \\
&S = U_N = \alpha_1^N T_N U_{N-1} + \alpha_2^N U_{N-1} + \alpha_3^N I.
\end{aligned}$$

This mapping is called an *S-mapping* generated by $T_1, T_2, \dots, T_N$ and $\alpha_1, \alpha_2, \dots, \alpha_N$.

Lemma 4.3 ([28]) *Let C be a nonempty closed convex subset of a real Hilbert space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1,2,\ldots,N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0,1)$ for all $j = 1,2,\ldots,N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$, $\alpha_2^j \in [0,1)$ for all $j = 1,2,\ldots,N$. Let S be the mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Then $F(S) = \bigcap_{i=1}^N F(T_i)$ and S is a quasi-nonexpansive mapping.*

By using these results, we obtain the following theorems.

Theorem 4.4 *Let C and Q be nonempty closed convex subsets of a real Hilbert space H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $D_1, D_2 : C \to H_1$ be α, β-inverse strongly monotone mappings, respectively. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be the bifunctions satisfying* (A1)–(A4). *Let $\{T_i\}_{i=1}^N$ be a finite family of nonspreading mappings of C into C with $\bigcap_{i=1}^N F(T_i) \neq \emptyset$, and let $\alpha_j = (\alpha_1^j, \alpha_2^j, \alpha_3^j) \in I \times I \times I$, $j = 1,2,\ldots,N$, where $I = [0,1]$, $\alpha_1^j + \alpha_2^j + \alpha_3^j = 1$, $\alpha_1^j, \alpha_3^j \in (0,1)$ for all $j = 1,2,\ldots,N-1$ and $\alpha_1^N \in (0,1]$, $\alpha_3^N \in [0,1)$, $\alpha_2^j \in [0,1)$ for all $j = 1,2,\ldots,N$. Let S be the mapping generated by $T_1, T_2, \ldots, T_N$ and $\alpha_1, \alpha_2, \ldots, \alpha_N$. Let $f_1 : H_1 \to H_1$ be a ρ-inverse strongly monotone mapping and $f_2 : H_2 \to H_2$ be a firmly nonexpansive mapping. Assume $\mathcal{F} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Omega \neq \emptyset$. For given $x_1, u \in C$ and let $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} u_n = T_r^{F_1}(I - rf_1)(x_n + \gamma A^*(T_s^{F_2}(I - sf_2) - I)Ax_n), \\ y_n = P_C(I - d_1 D_1)(au_n + (1-a)P_C(I - d_2 D_2)u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n P_C(I - \lambda_n(I - S))y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $d_1 \in (0, 2\alpha)$, $d_2 \in (0, 2\beta)$, $r \in (0, 2\rho)$, $s \in (0,1)$, $a \in [0,1]$, $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. Also $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \in \mathbb{N}$. Suppose the conditions (i)–(iv) *of Theorem* 3.4 *hold. Then $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ converge strongly to $z = P_{\mathcal{F}} u$.*

Proof By using Corollary 3.5 and Lemma 4.3, we obtain the conclusion. □

5 Example and numerical results

In this section, an example is given for supporting Theorem 3.4. In Example 5.1, we only instance an example in infinite dimensional Hilbert space for supporting Theorem 3.4. We omit the computer programming.

Example 5.1 Let $H_1 = H_2 = C = Q = \ell_2$ be the linear space whose elements consist of all 2-summable sequences $(x_1, x_2, \ldots, x_j, \ldots)$ of scalars, i.e.,

$$\ell_2 = \left\{ x : x = (x_1, x_2, \ldots, x_j, \ldots) \text{ and } \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\},$$

with an inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle x, y \rangle = \sum_{j=1}^{\infty} x_j y_j$ where $x = \{x_j\}_{j=1}^{\infty}$, $y = \{y_j\}_{j=1}^{\infty} \in \ell_2$ and a norm $\|\cdot\| : \ell_2 \to \mathbb{R}$ defined by $\|x\|_2 = (\sum_{j=1}^{\infty} |x_j|^2)^{\frac{1}{2}}$ where $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$. Let the mapping $A : \ell_2 \to \ell_2$ be defined by $Ax = (\frac{x_1}{3}, \frac{x_2}{3}, \ldots, \frac{x_j}{3}, \ldots)$ for all $x = \{x_j\}_{j=1}^{\infty} \in \ell_2$ and $A^* : \ell_2 \to \ell_2$ be defined by $A^*z = (\frac{z_1}{3}, \frac{z_2}{3}, \ldots, \frac{z_j}{3}, \ldots)$ for all $z = \{z_j\}_{j=1}^{\infty} \in \ell_2$. Let D_1, D_2 :

