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# A new sequence convergent to Euler–Mascheroni constant

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**Abstract**
In this paper, we provide a new sequence converging to the Euler–Mascheroni constant. Finally, we establish some inequalities for the Euler–Mascheroni constant by the new sequence.

**MSC:** 11Y60; 41A25; 41A20

**Keywords:** Euler–Mascheroni constant; Rate of convergence; Taylor's formula; Harmonic sequence

## 1 Introduction

The Euler–Mascheroni constant was first introduced by Leonhard Euler (1707–1783) in 1734 as the limit of the sequence

$$\gamma(n) := \sum_{m=1}^{n} \frac{1}{m} - \ln n. \tag{1.1}$$

There are many famous unsolved problems about the nature of this constant (see, e.g., the survey papers or books of Brent and Zimmermann [1], Dence and Dence [2], Havil [3], and Lagarias [4]). For example, it is a long-standing open problem if the Euler–Mascheroni constant is a rational number. A good part of its mystery comes from the fact that the known algorithms converging to $\gamma$ are not very fast, at least when they are compared to similar algorithms for $\pi$ and $e$.

The sequence $(\gamma(n))_{n\in\mathbb{N}}$ converges very slowly toward $\gamma$, like $(2n)^{-1}$. Up to now, many authors are preoccupied to improve its rate of convergence; see, for example, [2, 5–14] and references therein. We list some main results:

$$\sum_{m=1}^{n} \frac{1}{m} - \ln\left(n + \frac{1}{2}\right) = \gamma + O\left(n^{-2}\right) \quad \text{(DeTemple [6])},$$

$$\sum_{m=1}^{n} \frac{1}{m} - \ln \frac{n^3 + \frac{3}{2}n^2 + \frac{227}{240} + \frac{107}{480}}{n^2 + n + \frac{97}{240}} = \gamma + O\left(n^{-6}\right) \quad \text{(Mortici [13])},$$

$$\sum_{m=1}^{n} \frac{1}{m} - \ln\left(1 + \frac{1}{2n} + \frac{1}{24n^2} - \frac{1}{48n^3} + \frac{23}{5760n^4}\right) = \gamma + O\left(n^{-5}\right)$$

(Chen and Mortici [5]).

Recently, Mortici and Chen [14] provided a very interesting sequence

$$\nu(n) = \sum_{m=1}^{n} \frac{1}{m} - \frac{1}{2}\ln\left(n^2 + n + \frac{1}{3}\right) - \left(\frac{-\frac{1}{180}}{(n^2+n+\frac{1}{3})^2} + \frac{\frac{8}{2835}}{(n^2+n+\frac{1}{3})^3} + \frac{\frac{5}{1512}}{(n^2+n+\frac{1}{3})^4} + \frac{\frac{592}{93,555}}{(n^2+n+\frac{1}{3})^5}\right)$$

and proved that

$$\lim_{n\to\infty} n^{12}\bigl(\nu(n) - \gamma\bigr) = -\frac{796,801}{43,783,740}. \tag{1.2}$$

Hence the rate of the convergence of the sequence $(\nu(n))_{n\in\mathbb{N}}$ is $n^{-12}$.

Very recently, by inserting the continued fraction term into (1.1), Lu [9] introduced a class of sequences $(r_k(n))_{n\in\mathbb{N}}$ (see Theorem 1) and showed that

$$\frac{1}{72(n+1)^3} < \gamma - r_2(n) < \frac{1}{72n^3}, \tag{1.3}$$

$$\frac{1}{120(n+1)^4} < r_3(n) - \gamma < \frac{1}{120(n-1)^4}. \tag{1.4}$$

In fact, Lu [9] also found $a_4$ without proof, and his works motivate our study. In this paper, starting from the well-known sequence $\gamma_n$, based on the early works of Mortici, DeTemple, and Lu, we provide some new classes of convergent sequences for the Euler–Mascheroni constant.

