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Solving the multiple-set split equality common fixed-point problem of firmly quasi-nonexpansive operators

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Abstract

In this paper, we propose parallel and cyclic iterative algorithms for solving the multiple-set split equality common fixed-point problem of firmly quasi-nonexpansive operators. We also combine the process of cyclic and parallel iterative methods and propose two mixed iterative algorithms. Our several algorithms do not need any prior information about the operator norms. Under mild assumptions, we prove weak convergence of the proposed iterative sequences in Hilbert spaces. As applications, we obtain several iterative algorithms to solve the multiple-set split equality problem.

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1 Introduction

Let H_1, H_2, and H_3 be real Hilbert spaces. The multiple-set split equality common fixed-point problem (MSECFP) is to find x^*, y^* with the property

$$x^* \in \bigcap_{i=1}^{p} F(U_i), \qquad y^* \in \bigcap_{j=1}^{r} F(T_j) \quad \text{such that } Ax^* = By^*, \tag{1.1}$$

where $p, r \geq 1$ are integers, $\{U_i\}_{i=1}^{p} : H_1 \to H_1$ and $\{T_j\}_{j=1}^{r} : H_2 \to H_2$ are nonlinear operators, $A : H_1 \to H_2$ and $B : H_2 \to H_3$ are two bounded linear operators. If U_i $(1 \leq i \leq p)$ and T_j $(1 \leq j \leq r)$ are projection operators, then the MSECFP is reduced to the multiple-set split equality problem (MSEP):

$$\text{finding} \quad x^* \in \bigcap_{i=1}^{p} C_i \quad \text{and} \quad y^* \in \bigcap_{j=1}^{r} Q_j \quad \text{such that } Ax^* = By^*, \tag{1.2}$$

where $\{C_i\}_{i=1}^{p}$ and $\{Q_j\}_{j=1}^{r}$ are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively. When $p = r = 1$, the MSECFP and MSEP become the split equality common fixed-point problem (SECFP) and split equality problem (SEP), respectively, which were first put forward by Moudafi [1]. These allow asymmetric and partial relations between the variables x and y. They are applied in many situations, for instance, in game theory and in intensity-modulated radiation therapy (see [2] and [3]).

If $H_2 = H_3$ and $B = I$, then MSECFP (1.1) reduces to the multiple-set split common fixed-point problem (MSCFP):

$$\text{finding} \quad x^* \in \bigcap_{i=1}^{p} F(U_i) \quad \text{such that } Ax^* \in \bigcap_{j=1}^{r} F(T_j) \tag{1.3}$$

and MSEP (1.2) reduces to the multiple-set split feasibility problem (MSFP):

$$\text{finding} \quad x^* \in \bigcap_{i=1}^{p} C_i \quad \text{such that } Ax^* \in \bigcap_{j=1}^{r} Q_j. \tag{1.4}$$

They play significant roles in dealing with problems in image restoration, signal processing, and intensity-modulated radiation therapy [3–6]. With $p = r = 1$, MSCFP (1.3) is known as the split common fixed-point problem (SCFP) and MSFP (1.4) is known as the split feasibility problem (SFP). Many iterative algorithms have been developed to solve the MSCFP and the MSFP. See, for example, [7–14] and the references therein.

Note that the SFP can be formulated as a fixed-point equation

$$P_C\bigl(I - \gamma A^*(I - P_Q)A\bigr)x^* = x^*, \tag{1.5}$$

where P_C and P_Q are the (orthogonal) projections onto C and Q, respectively, $\gamma > 0$ is any positive constant, and A^* denotes the adjoint of A. This implies that we can use fixed-point algorithms (see [15–21]) to solve SFP. Byrne [22] proposed the so-called CQ algorithm which generates a sequence $\{x_k\}$:

$$x_{n+1} = P_C\bigl(x_n - \gamma A^*(I - P_Q)Ax_n\bigr), \tag{1.6}$$

where $\gamma \in (0, 2/\lambda)$ with λ being the spectral radius of the operator A^*A. The CQ algorithm is efficient when P_C and P_Q are easily calculated. However, if C and Q are complex sets, for example, the fixed-point sets, the efficiency of the CQ algorithm will be affected because the projections onto such convex sets are generally hard to be accurately calculated. To solve the SCFP of nonexpansive operators, Censor and Segal [23] proposed and proved, in finite-dimensional spaces, the convergence of the following algorithm:

$$x_{n+1} = U\bigl(x_n - \gamma A^*(I - T)Ax_n\bigr), \quad n \in N, \tag{1.7}$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the largest eigenvalue of the matrix A^*A.

For solving the constrained MSFP, Censor et al. [6] introduced the following proximity function:

$$g(x) := \frac{1}{2}\sum_{i=1}^{p} \alpha_i \|x - P_{C_i}x\|^2 + \frac{1}{2}\sum_{j=1}^{r} \beta_j \bigl\|Ax - P_{Q_j}(Ax)\bigr\|^2, \tag{1.8}$$

where $\alpha_i > 0$ $(1 \le i \le p)$, $\beta_j > 0$ $(1 \le j \le r)$, and $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{r} \beta_j = 1$. Then

$$\nabla g(x) = \sum_{i=1}^{p} \alpha_i (x - P_{C_i} x) + \sum_{j=1}^{r} \beta_j A^* \bigl(Ax - P_{Q_j}(Ax)\bigr),$$

and they proposed the following projection method:

$$x_{n+1} = P_\Omega \bigl(x_n - \gamma \nabla g(x_n)\bigr), \tag{1.9}$$

where Ω is the constrained set, $0 < \gamma_L \le \gamma \le \gamma_U < \frac{2}{L}$, and L is the Lipschitz constant of ∇g.

For solving MSCFP (1.3) of directed operators, Censor and Segal [23] introduced a parallel iterative algorithm as follows:

$$x_{n+1} = x_n - \gamma \left[\sum_{i=1}^{p} \alpha_i \bigl(x_n - U_i(x_n)\bigr) + \sum_{j=1}^{r} \beta_j A^* \bigl(Ax_n - T_j(Ax_n)\bigr) \right], \tag{1.10}$$

where $\{\alpha_i\}_{i=1}^{p}$, $\{\beta_j\}_{j=1}^{r}$ are nonnegative constants, $0 < \gamma < 2/L$ with $L = \sum_{i=1}^{p} \alpha_i + \lambda \sum_{j=1}^{r} \beta_j$ and λ being the largest eigenvalue of A^*A. They obtained the convergence of iterative algorithm (1.6).