$\ell_2 \to \ell_2$ be defined by $D_1x = (\frac{x_1}{6}, \frac{x_2}{6}, \ldots, \frac{x_j}{6}, \ldots)$ and $D_2x = (\frac{x_1}{5}, \frac{x_2}{5}, \ldots, \frac{x_j}{5}, \ldots)$, $\forall x = \{x_j\}_{j=1}^{\infty} \in \ell_2$, respectively. Let the mapping $T_i : \ell_2 \to \ell_2$ be defined by $T_ix = (\frac{3ix_1}{5i+1}, \frac{3ix_2}{5i+1}, \ldots, \frac{3ix_j}{5i+1}, \ldots)$, $\forall x = \{x_j\}_{j=1}^{\infty} \in \ell_2$ and $k_i = \frac{6}{7^i} + \frac{1}{N7^N}$ for every $i = 1, 2, \ldots, N$. Let the mapping $F_1, F_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F_1(x, y) = -x^2 + y^2, \quad \forall x = \{x_j\}_{j=1}^{\infty}, y = \{y_j\}_{j=1}^{\infty} \in \ell_2,$$

and

$$F_2(x, y) = -2x^2 + xy + y^2, \quad \forall x = \{x_j\}_{j=1}^{\infty}, y = \{y_j\}_{j=1}^{\infty} \in \ell_2.$$

Let the mapping $f_1 : \ell_2 \to \ell_2$ be defined by $f_1x = (\frac{x_1}{5}, \frac{x_2}{5}, \ldots, \frac{x_j}{5}, \ldots)$, $\forall x = \{x_j\}_{j=1}^{\infty} \in \ell_2$ and the mapping $f_2 : \ell_2 \to \ell_2$ be defined by $f_2x = (\frac{x_1}{7}, \frac{x_2}{7}, \ldots, \frac{x_j}{7}, \ldots)$, $\forall x = \{x_j\}_{j=1}^{\infty} \in \ell_2$. Let $r = 1$ and $s = 0.5$. Since $L = \frac{1}{9}$, we choose $\gamma = 0.5$. Let $x_1 = (x_1^1, x_1^2, \ldots, x_1^j, \ldots)$ and $u = (u_1, u_2, \ldots, u_j, \ldots) \in \ell_2$ and let the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be generated by (3.6) as follows:

$$\begin{cases} u_n = T_1^{F_1}(I - f_1)(x_n + 0.5A^*(T_{0.5}^{F_2}(I - 0.5f_2) - I)Ax_n), \\ y_n = (I - D_1)(0.5u_n + 0.5(I - D_2)u_n), \\ x_{n+1} = \frac{1}{2n}u + \frac{7n-4}{12n}x_n + \frac{5n-2}{12n}(y_n - ((\frac{1}{2n^2})(\sum_{i=1}^N(\frac{6}{7^i} + \frac{1}{N7^N})(y_n - T_iy_n)))), \end{cases}$$

for all $n \geq 1$, where $x_n = (x_n^1, x_n^2, \ldots, x_n^j, \ldots), y_n = (y_n^1, y_n^2, \ldots, y_n^j, \ldots)$ and $u_n = (u_n^1, u_n^2, \ldots, u_n^j, \ldots)$. It easy to see that $D_1, D_2, T_i, F_1, F_2, f_1$ and f_2 satisfy Theorem 3.4. Moreover, we have $VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Omega = \{0\}$, where $\rho = d_1 = d_2 = 1$. From Theorem 3.4, we can conclude that the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to 0.

In Example 5.2, we give computer programming to support our main result.