**Theorem 1** *For the Euler–Mascheroni constant, we have the following convergent sequence*:

$$r(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n - \ln\left(1 + \frac{a_1}{n}\right) - \ln\left(1 + \frac{a_2}{n^2}\right) - \cdots,$$

*where*

$$a_1 = \frac{1}{2}, \qquad a_2 = \frac{1}{24}, \qquad a_3 = -\frac{1}{24}, \qquad a_4 = \frac{143}{5760},$$
$$a_5 = -\frac{1}{160}, \qquad a_6 = -\frac{151}{290,304}, \qquad a_7 = -\frac{1}{896}, \qquad \ldots.$$

*Let*

$$r_k(n) := \sum_{m=1}^{n} \frac{1}{m} - \ln n - \sum_{m=1}^{k} \ln\left(1 + \frac{a_m}{n^m}\right).$$

*For $1 \le k \le 7$, we have*

$$\lim_{n\to\infty} n^{k+2}\bigl(r_k(n) - \gamma\bigr) = C_k, \tag{1.5}$$

*where*

$$C_1 = \frac{1}{24}, \qquad C_2 = -\frac{1}{24}, \qquad C_3 = \frac{143}{5760}, \qquad C_4 = -\frac{1}{160},$$
$$C_5 = -\frac{151}{290{,}304}, \qquad C_6 = -\frac{1}{896}, \qquad C_7 = \frac{109{,}793}{22{,}118{,}400}, \qquad \ldots.$$

Furthermore, for $r_2(n)$ and $r_3(n)$, we also have the following inequalities.

**Theorem 2** *Let $r_2(n)$ and $r_3(n)$ be as in Theorem* 1. *Then*

$$\frac{1}{24}\frac{1}{(n+1)^3} < \gamma - r_2(n) < \frac{1}{24}\frac{1}{n^3}, \tag{1.6}$$

$$\frac{143}{5760}\frac{1}{(n+1)^4} < r_3(n) - \gamma < \frac{143}{5760}\frac{1}{n^4}. \tag{1.7}$$

*Remark* 1 Certainly, there are similar inequalities for $r_k(n)$ $(1 \le k \le 7)$; we omit the details.

## 2 Proof of Theorem 1

The following lemma gives a method for measuring the rate of convergence. This lemma was first used by Mortici [15, 16] for constructing asymptotic expansions or accelerating some convergences. For a proof and other details, see, for example, [16].

**Lemma 1** *If the sequence $(x_n)_{n\in\mathbb{N}}$ converges to zero and there exists the limit*

$$\lim_{n\to+\infty} n^s(x_n - x_{n+1}) = l \in [-\infty, +\infty] \tag{2.1}$$

*with $s > 1$, then there exists the limit*

$$\lim_{n\to+\infty} n^{s-1}x_n = \frac{l}{s-1}. \tag{2.2}$$

We need to find the value $a_1 \in \mathbb{R}$ that produces the most accurate approximation of the form

$$r_1(n) = \sum_{m=1}^{n} \frac{1}{m} - \ln n - \ln\left(1 + \frac{a_1}{n}\right). \tag{2.3}$$

To measure the accuracy of this approximation, we usually say that an approximation (2.3) is better as $r_1(n) - \gamma$ faster converges to zero. Clearly,

$$r_1(n) - r_1(n+1) = \ln\left(1 + \frac{1}{n}\right) - \frac{1}{n+1} + \ln\left(1 + \frac{a_1}{n+1}\right) - \ln\left(1 + \frac{a_1}{n}\right). \tag{2.4}$$

Developing expression (2.4) into a power series expansion in $1/n$, we obtain

$$r_1(n) - r_1(n+1) = \left(\frac{1}{2} - a_1\right)\frac{1}{n^2} + \left(-\frac{2}{3} + a_1 + {a_1}^2\right)\frac{1}{n^3} + O\left(\frac{1}{n^4}\right). \tag{2.5}$$

From Lemma 1 we see that the rate of convergence of the sequence $(r_1(n) - \gamma)_{n\in\mathbb{N}}$ is even higher as the value $s$ satisfies (2.1). By Lemma 1 we have