Wang and Xu [24] proposed the following cyclic iterative algorithm for MSCFP (1.3) of directed operators:

$$x_{n+1} = U_{[n]_1}(x_n + \gamma A^*(T_{[n]_2} - I)Ax_n, \tag{1.11}$$

where $0 < \gamma < 2/\rho(A^*A)$, $[n]_1 := n(\operatorname{mod} p)$, and $[n]_2 := n(\operatorname{mod} r)$. They proved the weak convergence of the sequence $\{x_n\}$ generated by (1.7).

For solving MSCFP (1.3), Tang and Liu [25] introduced inner parallel and outer cyclic iterative algorithm:

$$x_{n+1} = U_{[n]_1} \left(x_n + \gamma_n \sum_{j=1}^{r} \eta_j A^* (T_j - I) Ax_n \right) \tag{1.12}$$

and outer parallel and inner cyclic iterative algorithm:

$$x_{n+1} = \sum_{i=1}^{p} \omega_i U_i \bigl(x_n + \gamma_n A^* (T_{[n]_2} - I) Ax_n\bigr) \tag{1.13}$$

for directed operators $\{U_i\}_{i=1}^{p}$ and $\{T_j\}_{j=1}^{r}$, where $[n]_1 = n(\operatorname{mod} p)$, $[n]_2 = n(\operatorname{mod} r)$, $0 < a \le \gamma_n \le b < 2/\rho(A^*A)$, $\{\eta_j\}_{j=1}^{r}$, $\{\omega_i\}_{i=1}^{p} \subset (0,1)$ with $\sum_{j=1}^{r} \eta_j = 1$ and $\sum_{i=1}^{p} \omega_i = 1$. They obtained the weak convergence of the above two mixed iterative sequences to solve MSCFP (1.3) of directed operators.

The SEP proposed by Moudafi [1] is to

$$\text{find} \quad x^* \in C, \qquad y^* \in Q \quad \text{such that } Ax^* = By^*, \tag{1.14}$$

which can be written as the following minimization problem:

$$\min_{x\in C,y\in Q} \frac{1}{2}\|Ax - By\|^2. \tag{1.15}$$

Assume that the solution set of the SEP is nonempty. By the optimality conditions, Moudafi [1] obtained the following fixed-point formulation: (x^*, y^*) solves the SEP if and only if

$$\begin{cases} x^* = P_C(x^* - \gamma A^*(Ax^* - By^*)), \\ y^* = P_Q(y^* + \beta B^*(Ax^* - By^*)), \end{cases} \tag{1.16}$$

where $\gamma, \beta > 0$. Therefore, for solving the SECP of firmly quasi-nonexpansive operators, Moudafi [1] introduced the following alternating algorithm:

$$\begin{cases} x_{n+1} = U(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = T(y_n + \gamma_n B^*(Ax_{n+1} - By_n)), \end{cases} \tag{1.17}$$

where a nondecreasing sequence $\gamma_n \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, λ_A, λ_B stand for the spectral radius of A^*A and B^*B, respectively. In [26], Moudafi and Al-Shemas introduced the following simultaneous iterative method:

$$\begin{cases} x_{n+1} = U(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = T(y_n + \gamma_n B^*(Ax_n - By_n)), \end{cases} \tag{1.18}$$

where $\gamma_n \in (\varepsilon, \frac{2}{\lambda_A+\lambda_B} - \varepsilon)$, λ_A, λ_B stand for the spectral radius of A^*A and B^*B, respectively. Recently, many iterative algorithms have been developed to solve the SEP, SECFP, and MSEP. See, for example, [27–34] and the references therein. Note that in algorithms (1.17) and (1.18), the determination of the step size $\{\gamma_n\}$ depends on the operator (matrix) norms $\|A\|$ and $\|B\|$ (or the largest eigenvalues of A^*A and B^*B). To overcome this shortage, we introduce parallel and cyclic iterative algorithms with self-adaptive step size to solve MSECFP (1.1) governed by firmly quasi-nonexpansive operators. We also propose two mixed iterative algorithms which combine the process of cyclic and parallel iterative methods and do not need the norms of bounded linear operators. As applications, we obtain several iterative algorithms to solve MSEP (1.2).

2 Preliminaries

2.1 Concepts

Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let I denote the identity operator on H. Denote the fixed-point set of an operator T by $F(T)$. We denote by $\to$ the strong convergence and by $\rightharpoonup$ the weak convergence. We use $\omega_w(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$ to stand for the weak ω-limit set of $\{x_k\}$ and use Γ to stand for the solution set of MSECFP (1.1).

Definition 2.1 An operator $T : H \to H$ is said to be

(i) nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in H$;

(ii) firmly nonexpansive if $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(x - y) - (Tx - Ty)\|^2$ for all $x, y \in H$;
(iii) firmly quasi-nonexpansive (i.e., directed operator) if $F(T) \neq \emptyset$ and

$$\|Tx - q\|^2 \leq \|x - q\|^2 - \|x - Tx\|^2$$

or equivalently

$$\langle x - q, x - Tx\rangle \geq \|x - Tx\|^2$$

for all $x \in H$ and $q \in F(T)$.

Definition 2.2 An operator $T : H \to H$ is called demiclosed at the origin if, for any sequence $\{x_n\}$ which weakly converges to x, and if the sequence $\{Tx_n\}$ strongly converges to 0, then $Tx = 0$.

Recall that the metric (nearest point) projection from H onto a nonempty closed convex subset C of H, denoted by P_C, is defined as follows: for each $x \in H$,

$$P_C(x) = \arg\min_{y \in C}\bigl\{\|x - y\|\bigr\}.$$

It is well known that $P_C x$ is characterized by the inequality

$$P_C x \in C, \qquad \langle x - P_C x, z - P_C x\rangle \leq 0, \quad z \in C.$$

Remark 2.1 It is easily seen that a firmly nonexpansive operator is nonexpansive. Firmly quasi-nonexpansive operators contain firmly nonexpansive operators with a nonempty fixed-point set. A projection operator is firmly nonexpansive.

2.2 Mathematical model

Recall that the SCFP is to find x^* with the property

$$x^* \in F(U) \quad \text{such that } Ax^* \in F(T) \tag{2.1}$$

and the SFP is to find x^* with the property:

$$x^* \in C \quad \text{such that } Ax^* \in Q, \tag{2.2}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are nonlinear operators, C and Q are closed convex sets of Hilbert spaces H_1 and H_2, respectively.