Example 5.2 Let $H_1 = H_2 = C = Q = \mathbb{R}^2$ be the two-dimensional Euclidean space of the real number with an inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $\langle x, y \rangle = x \cdot y = x_1y_1 + x_2y_2$ where $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ and a usual norm $\| \cdot \| : \mathbb{R}^2 \to \mathbb{R}$ be defined by $\|x\| = \sqrt{x_1^2 + x_2^2}$ where $x = (x_1, x_2) \in \mathbb{R}^2$. Let the mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Ax = (2x_1 - x_2, x_1 + 2x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $A^* : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $A^*z = (2z_1 - z_2, 2z_2 - z_1)$ for all $z = (z_1, z_2) \in \mathbb{R}^2$. Let $D_1, D_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $D_1x = (\frac{x_1}{6}, \frac{x_2}{6})$ and $D_2x = (\frac{x_1}{2}, \frac{x_2}{3})$, $\forall x = (x_1, x_2) \in \mathbb{R}^2$, respectively. Let the mapping $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T_ix = (\frac{3ix_1}{3i+1}, \frac{3ix_2}{3i+2})$, $\forall x = (x_1, x_2) \in \mathbb{R}^2$ and $k_i = \frac{6}{7^i} + \frac{1}{N7^N}$ for every $i = 1, 2, \ldots, N$. Let the mapping $F_1, F_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$F_1(x, y) = -x^2 + y^2, \quad \forall x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2,$$

and

$$F_2(x, y) = -2x^2 + xy + y^2, \quad \forall x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2.$$

Let the mapping $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f_1x = (\frac{x_1}{5}, \frac{x_2}{5})$, $\forall x = (x_1, x_2) \in \mathbb{R}^2$ and the mapping $f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f_2x = (\frac{x_1}{7}, \frac{x_2}{7})$, $\forall x = (x_1, x_2) \in \mathbb{R}^2$. Let $r = 1$ and $s = 0.5$, the sequences $z_n = (z_n^1, z_n^2)$, $x_n = (x_n^1, x_n^2)$, $u_n = (u_n^1, u_n^2)$, $y = (y_1, y_2) \in \mathbb{R}^2$. By the definition of f_1 and f_2, we get

$$\begin{aligned}
0 &\le F_1(z_n, y) + \langle f_1(z_n), y - z_n\rangle + \frac{1}{r}\langle y - z_n, z_n - x_n\rangle \\
&= -\left(z_n^1\right)^2 - \left(z_n^2\right)^2 + (y_1)^2 + (y_2)^2 + \frac{1}{5}z_n^1\left(-z_n^1 + y_1\right) + \frac{1}{5}z_n^2\left(-z_n^2 + y_2\right) \\
&\quad + \left(y_1 - z_n^1\right)\left(z_n^1 - x_n^1\right) + \left(y_2 - z_n^2\right)\left(z_n^2 - x_n^2\right) \\
&= \left((y_1)^2 + \left(-x_n^1 + \frac{6}{5}z_n^1\right)y_1 + x_n^1 z_n^1 - \frac{11}{5}\left(z_n^1\right)^2\right) \\
&\quad + \left((y_2)^2 + \left(-x_n^2 + \frac{6}{5}z_n^2\right)y_2 + x_n^2 z_n^2 - \frac{11}{5}\left(z_n^2\right)^2\right) \\
&= G_1(y_1) + G_2(y_2).
\end{aligned}$$

Let $G_1(y_1) = (y_1)^2 + (-x_n^1 + \frac{6}{5}z_n^1)y_1 + x_n^1 z_n^1 - \frac{11}{5}(z_n^1)^2$ and $G_2(y_2) = (y_2)^2 + (-x_n^2 + \frac{6}{5}z_n^2)y_2 + x_n^2 z_n^2 - \frac{11}{5}(z_n^2)^2$. $G_1(y_1)$ and $G_2(y_2)$ are quadratic functions with coefficients $a_1 = 1$, $b_1 = -x_n^1 + \frac{6}{5}z_n^1$, and $c_1 = x_n^1 z_n^1 - \frac{11}{5}(z_n^1)^2$ of $G_1(y_1)$ and coefficients $a_2 = 1$, $b_2 = -x_n^2 + \frac{6}{5}z_n^2$, and $c_2 = x_n^2 z_n^2 - \frac{11}{5}(z_n^2)^2$ of $G_2(y_2)$, respectively. Determine the discriminant Δ_1 of G_1 as follows:

$$\begin{aligned}
\Delta_1 &= b_1^2 - 4a_1 c_1 \\
&= \left(-x_n^1 + \frac{6}{5}z_n^1\right)^2 - 4(1)\left(x_n^1 z_n^1 - \frac{11}{5}\left(z_n^1\right)^2\right) = \frac{1}{25}\left(5x_n^1 - 16z_n^1\right)^2.
\end{aligned}$$

We know that $G_1(y_1) \ge 0$, $\forall y \in \mathbb{R}$. If it has most one solution in $\mathbb{R}$, then $\Delta_1 \le 0$, so we obtain $z_n^1 = \frac{5x_n^1}{16}$. Next, we determine the discriminant Δ_2 of G_2 by using the same method as above, we obtain $z_n^2 = \frac{5x_n^2}{16}$. That is $T_r^{F_1}(I - rf_1)z_n = (\frac{5x_n^1}{16}, \frac{5x_n^2}{16})$. After that, we find the solution of $u_n = (u_n^1, u_n^2)$ in this inequality $0 \le F_2(u_n, y) + \langle f_2(u_n), y - u_n\rangle + \frac{1}{s}\langle y - u_n, u_n - x_n\rangle$. By using the same method as $z_n = (z_n^1, z_n^2)$, we obtain

$$u_n = \left(u_n^1, u_n^2\right) = \left(\frac{7x_n^1}{51}, \frac{7x_n^2}{51}\right). \tag{5.1}$$

That is, $T_s^{F_2}(I - sf_2)u_n = (\frac{7x_n^1}{51}, \frac{7x_n^2}{51})$.

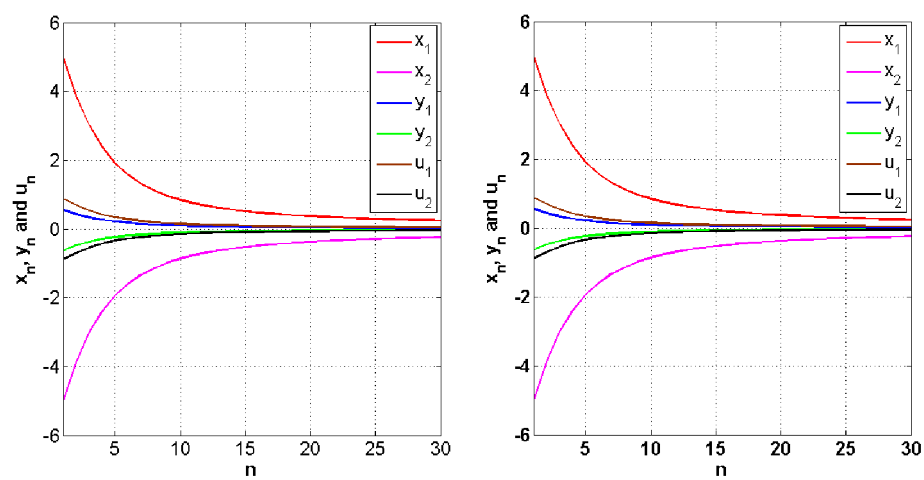
Let $x_1 = (x_1^1, x_1^2)$ and $u = (u_1, u_2) \in \mathbb{R}^2$. The sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ are generated by (3.6), where $k_i = \frac{6}{7^i} + \frac{1}{N7^N}$, $d_1 = 1$, $d_2 = 1$, $a = 0.5$, $\alpha_n = \frac{1}{2n}$, $\beta_n = \frac{7n-4}{12n}$, $\gamma_n = \frac{5n-2}{12n}$ and $\lambda_n = \frac{1}{2n^2}$ for all $n \in \mathbb{N}$. Since $L = 5$, we choose $\gamma = 0.1$. From the definition of D_1, D_2, T_i, F_1, F_2, f_1 and f_2, we have $VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Omega = \{0\}$. From Theorem 3.4, we can conclude that the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge strongly to 0. We can rewrite (3.6) as follows:

$$\begin{cases}
u_n = T_1^{F_1}(I - f_1)(x_n + 0.1A^*(T_{0.5}^{F_2}(I - 0.5f_2) - I)Ax_n), \\
y_n = (I - D_1)(0.5u_n + 0.5(I - D_2)u_n), \\
x_{n+1} = \frac{1}{2n}u + \frac{7n-4}{12n}x_n + \frac{5n-2}{12n}(y_n - ((\frac{1}{2n^2})(\sum_{i=1}^N(\frac{6}{7^i} + \frac{1}{N7^N})(y_n - T_i y_n)))),
\end{cases}$$

for all $n \ge 1$, where $x_n = (x_n^1, x_n^2)$, $y_n = (y_n^1, y_n^2)$ and $u_n = (u_n^1, u_n^2)$.

Table 1 The values of $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ where $u = (5,-5)$, $x_1 = (5,-5)$ and $n = 30$

	$N = 1$			$N = 20$		
n	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$	$u_n = (u_n^1, u_n^2)$	$x_n = (x_n^1, x_n^2)$	$y_n = (y_n^1, y_n^2)$	$u_n = (u_n^1, u_n^2)$
1	(5.0000, −5.0000)	(0.5553, −0.6170)	(0.8885, −0.8885)	(5.0000, −5.0000)	(0.5553, −0.6170)	(0.8885, −0.8885)
2	(3.8715, −3.8850)	(0.4300, −0.4794)	(0.6879, −0.6903)	(3.8700, −3.8833)	(0.4298, −0.4792)	(0.6877, −0.6901)
⋮	⋮	⋮	⋮	⋮	⋮	⋮
15	(0.5189, −0.5274)	(0.0576, −0.0651)	(0.0922, −0.0937)	(0.5189, −0.5274)	(0.0576, −0.0651)	(0.0922, −0.0937)
⋮	⋮	⋮	⋮	⋮	⋮	⋮
29	(0.2485, −0.2522)	(0.0276, −0.0311)	(0.0442, −0.0448)	(0.2485, −0.2522)	(0.0276, −0.0311)	(0.0442, −0.0448)
30	(0.2397, −0.2432)	(0.0266, −0.0300)	(0.0426, −0.0432)	(0.2397, −0.2432)	(0.0266, −0.0300)	(0.0426, −0.0432)



(a) $N = 1$ (b) $N = 20$

Figure 1 The convergence comparison with different values N

Table 1 shows the values of sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ where $u = (5,-5)$, $x_1 = (5,-5)$ and $n = 30$.

6 Conclusion

1. Example 5.1 is an example in infinite dimensional Hilbert space for supporting Theorem 3.4
2. Table 1 and Fig. 1 in Example 5.2 show that the sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ converge to 0, where $\{0\} = VI(C, D_1) \cap VI(C, D_2) \cap \bigcap_{i=1}^N F(T_i) \cap \Omega$.
3. Theorem 3.4 guarantees the convergence of $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ in Example 5.1 and Example 5.2.
4. By using the concept of Picard iteration, Wang [13] defined the iterative scheme $\{x_n\}$ for solving SCFPP as follows:

$$\begin{aligned} x_{n+1} &= x_n - \rho_n\big((I-U)x_n + A^*(I-T)Ax_n\big) \\ &= \big(I - \rho_n\big((I-U) + A^*(I-T)A\big)\big)x_n, \end{aligned} \tag{6.1}$$

where ρ_n is according to (1.4) and U and T are firmly quasi-nonexpansive mappings. Then the sequence $\{x_n\}$ converges weakly to z, where $z = \lim_{n\to\infty} P_\Phi x_n$. In Theorem 3.4, we use the concept of Halpern iteration and suitable conditions of the parameters d_1, d_2, r, s, a, γ, L, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$, the sequence $\{x_n\}$ defined by (3.6) converges strongly to $z = P_{\mathcal{F}}u$, which is a different method from (6.1).

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Competing interests
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Authors' contributions
The two authors contributed equally to the writing of this paper. Both authors read and approved the final manuscript.

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