(i) If $a_1 \neq 1/2$, then the rate of convergence of the $(r_1(n)-\gamma)_{n\in\mathbb{N}}$ is $n^{-2}$, since

$$\lim_{n\to\infty} n\big(r_1(n)-\gamma\big) = \frac{1}{2} - a_1 \neq 0.$$

(ii) If $a_1 = 1/2$, then from (2.5) we have

$$r_1(n) - r_1(n+1) = \frac{1}{12}\frac{1}{n^3} + O\left(\frac{1}{n^4}\right).$$

Hence the rate of convergence of the $(r_1(n)-\gamma)_{n\in\mathbb{N}}$ is $n^{-3}$, since

$$\lim_{n\to\infty} n^3\big(r_1(n)-\gamma\big) = \frac{1}{24} := C_1.$$

We also observe that the fastest possible sequence $(r_1(n))_{n\in\mathbb{N}}$ is obtained only for $a_1 = 1/2$.

We repeat our approach to determine $a_1$ to $a_7$ step by step. In fact, we can easily compute $a_k$, $k \leq 15$, by the *Mathematica* software. In this paper, we use the *Mathematica* software to manipulate symbolic computations.

Let

$$r_k(n) := \sum_{m=1}^{n} \frac{1}{m} - \ln n - \sum_{m=1}^{k} \ln\left(1 + \frac{a_m}{n^m}\right). \tag{2.6}$$

Then

$$\begin{aligned} &r_k(n) - r_k(n+1) \\ &\quad = \ln\left(1+\frac{1}{n}\right) - \frac{1}{n+1} + \sum_{m=1}^{k} \ln\left(1 + \frac{a_m}{(n+1)^m}\right) - \sum_{m=1}^{k} \ln\left(1+\frac{a_m}{n^m}\right). \end{aligned} \tag{2.7}$$

Hence the key step is to expand $r_k(n) - r_k(n+1)$ into power series in $1/n$. Here we use some examples to explain our method.

*Step 1*: For example, given $a_1$ to $a_4$, find $a_5$. Define

$$r_5(n) := \sum_{m=1}^{n} \frac{1}{m} - \ln n - \sum_{m=1}^{5} \ln\left(1 + \frac{a_m}{n^m}\right).$$

By using the *Mathematica* software (the *Mathematica Program* is very similar to that given further in Remark 2; however, it has the parameter $a_8$) we obtain

$$r_5(n) - r_5(n+1) = \left(-\frac{1}{32} - 5a_5\right)\frac{1}{n^6} + \left(\frac{4385}{48{,}384} + 15a_5\right)\frac{1}{n^7} + O\left(\frac{1}{n^8}\right). \tag{2.8}$$

The fastest possible sequence $(r_5(n))_{n\in\mathbb{N}}$ is obtained only for $a_5 = -\frac{1}{160}$. At the same time, it follows from (2.8) that

$$r_5(n) - r_5(n+1) = -\frac{151}{48{,}384}\frac{1}{n^7} + O\left(\frac{1}{n^8}\right). \tag{2.9}$$

The rate of convergence of $(r_8(n)-\gamma)_{n\in\mathbb{N}}$ is $n^{-7}$, since

$$\lim_{n\to\infty} n^7\bigl(r_5(n)-\gamma\bigr) = -\frac{151}{290{,}304} := C_5.$$

We can use this approach to find $a_k$ $(1\le k\le 15)$. From the computations we may the conjecture $a_{n+1}=C_n$, $n\ge 1$. Now, let us check it carefully.

*Step 2*: Check $a_6=-\frac{151}{290{,}304}$ to $a_7=-\frac{1}{896}$.