We can formulate SFP (2.2) as an optimization. First, we consider the following proximity function:

$$g(x) = \frac{1}{2}\|x - P_C x\|^2 + \frac{1}{2}\|Ax - P_Q Ax\|^2.$$

Then the proximity function $g(x)$ is convex and differentiable with gradient

$$\nabla g(x) = x - P_C x + A^*(I - P_Q)Ax,$$

where A^* denotes the adjoint of A. Assume that the solution set of the SFP is nonempty, then x^* is a solution of the SFP if and only if $x^* = \arg\min_{x\in H_1} g(x)$, i.e.,

$$\nabla g(x^*) = 0,$$

which is equivalent to

$$\begin{aligned} x^* &= x^* - \tau\nabla g(x^*) \\ &= x^* - \tau\left(x^* - P_C x^* + A^*(I - P_Q)Ax^*\right) \end{aligned} \tag{2.3}$$

for all $\tau > 0$. For solving the SCFP of directed operators (i.e., firmly quasi-nonexpansive operators), Wang [35] proposed the following algorithm:

$$x_{n+1} = x_n - \tau_n\left[(x_n - Ux_n) + A^*(I - T)Ax_n\right], \tag{2.4}$$

where the variable size step τ_n was chosen:

$$\tau_n = \frac{\|x_n - Ux_n\|^2 + \|(I - T)Ax_n\|^2}{\|(x_n - Ux_n) + A^*(I - T)Ax_n\|^2}.$$

This algorithm can be obtained by the fixed-point Eq. (2.3), where projection operators P_C and P_Q are replaced by U and T.

Setting

$$h(x,y) = \frac{1}{2}\sum_{i=1}^{p}\alpha_i\|x - P_{C_i}x\|^2 + \frac{1}{2}\sum_{j=1}^{r}\beta_j\|y - P_{Q_j}y\|^2 + \frac{1}{2}\|Ax - By\|^2, \tag{2.5}$$

MSEP (1.2) can be written as the following minimization problem:

$$\min_{x\in H_1, y\in H_2} h(x,y),$$

where $\alpha_i > 0$ $(1 \le i \le p)$, $\beta_j > 0$ $(1 \le j \le r)$, $\sum_{i=1}^{p}\alpha_i = 1$, and $\sum_{j=1}^{r}\beta_j = 1$. Assume that the solution set of the MSEP is nonempty, by the optimality conditions (x^*, y^*) solves the MSEP if and only if

$$\begin{cases} \nabla_x h(x^*, y^*) = 0, \\ \nabla_y h(x^*, y^*) = 0, \end{cases} \tag{2.6}$$

which is equivalent to

$$\begin{cases} x^* = x^* - \gamma\nabla_x h(x^*, y^*) = x^* - \gamma[x^* - \sum_{i=1}^{p}\alpha_i P_{C_i}(x^*) + A^*(Ax^* - By^*)], \\ y^* = y^* - \beta\nabla_y h(x^*, y^*) = y^* - \beta[y^* - \sum_{j=1}^{r}\beta_j P_{Q_j}(y^*) - B^*(Ax^* - By^*)] \end{cases} \tag{2.7}$$

for $\gamma, \beta > 0$. These motivate us to introduce several iterative algorithms with self-adaptive step size for solving MSECFP (1.1) governed by firmly quasi-nonexpansive mappings and MSEP (1.2).

2.3 The well-known lemmas

The following lemmas will be helpful for our main results in the next section.

Lemma 2.1 *Let H be a real Hilbert space. Then*

$$2\langle x,y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2 = \|x+y\|^2 - \|x\|^2 - \|y\|^2, \quad \forall x,y \in H. \tag{2.8}$$

Lemma 2.2 ([36]) *Let H be a real Hilbert space. Then, for all $t \in [0,1]$ and $x,y \in H$,*

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2.$$

Lemma 2.3 ([37]) *Let H be a real Hilbert space. Then*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \cdots + \alpha_r x_r\|^2 \le \sum_{i=0}^{r} \alpha_i \|x_i\|^2 - \alpha_s \alpha_t \|x_s - x_t\|^2$$

for any $s,t \in \{0,1,2,\ldots,r\}$ and for $x_i \in H, i = 0,1,2,\ldots,r$, with $\alpha_0 + \alpha_1 + \cdots + \alpha_r = 1$ and $0 \le \alpha_i \le 1$.

Lemma 2.4 ([38]) *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E, and $T : K \to K$ be a nonexpansive mapping. Then $I - T$ is demi-closed at origin.*

3 Parallel and cyclic iterative algorithms

In this section, we introduce parallel and cyclic iterative algorithms and prove the weak convergence for solving MSECFP (1.1) of firmly quasi-nonexpansive operators. In our algorithms, the selection of the step size does not need any prior information of the operator norms $\|A\|$ and $\|B\|$.

In what follows, we adopt the following assumptions:

(A1) The problem is consistent, namely its solution set Γ is nonempty;

(A2) Both U_i and T_j are firmly quasi-nonexpansive operators, and both $I - U_i$ and $I - T_j$ are demiclosed at origin ($1 \le i \le p$, $1 \le j \le r$).

(A3) The sequences $\{\alpha_n^i\}_{i=1}^p$, $\{\beta_n^j\}_{j=1}^r \subset [0,1]$ such that $\sum_{i=1}^p \alpha_n^i = 1$ and $\sum_{j=1}^r \beta_n^j = 1$ for every $n \ge 0$, $j(n) = n(\operatorname{mod} r) + 1$, $i(n) = n(\operatorname{mod} p) + 1$.

Algorithm 3.1 Let $x_0 \in H_1, y_0 \in H_2$ be arbitrary. For $n \ge 0$, let

$$\begin{cases} u_n = x_n - (\alpha_n^1 U_1(x_n) + \cdots + \alpha_n^p U_p(x_n)) + A^*(Ax_n - By_n), \\ x_{n+1} = x_n - \tau_n u_n, \\ v_n = y_n - (\beta_n^1 T_1(y_n) + \cdots + \beta_n^r T_r(y_n)) - B^*(Ax_n - By_n), \\ y_{n+1} = y_n - \tau_n v_n, \end{cases} \tag{3.1}$$

where the step size τ_n is chosen as

$$\tau_n \in \left(\epsilon, \min\left\{1, \frac{\|Ax_n - By_n\|^2}{\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2}\right\} - \epsilon\right), \quad n \in \Omega, \tag{3.2}$$

for small enough $\epsilon > 0$, otherwise, $\tau_n = \tau \in (0,1)$ (τ being any value in $(0,1)$), the set of indexes $\Omega = \{n \in N : Ax_n - By_n \neq 0\}$.

Remark 3.1 Note that in (3.2) the choice of the step size τ_n is independent of the norms $\|A\|$ and $\|B\|$. The value of τ does not influence the considered algorithm, it was introduced just for the sake of clarity.

Lemma 3.1 *τ_n defined by* (3.2) *is well defined.*

Proof Taking $(x,y) \in \Gamma$, i.e., $x \in \cap_{i=1}^{p} F(U_i)$, $y \in \cap_{j=1}^{r} F(T_j)$, and $Ax = By$, we have

$$\left\langle A^*(Ax_n - By_n), x_n - x\right\rangle = \langle Ax_n - By_n, Ax_n - Ax\rangle$$

and

$$\left\langle B^*(Ax_n - By_n), y - y_n\right\rangle = \langle Ax_n - By_n, By - By_n\rangle.$$

By adding the two above equalities and by taking into account the fact that $Ax = By$, we obtain

$$\begin{aligned} &\|Ax_n - By_n\|^2 \\ &\quad = \left\langle A^*(Ax_n - By_n), x_n - x\right\rangle + \left\langle B^*(Ax_n - By_n), y - y_n\right\rangle \\ &\quad \leq \left\|A^*(Ax_n - By_n)\right\| \cdot \|x_n - x\| + \left\|B^*(Ax_n - By_n)\right\| \cdot \|y - y_n\|. \end{aligned} \tag{3.3}$$

Consequently, for $n \in \Omega$, that is, $\|Ax_n - By_n\| > 0$, we have $\|A^*(Ax_n - By_n)\| \neq 0$ or $\|B^*(Ax_n - By_n)\| \neq 0$. This leads to the fact that τ_n is well defined. □

Theorem 3.1 *Assume that* $\liminf_{n\to\infty} \alpha_n^i > 0 (1 \leq i \leq p)$ *and* $\liminf_{n\to\infty} \beta_n^j > 0 (1 \leq j \leq r)$. *Then the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm* 3.1 *weakly converges to a solution* (x^*, y^*) *of MSECFP* (1.1). *Moreover,* $\|Ax_n - By_n\| \to 0$, $\|x_{n+1} - x_n\| \to 0$, *and* $\|y_{n+1} - y_n\| \to 0$ *as* $n \to \infty$.

Proof From the condition on $\{\tau_n\}$, we have $\{\tau_n\}_{n\geq 0}$ is bounded. It follows from Algorithm 3.1 and $\sum_{i=1}^{p} \alpha_n^i = 1$ that

$$u_n = \alpha_n^1\bigl(x_n - U_1(x_n)\bigr) + \cdots + \alpha_n^p\bigl(x_n - U_p(x_n)\bigr) + A^*(Ax_n - By_n). \tag{3.4}$$

Taking $(x^*, y^*) \in \Gamma$, i.e., $x^* \in \bigcap_{i=1}^{p} F(U_i)$, $y^* \in \bigcap_{j=1}^{r} F(T_j)$, and $Ax^* = By^*$, we have

$$\begin{aligned} &\left\langle u_n, x_n - x^*\right\rangle \\ &\quad = \alpha_n^1\left\langle x_n - U_1(x_n), x_n - x^*\right\rangle + \cdots + \alpha_n^p\left\langle x_n - U_p(x_n), x_n - x^*\right\rangle \\ &\qquad + \left\langle A^*(Ax_n - By_n), x_n - x^*\right\rangle \\ &\quad \geq \alpha_n^1\left\|x_n - U_1(x_n)\right\|^2 + \cdots + \alpha_n^p\left\|x_n - U_p(x_n)\right\|^2 + \left\langle Ax_n - By_n, Ax_n - Ax^*\right\rangle. \end{aligned} \tag{3.5}$$

Similarly, we have

$$\begin{aligned}\langle v_n, y_n - y^*\rangle &\geq \beta_n^1 \|y_n - T_1(y_n)\|^2 + \cdots + \beta_n^r \|y_n - T_r(y_n)\|^2 \\ &\quad - \langle Ax_n - By_n, By_n - By^*\rangle.\end{aligned} \tag{3.6}$$

By adding the two inequalities (3.5)–(3.6) and taking into account the fact that $Ax^* = By^*$, we obtain

$$\begin{aligned}&\langle u_n, x_n - x^*\rangle + \langle v_n, y_n - y^*\rangle \\ &\quad \geq \sum_{i=1}^{p} \alpha_n^i \|x_n - U_i(x_n)\|^2 + \sum_{j=1}^{r} \beta_n^j \|y_n - T_j(y_n)\|^2 + \|Ax_n - By_n\|^2.\end{aligned} \tag{3.7}$$

From Algorithm 3.1 we also have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 \\ &\quad = \|x_n - \tau_n u_n - x^*\|^2 + \|y_n - \tau_n v_n - y^*\|^2 \\ &\quad = \|x_n - x^*\|^2 + \|y_n - y^*\|^2 - 2\tau_n(\langle u_n, x_n - x^*\rangle + \langle v_n, y_n - y^*\rangle) + \tau_n^2(\|u_n\|^2 + \|v_n\|^2).\end{aligned} \tag{3.8}$$

By Lemma 2.3 we get

$$\begin{aligned}\|u_n\|^2 &= \|\alpha_n^1(x_n - U_1(x_n)) + \cdots + \alpha_n^p(x_n - U_p(x_n)) + A^*(Ax_n - By_n)\|^2 \\ &\leq 2\|\alpha_n^1(x_n - U_1(x_n)) + \cdots + \alpha_n^p(x_n - U_p(x_n))\|^2 + 2\|A^*(Ax_n - By_n)\|^2 \\ &\leq 2\sum_{i=1}^{p} \alpha_n^i \|x_n - U_i(x_n)\|^2 + 2\|A^*(Ax_n - By_n)\|^2\end{aligned} \tag{3.9}$$

and

$$\|v_n\|^2 \leq 2\sum_{j=1}^{r} \beta_n^j \|y_n - T_j(y_n)\|^2 + 2\|B^*(Ax_n - By_n)\|^2. \tag{3.10}$$