Let $a_1,\dots,a_6$, and $r_6(n)$ be defined as in Theorem 1. Applying the *Mathematica* software, we obtain

$$r_6(n)-r_6(n+1) = -\frac{1}{128}\frac{1}{n^8}+O\left(\frac{1}{n^9}\right). \tag{2.10}$$

The rate of convergence of $(r_6(n)-\gamma)_{n\in\mathbb{N}}$ is $n^{-8}$, since

$$\lim_{n\to\infty} n^8\bigl(r_6(n)-\gamma\bigr) = -\frac{1}{896} := C_6.$$

Finally, we check that $a_7=-\frac{1}{896}$:

$$r_7(n)-r_7(n+1) = \left(-\frac{1}{128}-7a_7\right)\frac{1}{n^8}+\left(\frac{196{,}193}{2{,}764{,}800}+28a_7\right)\frac{1}{n^9}+O\left(\frac{1}{n^{10}}\right). \tag{2.11}$$

Since $a_7=-\frac{1}{896}$ and

$$\lim_{n\to\infty} n^9\bigl(r_7(n)-\gamma\bigr) = \frac{109{,}793}{22{,}118{,}400} := C_7,$$

the rate of convergence of the $(r_7(n)-\gamma)_{n\in\mathbb{N}}$ is $n^{-9}$.

This completes the proof of Theorem 1.

*Remark* 2 From the computations we can guess that $a_{n+1}=C_n$, $n\ge 1$. It is a very interesting problem to prove this. However, it seems impossible by the provided method.

## 3 Proof of Theorem 2

Before we prove Theorem 2, let us give a simple inequality, which follows from the Hermite–Hadamard inequality and plays an important role in the proof.

**Lemma 2** *Let $f$ be a twice continuously differentiable function. If $f''(x)>0$, then*

$$\int_a^{a+1} f(x)\,dx > f(a+1/2). \tag{3.1}$$

By $P_k(x)$ we denote polynomials of degree $k$ in $x$ such that all its nonzero coefficients are positive; it may be different at each occurrence.

Let us prove Theorem 2. Noting that $r_2(\infty)=0$, we easily see that

$$\gamma-r_2(n) = \sum_{m=n}^{\infty}\bigl(r_2(m+1)-r_2(m)\bigr) = \sum_{m=n}^{\infty} f(m), \tag{3.2}$$

where

$$f(m) = \frac{1}{m+1} - \ln\left(1 + \frac{1}{m}\right) - \ln\left(1 + \frac{a_1}{m+1}\right) - \ln\left(1 + \frac{a_2}{(m+1)^2}\right)$$
$$+ \ln\left(1 + \frac{a_1}{m}\right) + \ln\left(1 + \frac{a_2}{m^2}\right).$$

Let $D_1 = 1/2$. By using the *Mathematica* software we have

$$-f'(x) - D_1 \frac{1}{(x+1)^5}$$
$$= \frac{300 + 2739x + 11{,}434x^2 + 24{,}870x^3 + 28{,}314x^4 + 15{,}936x^5 + 3472x^6}{2x(1+x)^5(1+2x)(3+2x)(1+24x^2)(25+48x+24x^2)} > 0$$

and

$$-f'(x) - D_1 \frac{1}{(x+\frac{1}{2})^5}$$
$$= -\frac{P_6(x)(x-1) + 151{,}085}{2x^5(1+x)^2(1+2x)(3+2x)(1+24x^2)(25+48x+24x^2)} < 0.$$

Hence, we get the following inequalities for $x \geq 1$:

$$D_1 \frac{1}{(x+1)^5} < -f'(x) < D_1 \frac{1}{(x+\frac{1}{2})^5}. \tag{3.3}$$

Since $f(\infty) = 0$, from the right-hand side of (3.3) and Lemma 2 we get

$$f(m) = -\int_m^\infty f'(x)\,dx \leq D_1 \int_m^\infty \left(x + \frac{1}{2}\right)^{-5} dx$$
$$= \frac{D_1}{4}\left(m + \frac{1}{2}\right)^{-4} \leq \frac{D_1}{4} \int_m^{m+1} x^{-4}\,dx. \tag{3.4}$$

From (3.1) and (3.4) we obtain

$$\gamma - r_2(n) \leq \sum_{m=n}^\infty \frac{D_1}{4} \int_m^{m+1} x^{-4}\,dx$$
$$= \frac{D_1}{4} \int_n^\infty x^{-4}\,dx = \frac{D_1}{12} \frac{1}{n^3}. \tag{3.5}$$