Setting $s_n(x^*, y^*) = \|x_n - x^*\|^2 + \|y_n - y^*\|^2$ and using (3.7), (3.9)–(3.10), (3.8) can be written as

$$\begin{aligned}&s_{n+1}(x^*, y^*) \\ &\quad \leq s_n(x^*, y^*) \\ &\qquad - 2\tau_n\left[\sum_{i=1}^{p} \alpha_n^i \|x_n - U_i(x_n)\|^2 + \sum_{j=1}^{r} \beta_n^j \|y_n - T_j(y_n)\|^2 + \|Ax_n - By_n\|^2\right] \\ &\qquad + 2\tau_n^2\left[\sum_{i=1}^{p} \alpha_n^i \|x_n - U_i(x_n)\|^2 + \sum_{j=1}^{r} \beta_n^j \|y_n - T_j(y_n)\|^2\right.\end{aligned}$$

$$
\begin{aligned}
&+ \|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2 \Bigg] \\
&= s_n(x^*, y^*) - 2\tau_n(1-\tau_n)\Bigg[\sum_{i=1}^{p} \alpha_n^i \|x_n - U_i(x_n)\|^2 + \sum_{j=1}^{r} \beta_n^j \|y_n - T_j(y_n)\|^2\Bigg] \\
&\quad - 2\tau_n\big[\|Ax_n - By_n\|^2 - \tau_n\big(\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2\big)\big].
\end{aligned} \tag{3.11}
$$

We see that the sequence $\{s_n(x^*, y^*)\}$ is decreasing and lower bounded by 0; consequently, it converges to some finite limit which is denoted by $s(x^*, y^*)$. So the sequences $\{x_n\}$ and $\{y_n\}$ are bounded.

By the conditions on $\{\tau_n\}$, $\{\alpha_n^i\}$ $(1 \le i \le p)$ and $\{\beta_n^j\}$ $(1 \le j \le r)$, from (3.11) we obtain, for all i $(1 \le i \le p)$ and j $(1 \le j \le r)$,

$$
\lim_{n\to\infty} \|x_n - U_i(x_n)\| = \lim_{n\to\infty} \|y_n - T_j(y_n)\| = 0 \tag{3.12}
$$

and

$$
\lim_{n\to\infty} \|A^*(Ax_n - By_n)\| = \lim_{n\to\infty} \|B^*(Ax_n - By_n)\| = 0. \tag{3.13}
$$

It follows from (3.3) and (3.13) that

$$
\lim_{n\to\infty} \|Ax_n - By_n\| = 0. \tag{3.14}
$$

Since

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad = \|x_n - \tau_n u_n - x_n\| \\
&\quad = \tau_n \big\|\alpha_n^1\big(x_n - U_1(x_n)\big) + \cdots + \alpha_n^p\big(x_n - U_p(x_n)\big) + A^*(Ax_n - By_n)\big\| \\
&\quad \le \tau_n\big(\alpha_n^1\|x_n - U_1(x_n)\| + \cdots + \alpha_n^p\|x_n - U_p(x_n)\| + \|A^*(Ax_n - By_n)\|\big),
\end{aligned} \tag{3.15}
$$

we get

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \tag{3.16}
$$

which infers that $\{x_n\}$ is asymptotically regular. Similarly, we also have that $\{y_n\}$ is asymptotically regular, namely $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$.

Take $(\tilde{x}, \tilde{y}) \in \omega_\omega(x_n, y_n)$, i.e., there exists a subsequence $\{(x_{n_k}, y_{n_k})\}$ of $\{(x_n, y_n)\}$ such that $(x_{n_k}, y_{n_k}) \rightharpoonup (\tilde{x}, \tilde{y})$ as $k \to \infty$. Combined with the demiclosedness of $U_i - I$ and $T_j - I$ at 0, it follows from (3.12) that $U_i(\tilde{x}) = \tilde{x}$ and $T_j(\tilde{y}) = \tilde{y}$ for $1 \le i \le p$ and $1 \le j \le r$. So, $\tilde{x} \in \bigcap_{i=1}^{p} F(U_i)$ and $\tilde{y} \in \bigcap_{j=1}^{r} F(T_j)$. On the other hand, $A\tilde{x} - B\tilde{y} \in \omega_w(Ax_n - By_n)$ and weakly lower semicontinuity of the norm imply that

$$
\|A\tilde{x} - B\tilde{y}\| \le \liminf_{n\to\infty} \|Ax_n - By_n\| = 0,
$$

hence $(\tilde{x}, \tilde{y}) \in \Gamma$. So $\omega_w(x_n, y_n) \subseteq \Gamma$.

Next, we will show the uniqueness of the weak cluster point $\{(x_n, y_n)\}$. Indeed, let $(\bar{x}, \bar{y})$ be another weak cluster point of $\{(x_n, y_n)\}$, then $(\bar{x}, \bar{y}) \in \Gamma$. From the definition of $s_n(x^*, y^*)$, we have

$$\begin{aligned} s_n(\tilde{x}, \tilde{y}) &= \|x_n - \bar{x}\|^2 + \|\bar{x} - \tilde{x}\|^2 + 2\langle x_n - \bar{x}, \bar{x} - \tilde{x}\rangle + \|y_n - \bar{y}\|^2 + \|\bar{y} - \tilde{y}\|^2 + 2\langle y_n - \bar{y}, \bar{y} - \tilde{y}\rangle \\ &= s_n(\bar{x}, \bar{y}) + \|\bar{x} - \tilde{x}\|^2 + \|\bar{y} - \tilde{y}\|^2 + 2\langle x_n - \bar{x}, \bar{x} - \tilde{x}\rangle + 2\langle y_n - \bar{y}, \bar{y} - \tilde{y}\rangle. \end{aligned} \tag{3.17}$$

Without loss of generality, we may assume that $x_n \rightharpoonup \bar{x}$ and $y_n \rightharpoonup \bar{y}$. By passing to the limit in relation (3.17), we obtain

$$s(\tilde{x}, \tilde{y}) = s(\bar{x}, \bar{y}) + \|\bar{x} - \tilde{x}\|^2 + \|\bar{y} - \tilde{y}\|^2.$$

Reversing the role of $(\tilde{x}, \tilde{y})$ and $(\bar{x}, \bar{y})$, we also have

$$s(\bar{x}, \bar{y}) = s(\tilde{x}, \tilde{y}) + \|\tilde{x} - \bar{x}\|^2 + \|\tilde{y} - \bar{y}\|^2.$$

By adding the two last equalities, we obtain $\tilde{x} = \bar{x}$ and $\tilde{y} = \bar{y}$, which implies that $\{(x_n, y_n)\}$ weakly converges to the solution of (1.1). This completes the proof. □

Next, we propose the cyclic iterative algorithm for solving MSECFP (1.1) of firmly quasi-nonexpansive operators.

Algorithm 3.2 Let $x_0 \in H_1, y_0 \in H_2$ be arbitrary. For $n \geq 0$, let

$$\begin{cases} u_n = x_n - U_{i(n)}(x_n) + A^*(Ax_n - By_n), \\ x_{n+1} = x_n - \tau_n u_n, \\ v_n = y_n - T_{j(n)}(y_n) - B^*(Ax_n - By_n), \\ y_{n+1} = y_n - \tau_n v_n, \end{cases} \tag{3.18}$$

where the step size τ_n is chosen as in Algorithm 3.1.

Theorem 3.2 *The sequence* $\{(x_n, y_n)\}$ *generated by Algorithm* 3.2 *weakly converges to a solution* (x^*, y^*) *of MSECFP* (1.1). *Moreover,* $\|Ax_n - By_n\| \to 0$, $\|x_n - x_{n+1}\| \to 0$, *and* $\|y_n - y_{n+1}\| \to 0$ *as* $n \to \infty$.

Proof Let $(x^*, y^*) \in \Gamma$, we have

$$\begin{aligned} \langle u_n, x_n - x^*\rangle &= \langle x_n - U_{i(n)}(x_n), x_n - x^*\rangle + \langle A^*(Ax_n - By_n), x_n - x^*\rangle \\ &\geq \|x_n - U_{i(n)}(x_n)\|^2 + \langle Ax_n - By_n, Ax_n - Ax^*\rangle \end{aligned} \tag{3.19}$$

and

$$\langle v_n, y_n - y^*\rangle \geq \|y_n - T_{j(n)}(y_n)\|^2 - \langle Ax_n - By_n, By_n - By^*\rangle. \tag{3.20}$$

By adding the two inequalities (3.19)–(3.20) and taking into account the fact that $Ax^* = By^*$, we obtain

$$\begin{aligned}&\langle u_n, x_n - x^*\rangle + \langle v_n, y_n - y^*\rangle \\ &\quad \geq \|x_n - U_{i(n)}(x_n)\|^2 + \|y_n - T_{j(n)}(y_n)\|^2 + \|Ax_n - By_n\|^2.\end{aligned} \tag{3.21}$$

Similar to (3.8), we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 + \|y_{n+1} - y^*\|^2 \\ &\quad = \|x_n - x^*\|^2 + \|y_n - y^*\|^2 - 2\tau_n\big(\langle u_n, x_n - x^*\rangle + \langle v_n, y_n - y^*\rangle\big) \\ &\qquad + \tau_n^2\big(\|u_n\|^2 + \|v_n\|^2\big).\end{aligned} \tag{3.22}$$

We also have

$$\begin{aligned}\|u_n\|^2 &= \|x_n - U_{i(n)}(x_n) + A^*(Ax_n - By_n)\|^2 \\ &\leq 2\|x_n - U_{i(n)}(x_n)\|^2 + 2\|A^*(Ax_n - By_n)\|^2\end{aligned} \tag{3.23}$$

and

$$\|v_n\|^2 \leq 2\|y_n - T_{j(n)}(y_n)\|^2 + 2\|B^*(Ax_n - By_n)\|^2. \tag{3.24}$$

Setting $s_n(x^*, y^*) = \|x_n - x^*\|^2 + \|y_n - y^*\|^2$ and using (3.21), (3.23)–(3.24), (3.22) can be written as

$$\begin{aligned}&s_{n+1}(x^*, y^*) \\ &\quad \leq s_n(x^*, y^*) - 2\tau_n\big[\|x_n - U_{i(n)}(x_n)\|^2 + \|y_n - T_{j(n)}(y_n)\|^2 + \|Ax_n - By_n\|^2\big] \\ &\qquad + 2\tau_n^2\big[\|x_n - U_{i(n)}(x_n)\|^2 + \|y_n - T_{j(n)}(y_n)\|^2 + \|A^*(Ax_n - By_n)\|^2 \\ &\qquad + \|B^*(Ax_n - By_n)\|^2\big] \\ &\quad = s_n(x^*, y^*) - 2\tau_n(1 - \tau_n)\big[\|x_n - U_{i(n)}(x_n)\|^2 + \|y_n - T_{j(n)}(y_n)\|^2\big] \\ &\qquad - 2\tau_n\big[\|Ax_n - By_n\|^2 - \tau_n\big(\|A^*(Ax_n - By_n)\|^2 + \|B^*(Ax_n - By_n)\|^2\big)\big].\end{aligned} \tag{3.25}$$

Similar to the proof of Theorem 3.1, we have

$$\lim_{n\to\infty}\|x_n - U_{i(n)}(x_n)\| = \lim_{n\to\infty}\|y_n - T_{j(n)}(y_n)\| = 0 \tag{3.26}$$

and

$$\lim_{n\to\infty}\|Ax_n - By_n\| = 0. \tag{3.27}$$

Since

$$\begin{aligned}\|x_{n+1} - x_n\| &= \tau_n\|x_n - U_{i(n)}(x_n) + A^*(Ax_n - By_n)\| \\ &\leq \tau_n\big(\|x_n - U_{i(n)}(x_n)\| + \|A^*(Ax_n - By_n)\|\big),\end{aligned} \tag{3.28}$$

we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \tag{3.29}$$

which infers that $\{x_n\}$ is asymptotically regular. Similarly, we also have that $\{y_n\}$ is asymptotically regular, namely $\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0$.

Take $(\tilde{x}, \tilde{y}) \in \omega_\omega(x_n, y_n)$, i.e., there exists a subsequence $\{(x_{n_k}, y_{n_k})\}$ of $\{(x_n, y_n)\}$ such that $(x_{n_k}, y_{n_k}) \rightharpoonup (\tilde{x}, \tilde{y})$ as $k \to \infty$. Noting that the pool of indexes is finite and $\{x_n\}$ is asymptotically regular, for any $i \in \{1, 2, \dots, p\}$, we can choose a subsequence $\{n_{i_l}\} \subset \{n\}$ such that $x_{n_{i_l}} \rightharpoonup \tilde{x}$ as $l \to \infty$ and $i(n_{i_l}) = i$ for all l. It turns out that

$$\lim_{l\to\infty} \left\|x_{n_{i_l}} - U_i(x_{n_{i_l}})\right\| = \lim_{l\to\infty} \left\|x_{n_{i_l}} - U_{i(n_{i_l})}(x_{n_{i_l}})\right\| = 0. \tag{3.30}$$