Similarly, we also have

$$f(m) = -\int_m^\infty f'(x)\,dx \geq D_1 \int_m^\infty (x+1)^{-5}\,dx$$
$$= \frac{D_1}{4}(m+1)^{-4} \geq \frac{D_1}{4} \int_{m+1}^{m+2} x^{-4}\,dx$$

and

$$\begin{aligned}\gamma - r_2(n) &\geq \sum_{m=n}^{\infty} \frac{D_1}{4} \int_{m+1}^{m+2} x^{-4}\,dx \\ &= \frac{D_1}{4} \int_{n+1}^{\infty} x^{-4}\,dx = \frac{D_1}{12} \frac{1}{(n+1)^3}.\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) completes the proof of (1.6).

Noting that $r_3(\infty) = 0$, we easily deduce

$$r_3(n) - \gamma = \sum_{m=n}^{\infty} \bigl(r_3(m) - r_3(m+1)\bigr) = \sum_{m=n}^{\infty} g(m), \tag{3.7}$$

where

$$g(m) = r_3(m) - r_3(m+1).$$

Let $D_2 = \frac{143}{288}$. By using the *Mathematica* software we have

$$\begin{aligned}&-g'(x) - D_2 \frac{1}{(x+1)^6} \\ &\quad= \frac{P_{12}(x)}{288n(1+n)^6(1+2n)(3+2n)(1+24n^2)(25+48n+24n^2)(-1+24n^3)P_3(x)} > 0\end{aligned}$$

and

$$\begin{aligned}&-g'(x) - D_2 \frac{1}{(x+\frac{1}{2})^6} \\ &\quad= -\frac{P_{12}(x)(x-4) + 2{,}052{,}948{,}001{,}087{,}775}{9x(1+x)^2(1+2x)^6(3+2x)(1+24x^2)(25+48x+24x^2)(-1+24x^3)P_3(x)} < 0.\end{aligned}$$

Hence, for $x \geq 4$,

$$D_2 \frac{1}{(n+1)^6} < -g'(x) < D_2 \frac{1}{(x+\frac{1}{2})^6}. \tag{3.8}$$

Since $g(\infty) = 0$, by (3.8) we get

$$\begin{aligned}g(m) &= -\int_m^{\infty} g'(x)\,dx \leq D_2 \int_m^{\infty} \left(x + \frac{1}{2}\right)^{-6} dx \\ &= \frac{D_2}{5} \left(m + \frac{1}{2}\right)^{-5} \leq \frac{D_2}{5} \int_m^{m+1} x^{-5}\,dx.\end{aligned} \tag{3.9}$$

It follows from (3.7), (3.9), and Lemma 2 that

$$\begin{aligned}r_3(n) - \gamma &\leq \sum_{m=n}^{\infty} \frac{D_2}{5} \int_m^{m+1} x^{-5}\,dx \\ &= \frac{D_2}{5} \int_n^{\infty} x^{-5}\,dx = \frac{D_2}{20} \frac{1}{n^4}.\end{aligned} \tag{3.10}$$

Finally,

$$g(m) = -\int_m^\infty g'(x)\,dx \ge D_2 \int_m^\infty (x+1)^{-6}\,dx$$
$$= \frac{D_2}{5}(m+1)^{-5} \ge \frac{D_2}{5}\int_{m+1}^{m+2} x^{-5}\,dx$$

and

$$r_3(n) - \gamma \ge \sum_{m=n}^{\infty} \frac{D_2}{5}\int_{m+1}^{m+2} x^{-5}\,dx$$
$$= \frac{D_2}{5}\int_{n+1}^{\infty} x^{-5}\,dx = \frac{D_2}{20}\frac{1}{(n+1)^4}. \tag{3.11}$$

Combining (3.10) and (3.11) completes the proof of (1.7).

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**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
The authors read and approved the final manuscript.

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