By the same reason, for any $j \in \{1, 2, \dots, r\}$, we can choose a subsequence $\{n_{j_m}\} \subset \{n\}$ such that $y_{n_{j_m}} \rightharpoonup \tilde{y}$ as $m \to \infty$ and $j(n_{j_m}) = j$ for all m. So,

$$\lim_{m\to\infty} \left\|y_{n_{j_m}} - U_j(y_{n_{j_m}})\right\| = 0. \tag{3.31}$$

Combined with the demiclosedness of $U_i - I$ and $T_j - I$ at 0, it follows from (3.30) and (3.31) that $U_i(\tilde{x}) = \tilde{x}$ and $T_j(\tilde{y}) = \tilde{y}$ for $1 \le i \le p$ and $1 \le j \le r$. So, $\tilde{x} \in \bigcap_{i=1}^p F(U_i)$ and $\tilde{y} \in \bigcap_{j=1}^r F(T_j)$. Similar to the proof of Theorem 3.1, we can complete the proof. □

Now, we give applications of Theorem 3.1 and Theorem 3.2 to solve MSEP (1.2). Assume that the solution set S of MSEP (1.2) is nonempty. Since the orthogonal projection operator is firmly nonexpansive, by Lemma 2.4 we have the following results for solving MSEP (1.2).

Corollary 3.1 *For any given $x_0 \in H_1, y_0 \in H_2$, define a sequence $\{(x_n, y_n)\}$ by the following procedure*:

$$\begin{cases} u_n = x_n - (\alpha_n^1 P_{C_1}(x_n) + \cdots + \alpha_n^p P_{C_p}(x_n)) + A^*(Ax_n - By_n), \\ x_{n+1} = x_n - \tau_n u_n, \\ v_n = y_n - (\beta_n^1 P_{Q_1}(y_n) + \cdots + \beta_n^r P_{Q_r}(y_n)) - B^*(Ax_n - By_n), \\ y_{n+1} = y_n - \tau_n v_n, \end{cases} \tag{3.32}$$

where the step size τ_n is chosen as in Algorithm 3.1. *If* $\liminf_{n\to\infty} \alpha_n^i > 0$ $(1 \le i \le p)$ *and* $\liminf_{n\to\infty} \beta_n^j > 0$ $(1 \le j \le r)$, *then the sequence $\{(x_n, y_n)\}$ weakly converges to a solution (x^*, y^*) of MSEP* (1.2). *Moreover, $\|Ax_n - By_n\| \to 0$, $\|x_{n+1} - x_n\| \to 0$, and $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$.*

Corollary 3.2 *For any given $x_0 \in H_1, y_0 \in H_2$, define a sequence $\{(x_n, y_n)\}$ by the following procedure*:

$$\begin{cases} u_n = x_n - P_{C_{i(n)}}(x_n) + A^*(Ax_n - By_n), \\ x_{n+1} = x_n - \tau_n u_n, \\ v_n = y_n - P_{Q_{j(n)}}(y_n) - B^*(Ax_n - By_n), \\ y_{n+1} = y_n - \tau_n v_n, \end{cases} \tag{3.33}$$

where the step size τ_n is chosen as in Algorithm 3.1. *Then the sequence* $\{(x_n, y_n)\}$ *weakly converges to a solution* (x^*, y^*) *of MSEP* (1.2). *Moreover,* $\|Ax_n - By_n\| \to 0$, $\|x_{n+1} - x_n\| \to 0$, *and* $\|y_{n+1} - y_n\| \to 0$ *as* $n \to \infty$.

4 Mixed cyclic and parallel iterative algorithms

Now, for solving MSECFP (1.1) of firmly quasi-nonexpansive operators, we introduce two mixed iterative algorithms which combine the process of cyclic and simultaneous iterative methods. In our algorithms, the selection of the step size does not need any prior information of the operator norms $\|A\|$ and $\|B\|$, and the weak convergence is proved. We go on making use of assumptions (A1)–(A3).

Algorithm 4.1 Let $x_0 \in H_1, y_0 \in H_2$ be arbitrary. For $n \geq 0$, let

$$\begin{cases} u_n = x_n - (\alpha_n^1 U_1(x_n) + \cdots + \alpha_n^p U_p(x_n)) + A^*(Ax_n - By_n), \\ x_{n+1} = x_n - \tau_n u_n, \\ v_n = y_n - T_{j(n)}(y_n) - B^*(Ax_n - By_n), \\ y_{n+1} = y_n - \tau_n v_n, \end{cases} \tag{4.1}$$

where the step size τ_n is chosen in the same way as in Algorithm 3.1.

Theorem 4.1 *Assume that* $\liminf_{n\to\infty} \alpha_n^i > 0$ $(1 \leq i \leq p)$. *Then the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm* 4.1 *weakly converges to a solution* (x^*, y^*) *of MSECFP* (1.1). *Moreover,* $\|Ax_n - By_n\| \to 0$, $\|x_{n+1} - x_n\| \to 0$, *and* $\|y_{n+1} - y_n\| \to 0$ *as* $n \to \infty$.

Proof Let $(x^*, y^*) \in \Gamma$. We can get (3.5) and (3.20), so

$$\begin{aligned} &\langle u_n, x_n - x^* \rangle + \langle v_n, y_n - y^* \rangle \\ &\quad \geq \sum_{i=1}^p \alpha_n^i \| x_n - U_i(x_n) \|^2 + \| y_n - T_{j(n)}(y_n) \|^2 + \|Ax_n - By_n\|^2. \end{aligned} \tag{4.2}$$

It follows from Algorithm 4.1 that (3.8)–(3.9) and (3.24) are true. Setting $s_n(x^*, y^*) = \|x_n - x^*\|^2 + \|y_n - y^*\|^2$, we have

$$\begin{aligned} &s_{n+1}(x^*, y^*) \\ &\quad \leq s_n(x^*, y^*) - 2\tau_n(1 - \tau_n)\big[\alpha_n^1 \| x_n - U_1(x_n) \|^2 + \cdots + \alpha_n^p \| x_n - U_p(x_n) \|^2 \\ &\qquad + \| y_n - T_{j(n)}(y_n) \|^2\big] - 2\tau_n\big[\|Ax_n - By_n\|^2 - \tau_n\big(\| A^*(Ax_n - By_n) \|^2 \\ &\qquad + \| B^*(Ax_n - By_n) \|^2\big)\big]. \end{aligned} \tag{4.3}$$

By the same reason as in Theorem 3.1, we obtain that, for all i $(1 \leq i \leq p)$,

$$\lim_{n\to\infty} \| x_n - U_i(x_n) \| = 0, \tag{4.4}$$

$$\lim_{n\to\infty} \| y_n - T_{j(n)}(y_n) \| = 0 \tag{4.5}$$

and

$$\lim_{n\to\infty} \|Ax_n - By_n\| = 0. \tag{4.6}$$

So

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|y_{n+1} - y_n\| = 0, \tag{4.7}$$

which infers that $\{x_n\}$ and $\{y_n\}$ are asymptotically regular.

Take $(\tilde{x}, \tilde{y}) \in \omega_\omega(x_n, y_n)$, i.e., there exists a subsequence $\{(x_{n_k}, y_{n_k})\}$ of $\{(x_n, y_n)\}$ such that $(x_{n_k}, y_{n_k}) \rightharpoonup (\tilde{x}, \tilde{y})$ as $k \to \infty$. Noting that the pool of indexes is finite and $\{y_n\}$ is asymptotically regular, for any $j \in \{1, 2, \ldots, r\}$, we can choose a subsequence $\{n_{j_l}\} \subset \{n\}$ such that $y_{n_{j_l}} \rightharpoonup \tilde{y}$ as $l \to \infty$ and $j(n_{j_l}) = j$ for all l. It turns out that

$$\lim_{l\to\infty} \left\| y_{n_{j_l}} - T_j(y_{n_{j_l}}) \right\| = \lim_{l\to\infty} \left\| y_{n_{j_l}} - T_{j(n_{j_l})}(y_{n_{j_l}}) \right\| = 0. \tag{4.8}$$

Combined with the demiclosedness of $U_i - I$ and $T_j - I$ at 0, it follows from (4.4) and (4.7) that $U_i(\tilde{x}) = \tilde{x}$ and $T_j(\tilde{y}) = \tilde{y}$ for $1 \le i \le p$ and $1 \le j \le r$. So, $\tilde{x} \in \bigcap_{i=1}^{p} F(U_i)$ and $\tilde{y} \in \bigcap_{j=1}^{r} F(T_j)$. Similar to the proof of Theorem 3.1, we can complete the proof. □

Next, we propose another mixed cyclic and parallel iterative algorithm for solving MSECFP (1.1) of firmly quasi-nonexpansive operators.

Algorithm 4.2 Let $x_0 \in H_1, y_0 \in H_2$ be arbitrary. For $n \ge 0$, let

$$\begin{cases} u_n = x_n - U_{i(n)}(x_n) + A^*(Ax_n - By_n), \\ x_{n+1} = x_n - \tau_n u_n, \\ v_n = y_n - (\beta_n^1 T_1(y_n) + \cdots + \beta_n^r T_r(y_n)) - B^*(Ax_n - By_n), \\ y_{n+1} = y_n - \tau_n v_n, \end{cases} \tag{4.9}$$

where the step size τ_n is chosen as in Algorithm 3.1.

Similar to the proof of Theorem 4.1, we can get the following result.

Theorem 4.2 *Assume that* $\liminf_{n\to\infty} \beta_n^j > 0$ $(1 \le j \le r)$. *Then the sequence* $\{(x_n, y_n)\}$ *generated by Algorithm* 4.2 *weakly converges to a solution* (x^*, y^*) *of MSECFP* (1.1) *of firmly quasi-nonexpansive operators. Moreover,* $\|Ax_n - By_n\| \to 0$, $\|x_n - x_{n+1}\| \to 0$ *and* $\|y_n - y_{n+1}\| \to 0$ *as* $n \to \infty$.

Finally, we obtain two mixed iterative algorithms to solve MSEP (1.2). Assume that the solution set S of MSEP (1.2) is nonempty.

Corollary 4.1 *For any given $x_0 \in H_1, y_0 \in H_2$, define a sequence $\{(x_n, y_n)\}$ by the following procedure*:

$$\begin{cases} u_n = x_n - (\alpha_n^1 P_{C_1}(x_n) + \cdots + \alpha_n^p P_{C_p}(x_n)) + A^*(Ax_n - By_n), \\ x_{n+1} = x_n - \tau_n u_n, \\ v_n = y_n - P_{Q_{j(n)}}(y_n) - B^*(Ax_n - By_n), \\ y_{n+1} = y_n - \tau_n v_n, \end{cases} \tag{4.10}$$

where the step size τ_n is chosen as in Algorithm 3.1. *If* $\liminf_{n\to\infty} \alpha_n^i > 0$ $(1 \le i \le p)$, *then the sequence $\{(x_n, y_n)\}$ weakly converges to a solution (x^*, y^*) of MSEP* (1.2). *Moreover, $\|Ax_n - By_n\| \to 0$, $\|x_{n+1} - x_n\| \to 0$, and $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$.*

Corollary 4.2 *For any given $x_0 \in H_1$, $y_0 \in H_2$, define a sequence $\{(x_n, y_n)\}$ by the following procedure*:

$$\begin{cases} {}_n = x_n - P_{C_{i(n)}}(x_n) + A^*(Ax_n - By_n), \\ x_{n+1} = x_n - \tau_n u_n, \\ v_n = y_n - (\beta_n^1 P_{Q_1}(y_n) + \cdots + \beta_n^r P_{Q_r}(y_n)) - B^*(Ax_n - By_n), \\ y_{n+1} = y_n - \tau_n v_n, \end{cases} \tag{4.11}$$

where the step size τ_n is chosen as in Algorithm 3.1. *If* $\liminf_{n\to\infty} \beta_n^j > 0 (1 \le j \le r)$, *then the sequence $\{(x_n, y_n)\}$ weakly converges to a solution (x^*, y^*) of MSEP* (1.2). *Moreover, $\|Ax_n - By_n\| \to 0$, $\|x_{n+1} - x_n\| \to 0$, and $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$.*

5 Results and discussion

To avoid computing the norms of the bounded linear operators, we introduce parallel and cyclic iterative algorithms with self-adaptive step size to solve MSECFP (1.1) governed by firmly quasi-nonexpansive operators. We also propose two mixed iterative algorithms and do not need the norms of bounded linear operators. As applications, we obtain several iterative algorithms to solve MSEP (1.2).

6 Conclusion

In this paper, we have considered MSECFP (1.1) of firmly quasi-nonexpansive operators. Inspired by the methods for solving SCFP (2.1) and MSCFP (1.3), we introduce parallel and cyclic iterative algorithms for solving MSECFP (1.1). We also present two mixed iterative algorithms which combine the process of parallel and cyclic iterative methods. In our several iterative algorithms, the step size is chosen in a self-adaptive way and the weak convergence is proved.

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